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UPDATING PRIOR PARAMETERS BASED ON LIKELIHOOD FUNCTION-BAYESIAN METHOD FOR PARAMETER ESTIMATION AT HIGH MEASUREMENT UNCERTAINTY

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Abstract. *One approach in statistical analysis that distinguishes between frequentist and Bayesian is the inclusion of available prior information about the data even before measuring/ surveying the data. Many researchers argued that the inclusion of prior information resulted in a better model prediction/parameter estimation. Bayesian inference is repeatedly used in inverse problems to retrieve parameters due to the development of high efficient sampling algorithms such as Markov Chain Monte-Carlo (MCMC). Inverse problems are generally ill-posed in nature. Nevertheless, the inclusion of prior information reduces the ill-posedness of the problem to an extent. Any inverse problem relies on measured data by physical sensors, therefore induced random errors greatly affects the quality of estimation. When the uncertainty of the measured data is high, the inferences made from the resulting sampling distributions are nearly the same as the supplied prior information. The reason is that, probability of samples nearer to prior information is more at higher uncertainties of measurement and has more chance to get repeatedly accepted than the samples that are close to actual value. Therefore, in this work an effort is made to update the prior hyper parameters in each iterations of the MCMC algorithm based on the history of the likelihood function. The applicability of method is demonstrated by retrieving parameters from 1-D fin experiment. Three thermal properties such as thermal conductivity, heat transfer coefficient and emissivity are retrieved simultaneously. The estimation is carried out for both error and errorless temperature measurements and the results show that, the estimated parameters with the proposed method are in excellent agreement with the true parameter value and a maximum of 7% deviation occurs in estimating heat transfer coefficient at a measurement error of* $\pm 0.3K$

1 INTRODUCTION

In recent years, Bayesian inference is predominantly used in many engineering, science, social and behavioral sciences due to the advancement in high computing resources and efficient sampling algorithms. The result of Bayesian is a probabilistic statement regarding the unknown quantities unlike a single point estimate in other methods e.g., least-square parameter estimation. Bayesian treats the measured/surveyed data as a random variable and hence the inferences made by Bayesian are random variables. The uniqueness about Bayesian inference lies in that, a prior knowledge about the unknown parameters could be included in the statistical analysis of the data. The inclusion of prior information in inverse problems reduces the ill-posed nature of the problem and improves the quality of estimation. In general, prior information in the form of Gaussian distribution is commonly employed in inverse heat transfer literatures. Parathasarathy and Balaji [1] studied the different forms of prior model and its effects in the estimation of single and multiple parameters. In their work, they demonstrated that in multi-parameter estimation, Bayesian inference tends to point to alternate feasible solutions when highly correlated parameters are retrieved using non-informative prior models.

Monte-Carlo sampling techniques are widely practiced in order to find the expectations of the probability distribution functions. Among several Monte-Carlo techniques, Markov Chain Monte-Carlo (MCMC) is more frequently used in inverse heat transfer problems [2]-[4]. The hyper parameters (mean and standard deviation) of the prior model are usually taken from previously published results/previous experiments. More frequently these prior parameters are known vaguely. Under such case, when correlated parameters are estimated or when the measurement uncertainty is high (more likely to occur due to unavoidable random errors) the estimated parameters using sampling algorithms are close to the subjected prior parameters.

Single parameter estimation (inverse estimation of thermal conductivity) is carried out at different levels of measurement uncertainty. Refer section 2 and 3 for problem specification and inverse formulation. MCMC sampler is used to sample through the posterior. The estimation results reveal that, as the measurement uncertainty increases, the estimated parameter (mean of the posterior distribution) becomes closer and closer to the subjected prior mean. The reason for this could be explained intuitively as follows: consider the case (a) wherein the generated sample is close to actual parameter. In this case, at low uncertainty the likelihood probability is relatively larger than prior probability, so the combined probability (posterior probability) is more likely greater than the acceptance ratio (refer MCMC algorithm) and hence the sample is more frequently accepted. However, at high uncertainties the likelihood probability is slightly lesser or equal to prior probability, so the combined probability is likely lesser than the acceptance ratio and hence the sample is frequently rejected even though it is close to actual parameter. Now, consider the case (b) wherein the generated sample is close to prior parameter (mean). In this case, at low uncertainty the likelihood probability is relatively less compared to prior probability, so the combined probability is more likely lesser than acceptance ratio and hence the sample is more frequently rejected. However, at higher uncertainties the likelihood probability is slightly lesser or equal to prior probability, so the combined probability is likely greater than the acceptance ratio and hence the sample is accepted frequently even though it is away from actual parameter (nearer to prior mean). Therefore, in this work an attempt is made to modify the prior parameters in each iteration of the MCMC algorithm based on the history of likelihood function to overcome the aforesaid difficulty and study the applicability of modified algorithm to multi-parameter estimation related to inverse heat transfer applications.

