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Tucker's theorem for almost skew-symmetric matrices and a proof of Farkas' lemma

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ABSTRACT

A real square matrix A is said to be almost skew-symmetric if its symmetric part has rank one. In this article certain fundamental questions on almost skew-symmetric matrices are considered. Among other things, necessary and sufficient conditions on the entries of a matrix in order for it to be almost skew-symmetric are presented. Sums and subdirect sums are studied. Certain new results for the Moore–Penrose inverse of an almost skew-symmetric matrix are proved. An interesting analogue of Tucker's theorem for skew-symmetric matrices is derived for almost skew-symmetric matrices. Surprisingly, this analogue leads to a proof of Farkas' lemma.

1. Introduction

In this article, we study matrices $A \in \mathbb{R}^{n\times n}$ which have the property that their symmetric part is of rank one. Such matrices are called almost skew-symmetric. The motivation for this notion seems to have come from tournament matrices and their

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extensions. The spectra of almost skew-symmetric matrices were also considered in the literature. We refer the reader to [8] and the references cited therein, for these details. There, certain interesting results on eigenvalues and numerical range of almost skew-symmetric matrices are also derived. In [9], the authors study inheritance properties of almost skew-symmetry by the Schur complement and a generalized principal pivot transform.

The outline of the present work is as follows: This introductory section is followed by a short section on the preliminary results that will be required in the rest of the article. This includes some terminology. In Section 3, first we derive a fundamental result on the structure of an almost skew-symmetric matrix by giving certain conditions for the entries of the matrix. This is presented in Theorem 3.3. We then present a characterization for the sum of two almost skew-symmetric matrices to be almost skew-symmetric, in Theorem 3.4. A similar question for the subdirect sum is considered next. An answer is provided in Theorem 3.5. A partial converse is proved in Theorem 3.6. Section 4 presents generalizations of some results for invertible almost skew-symmetric matrices to the case of the Moore–Penrose inverse. These are given in Theorem 4.1, Theorem 4.2 and Theorem 4.3. For a skew-symmetric matrix A, a result of Tucker [11, Theorem 5] asserts that, there exists a nonnegative vector u such that Au is nonnegative and the vector $Au + u$ is strictly positive. This is widely referred to as Tucker's theorem in the literature. Broyden has shown that this theorem is equivalent to Farkas' lemma, a very well known theorem of alternative. In Section 5, a version of Tucker's theorem for almost skew-symmetric matrices is presented in Theorem 5.2. Quite surprisingly, we are able to prove Farkas' lemma from this result. We conclude the article by mentioning certain preservers of almost skew-symmetry.

2. Preliminaries

Let $\mathbb{R}^{n\times n}$ denote the set of all $m \times n$ matrices over the real numbers. For $A \in \mathbb{R}^{n\times n}$ let $S(A)$, $K(A)$, $R(A)$, $N(A)$ and $rk(A)$ denote the symmetric part of A ($\frac{1}{2}(A+A^t)$), the skew-symmetric part of A ($\frac{1}{2}(A-A^t)$), the range space of A, the null space of A and the rank of A. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ let $diag(A)$ denote the column vector of diagonal entries of A: $diag(A) = (a_{11}, a_{22}, \dots, a_{nn})^t$. The Moore–Penrose inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying $A = AXA$, $X = XAX$, $(AX)^T = AX$ and $(XA)^T = XA$ and is denoted by $A^\dagger$. The group inverse of a matrix $A \in \mathbb{R}^{n\times n}$, if it exists, is the unique matrix $X \in \mathbb{R}^{n\times n}$ satisfying $A = AXA$, $X = XAX$ and $AX = XA$ and is denoted by $A^\#$. If A is nonsingular, then $A^{-1} = A^\dagger = A^\#$. Recall that $A \in \mathbb{R}^{n\times n}$ is called range-symmetric if $R(A) = R(A^t)$. If A is range-symmetric, then $A^\dagger = A^\#$ [2, Theorem 4, p. 157]. $A \in \mathbb{R}^{n\times n}$, $n \geq 2$ is called an almost skew-symmetric matrix if $rk(S(A)) = 1$, where $S(A)$ is the symmetric part of A. It follows at once that A is almost skew-symmetric if and only if A^T is almost skew-symmetric. The nonzero eigenvalue of $S(A)$ is denoted by $\delta(A)$. In the remainder of the article we assume that $\delta(A) > 0$; otherwise, our results are applicable to $-A$. It follows that if A is an almost skew-symmetric matrix then

$S(A) = ww^t$ for some $w \in \mathbb{R}^n$. It then follows that $x^tS(A)x = x^tww^tx = (w^tx)^2 \geq 0$ for all $x \in \mathbb{R}^n$. Thus the symmetric part of an almost skew-symmetric matrix is positive semidefinite. Given an almost skew-symmetric matrix $A \in \mathbb{R}^{n\times n}$ with $S(A) = ww^t$, the variance of A is denoted by $V(A) = \frac{\|K(A)w\|^2}{\|w\|^2}$, where $\| . \|$ denotes the Euclidean norm. Given $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, let $A[i, j]$ denote the 2×2 submatrix of A obtained by excluding all the elements of the rows and columns of A indexed by $1, 2, \dots, i-1, i+1, \dots, n$:

$$A[i,j] = \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}.$$

The *k-subdirect* sum of two matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}^{m\times m} \text{ and } B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathbb{R}^{n\times n},$$

in which $A_{22}, B_{11} \in \mathbb{R}^{k\times k}$ and $0 \leq k \leq m, n$ is denoted and defined as

$$A \oplus_k B = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{pmatrix}.$$

Next, we list a couple of theorems to be used later. The first result is Tucker's theorem for a skew-symmetric matrix. We use the notation $x \geq 0$ for $x \in \mathbb{R}^n$ to denote the fact that the coordinates of x are nonnegative. $x > 0$ will signify that all the coordinates are positive.