2 DEFINITION OF FORWARD PROBLEM

A simple 1-D fin losing heat to the surrounding both by convection and radiation is considered as the forward problem. Mathematically, the forward problem is represented as,

$$k\frac{\partial^2 T}{\partial x^2} - hA_s(T - T_\infty) - \epsilon\sigma A_s(T^4 - T_\infty^4) = C_p\frac{\partial T}{\partial t} \qquad in\ 0 \le x \le L \tag{1}$$

subjected to the following initial and boundary conditions:

$$T(x,0) = T_i = T_\infty \qquad 0 \le x \le L$$

at $x = 0$,

$$q = \begin{cases} q_0 & 0 \le t \le t_0 \\ 0 & t > t_0 \end{cases}$$

In equation 1 k, ϵ and C_p are fin material properties. i.e., thermal conductivity (W/mK), surface emissivity and heat capacity (J/K) respectively. A_s (m^2) is lateral surface of the fin and L (m) is the length of the fin. A step heat input of magnitude q_0 is given at one end ($x = 0$) for time t_0. The experiment is carried out in an ambient of temperature T_∞(K) and heat transfer coefficient, h (W/m^{2}K)

The solution to equation 1 is the temperature distribution along the length of the fin w.r.t. time. In this work, Finite Volume Method (FVM) is employed to solve the forward problem [5] in which energy balance is applied to each discretized control volume in order to convert the above partial differential equation into set of algebraic equations. Because of the radiation heat loss term, equation 1 is non-linear and hence the discretized equations are also non-linear. Therefore, they are solved simultaneously using Gauss-Seidel iterative technique.

3 BAYESIAN INVERSE FORMULATION FOR PARAMETER ESTIMATION

The objective of the Bayesian inference in inverse problems is to formulate the *posterior probability distribution (ppdf)*, which in Bayesian context of linear/non-linear parameter estimation is *proportional* to product of *likelihood function and prior probability distribution*.

$$p(X|Y) \propto p(Y|X)p(X) \tag{2}$$

In equation 2 X is the unknown parameter vector to be estimated and Y is the measured/ surveyed data. The LHS of equation 2 is the required *ppdf*, whereas the first and second term of RHS is the *likelihood function* and *prior distribution* respectively. Without loss generosity, it is assumed that the errors between the measured and simulated variable is *additive, uncorrelated, normally distributed with zero mean* and *constant standard deviation* and hence the *likelihood function* is given by [6, 7].

$$p(Y|X) = \frac{1}{\sqrt{2\pi}}|V|^{-1/2}exp\left\{\frac{-1}{2}[Y - Y(X)]^T V^{-1}[Y - Y(X)]\right\} \tag{3}$$

In equation 3 Y is the measured quantity, $Y(X)$ is the simulated quantity for the assumed unknown parameter X and V is the covariance matrix of the measurement. As previously mentioned, Bayesian exploits all the information that are available even before the measurements are taken, this information is usually included in the analysis in the form of *prior distribution p(X)* as given in equation 2. In this work, a normal prior model is assumed and hence it is given by,

$$p(X) = \frac{1}{\sqrt{2\pi}}|V_p|^{-1/2}exp\left\{\frac{-1}{2}[X-\mu_p]^T V_p^{-1}[X-\mu_p]\right\} \tag{4}$$

In equation 4 μ_p and V_p are hyper parameters of the prior distribution (mean and covariance) respectively. Substituting 3 and 4 in equation 2 we get the desired *ppdf*. For the present parameter estimation problem the unknown parameters are thermal conductivity, heat transfer coefficient and emissivity i.e., $X = \{k, h, \epsilon\}$ and the measurable quantity is the temperature as a function of space and time $T(x, t)$.

3.1 MCMC algorithm for updating prior parameter

The next step in Bayesian inference is to calculate the appropriate statistic (mean/mode/ median/maximum a posterior/and standard deviation) of the unknown parameters using the posterior distribution. However, in inverse heat transfer literatures *mean/maximum a posterior* and *standard deviation* of the unknown parameters are more commonly reported. Monte-Carlo sampling methods are being widely used for this purpose, among which *Markov Chain Monte-Carlo (MCMC)* is more repeatedly used in the literatures. An excellent discussion is presented in [8] as in why to use sampling techniques over the analytical/numerical methods in order to find the *expectations of ppdf*. In the present work, the simple MCMC algorithm is slightly modified to update prior hyper parameters based on the history of likelihood function to overcome the convergence of estimated parameters to the prior mean at high measurement uncertainties. One iteration of the algorithm consist of the following steps,

1. Select a new sample, X^* from the proposal/jumping distribution $q(X^*|X^{i-1})$
2. Find the *ppdf* for new sample, p^* using 2
3. Accept the new sample based on *Metropolis Hasting (MH)* ratio, if $r \leq U$ otherwise reject the sample. Here U is a uniformly distribute random number and
$$r = min\left\{1, \frac{p^* \times q(X^*|X^{i-1})}{p^{i-1} \times q(X^{i-1}|X^*)}\right\}$$
4. Find the parameter vector X, corresponding to *minimum likelihood function* from the chain length of 1 to i-1 iterations and add a normally distributed random number (-1 to 1). Assign it as the new hyper parameter for the prior distribution.
5. Return to step 2 and continue until the number of iterations exceeds the specified limit

It is worth to mention here that in the above algorithm except step 4, all other steps are same as simple MCMC algorithm.

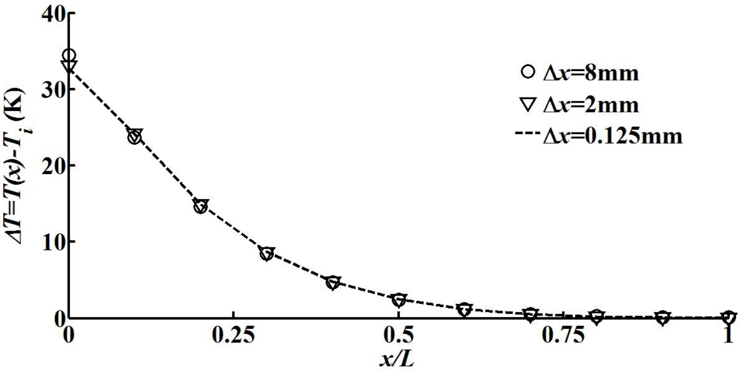

Figure 1: Grid independent study.

4 RESULTS AND DISCUSSION

The solution to the forward problem using FVM is written in Matlab and it is validated with commercial FVM based software package Fluent. A grid independent study is carried out and based on the study, a grid size of $\Delta x = 2$mm is chosen for further studies (refer Fig. 1).

The fin material is assumed to be steel and hence its thermo-physical properties are considered in the analysis for demonstration. It is also assumed that the temperature measurement is carried out in a natural environment; therefore the convective heat transfer coefficient is taken as 10W/m^{2}K. The duration and magnitude of heat input, total time of experiment and number of temperature measurements are decided based on D-optimal test [9]. The results of these preliminary studies are not presented here, since it is not the objective of the present work.

The results of multi-parameter estimation (k=25W/mk, h=10W/m^{2}K and ϵ=0.85) using both simple and present modified MCMC algorithm are presented for an error level of $\pm$0.03K and $\pm$0.3K respectively. The noisy temperature measurements are obtained by adding random errors with zero mean and standard deviation ($\pm$0.03/0.3K) to the solution of forward model.

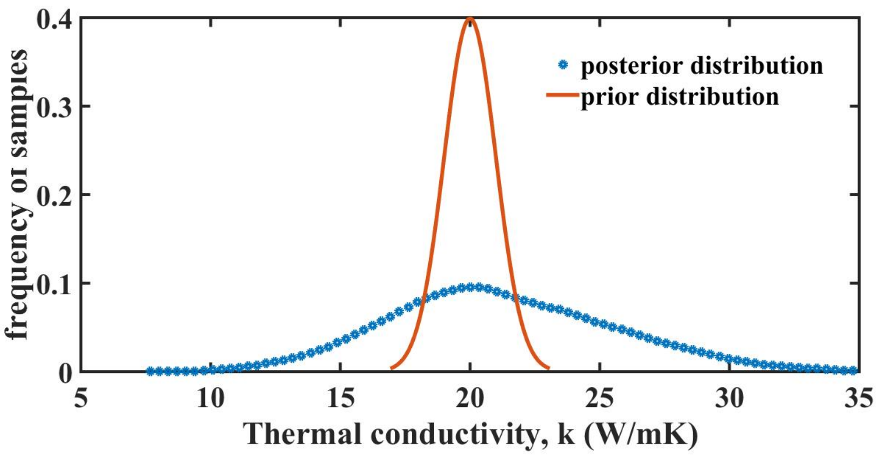

Figure 2: Sampling distribution Vs prior distribution for k (W/mK) obtained using simple MCMC algorithm.

Throughout the study the considered prior mean (μ_p) for k, h and ϵ are 20W/mK, 5W/m^{2}K and 0.6 respectively with standard deviation equal to $0.15\mu_p$. This multi-parameter estimation is especially difficult due to two reasons: (1) the posterior distribution is correlated and (2) due to high measurement errors the inverse problem becomes more ill-posed. Also the combination of above two makes the estimation highly difficult.

Figure (2-4) shows the sampling distribution for k, h and ϵ respectively at standard deviation $\pm$0.3K obtained using simple MCMC algorithm and from these figures it is clear that the *sampling distribution* for all parameters are very close to prior distribution.

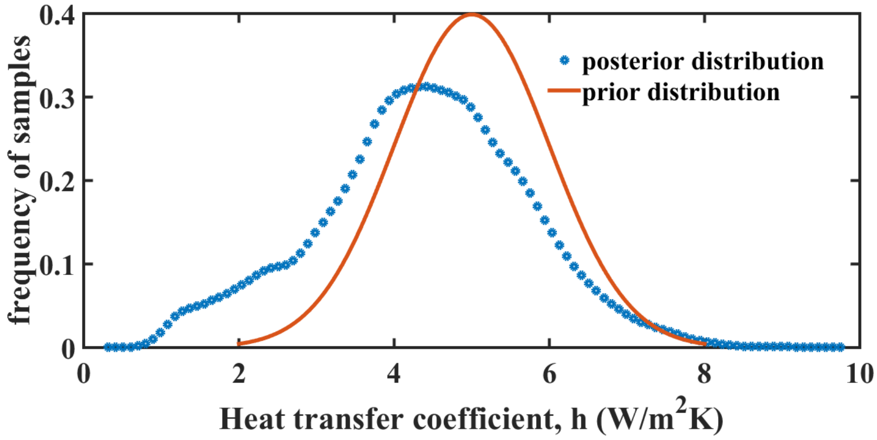

Figure 3: Sampling distribution Vs prior distribution for h (W/m^{2}K) obtained using simple MCMC algorithm.

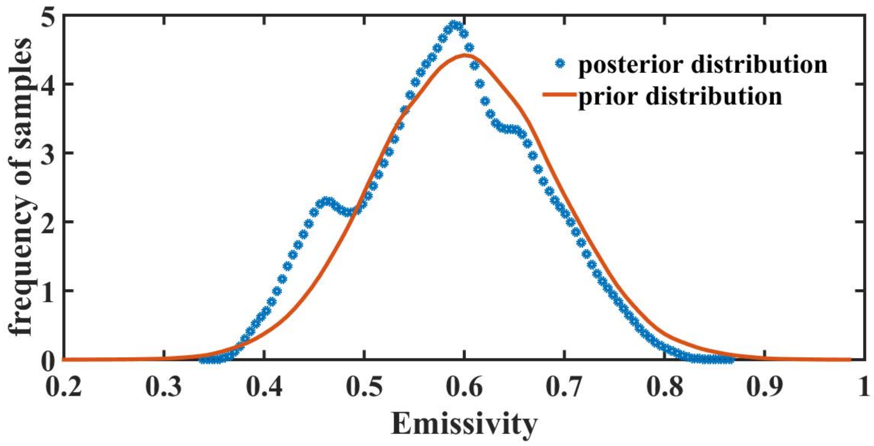

Figure 4: Sampling distribution Vs prior distribution for ϵ obtained using simple MCMC algorithm.

This parameter estimation problem is repeated again using the present modified MCMC algorithm and the resulting sampling distributions for all parameters at standard deviation $\pm$0.3K are shown in Fig. (5-7) and from these figures it is clear that the resulting *sampling distribution* for all parameters are normally distributed with mean close to the actual value for all the parameters. The estimates of the parameters ($\hat{k}$,$\hat{h}$ and $\hat{\epsilon}$) and spread of the samples ($\hat{\sigma}$) are calculated from the sampling distribution that are obtained using both simple and modified MCMC algorithms and are listed in table 1 and 2 respectively. Table 2 clearly shows that the estimates obtained using modified MCMC algorithm are in good agreement with the exact value even at higher level of measurement error (±0.3K). In contrary, the estimates obtained using simple MCMC are in good agreement with the prior parameter, but far away from the true parameter (refer table 1)

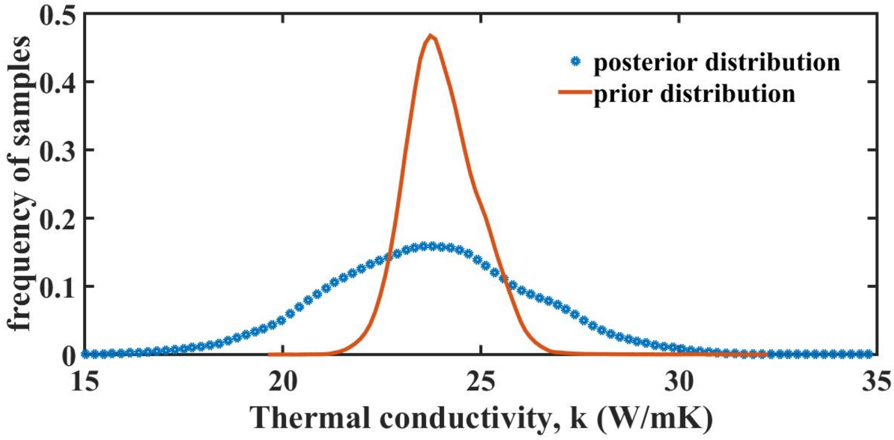

Figure 5: Sampling distribution Vs prior distribution for k (W/mK) obtained using present modified MCMC algorithm.

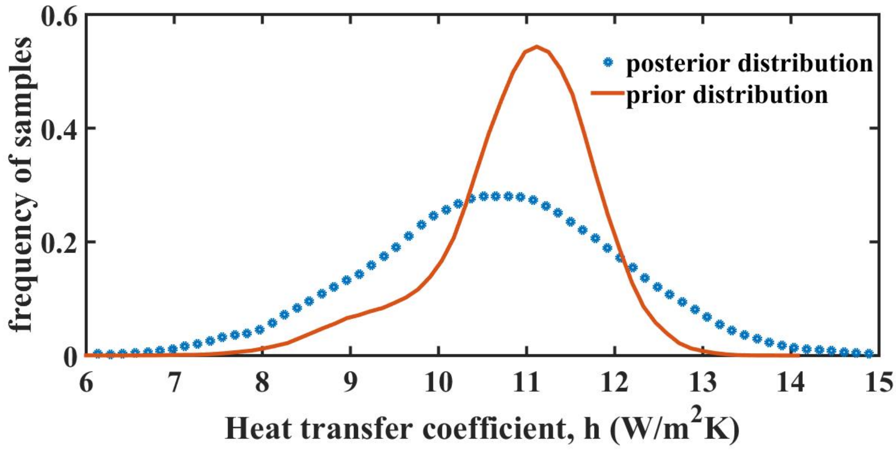

Figure 6: Sampling distribution Vs prior distribution for h (W/m^{2}K) obtained using present modified MCMC algorithm.

	k=25W/mK			h=10W/m^{2}K			ϵ=0.85		
	$\hat{k}$	$\hat{\sigma}_k$	% error	$\hat{h}$	$\hat{\sigma}_h$	% error	$\hat{\epsilon}$	$\hat{\sigma}_\epsilon$	% error
±0.03K	26.78	1.65	7.12	7.54	1.03	24.6	0.811	0.066	4.59
±0.3K	21.25	4.39	15	4.38	1.34	56.2	0.583	0.088	31.41

Table 1: Mean and standard deviation of the parameters calculated from sampling distribution obtained using simple MCMC.

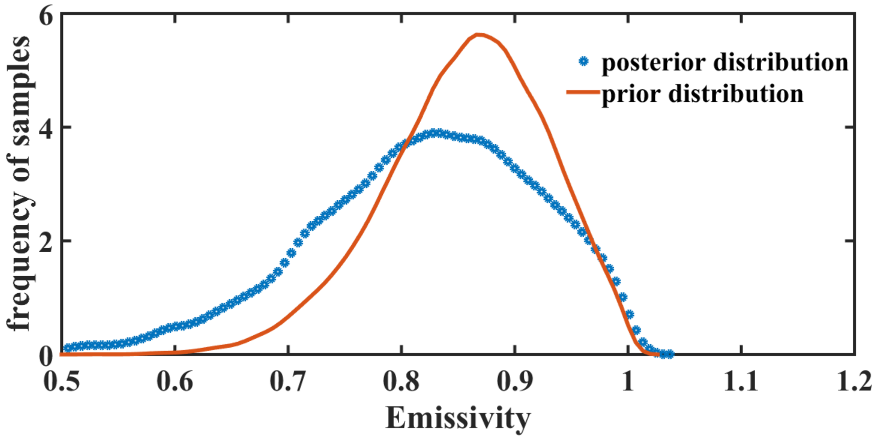

Figure 7: Sampling distribution Vs prior distribution for ϵ obtained using present modified MCMC algorithm.

	k=25W/mK			h=10W/m^{2}K			ϵ=0.85		
	$\hat{k}$	$\hat{\sigma}_k$	% error	$\hat{h}$	$\hat{\sigma}_h$	% error	$\hat{\epsilon}$	$\hat{\sigma}_\epsilon$	% error
±0.03K	24.92	1.4	0.32	9.38	1.03	6.2	0.862	0.066	1.41
±0.3K	23.71	2.6	5.16	10.61	1.5	6.1	0.821	0.099	3.41

Table 2: Mean and standard deviation of the parameters calculated from sampling distribution obtained using modified MCMC.

5 CONCLUSIONS

In this work, a modified MCMC algorithm was proposed in which the hyper parameter of the prior distribution was updated in each iteration based on the history of likelihood function. The idea of modifying prior parameter was put forth in order to restrict the sampling distribution moving close to the prior distribution at higher measurement uncertainties.

The proposed modified MCMC algorithm was tested for its reliability by employing it in inverse heat transfer problem, wherein three thermal properties were estimated simultaneously (the posterior for which is correlated). The estimation was carried out for both error and error less temperature measurements. The estimated parameters with modified MCMC were in good agreement with the true parameters, whereas with simple MCMC the estimated parameters were close to the supplied prior information at ±0.3K measurement uncertainty. Since the posterior is correlated, the error associated with the parameter estimation using simple MCMC are more and a maximum deviation of 24.6% occurred in estimating h even at zero error level. Nevertheless, with modified MCMC, all the parameters were estimated with less than 7% deviation from the actual parameter value. Therefore, modified MCMC algorithm could be effectively used in inverse problems to estimate parameters from correlated posterior distribution even at high level of measurement errors.

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