Theorem 2.1. *(See [11, Theorem], [4, Theorem 3.4].) Let $M \in \mathbb{R}^{n\times n}$ be an arbitrary skew-symmetric matrix. Then there exists $x \geq 0$ such that $Mx \geq 0$ and $Mx + x > 0$.*

The next result recalls the formula for the Moore–Penrose inverse of a structured rank-one perturbation. We shall refer to this as the generalized Sherman–Morrison–Woodbury formula.

Theorem 2.2. *(See [1, Theorem 2.1].) Let $A \in \mathbb{R}^{m\times n}$, $b \in R(A)$, $c \in R(A^t)$. Let M be a rank-one perturbation of A in the form $M = A + bc^t$ and let $\lambda = 1 + c^tA^\dagger b \neq 0$. Then $M^\dagger = A^\dagger - \lambda^{-1}A^\dagger bc^tA^\dagger$.*

3. Structure of almost skew-symmetric matrices, sums and subdirect sums

If $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix then A is invertible. We have the following generalization for a positive semidefinite matrix. This extension has been proved in [3] and [7]. However due to its importance and for the sake of completeness, we provide a proof of it.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a positive semidefinite matrix. Then A is range-symmetric. In particular, $A^{\#}$ exists and $A^{\#} = A^{\dagger}$.*

Proof. Let $Ax = 0$. Then $x^tAx = 0$ so that $x^tAx + x^tA^tx = 0$. Thus $x^t(A + A^t)x = 0$. Let $C = A + A^t$. Then C is symmetric and positive semidefinite, since A is positive semidefinite. Therefore, there exists $S \in \mathbb{R}^{n\times n}$ such that $C = SS^t$. So, $x^tSS^tx = 0$ so that $S^tx = 0$. Thus $0 = Cx = (A + A^t)x$. Thus $A^tx = 0$. So $N(A) \subseteq N(A^t)$. Similarly, we can show that $N(A^t) \subseteq N(A)$. So $N(A) = N(A^t)$. Hence A is range-symmetric. It now follows that $A^{\#}$ exists and $A^{\#} = A^{\dagger}$. □

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$ be an almost skew-symmetric matrix. Then A is positive semidefinite.*

Proof. Suppose that A is almost skew-symmetric. Then $S(A) = ww^t$ for some $w \in \mathbb{R}^n$. For every $x \in \mathbb{R}^n$, since $K(A)$ is skew-symmetric, we have $x^tK(A)x = 0$. So, for every $x \in \mathbb{R}^n$,

$$x^tAx = x^tww^tx + x^tK(A)x = (w^tx)^t(w^tx) \geq 0.$$

Hence A is positive semidefinite. □

Remark 3.1. From the results above, it follows that the group inverse and the Moore–Penrose inverse of an almost skew-symmetric matrix coincide.

Theorem 3.3. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. If A is an almost skew-symmetric matrix, then the following hold:*

(i) All the diagonal elements of A are nonnegative.
(ii) At least one diagonal element of A is positive.
(iii) If a diagonal element $a_{ii} = 0$ then $a_{ik}+a_{ki} = 0$ for all $k = 1, 2, \ldots, i-1, i+1, \ldots, n$.
(iv) If two diagonal elements are nonzero, say $a_{ii} \neq 0$ and $a_{jj} \neq 0$, then $a_{ij}+a_{ji} = 2ka_{ii}$ where $k^2 = a_{jj}/a_{ii}$.

Conversely, if A satisfies all the conditions as above, then A is an almost skew-symmetric matrix.

Proof. Suppose $A = [a_{ij}]$ is almost skew-symmetric. Then A is positive semidefinite. So, $a_{ii} = e_i^tAe_i \geq 0$. Hence (i) is true.

(ii) Let us suppose that all the diagonal elements are zero. Then $tr(S(A)) = tr(A) = 0$. But $\delta(A) = tr(S(A))$ and $\delta(A) > 0$, a contradiction. Hence at least one diagonal element of A is positive.

(iii) Suppose that the $a_{ii} = 0$ for some i and suppose that there exists $k \in \{1, 2, \dots, i-1, i+1, \dots, n\}$ such that $a_{ik} + a_{ki} \neq 0$. Consider the submatrix

$$A[i,k] = \begin{pmatrix} a_{ii} & a_{ik} \\ a_{ki} & a_{kk} \end{pmatrix}.$$

The symmetric part of this submatrix is

$$S(A[i,k]) = \begin{pmatrix} a_{ii} & \frac{a_{ik}+a_{ki}}{2} \\ \frac{a_{ik}+a_{ki}}{2} & a_{kk} \end{pmatrix}.$$

Since $a_{ii} = 0$ and $a_{ik} + a_{ki} \neq 0$, we have $rk(S(A[i,j])) = 2$. Then $rk(S(A)) \geq 2$, a contradiction. Hence (iii) holds.

(iv) Suppose that $a_{ii} \neq 0$ and $a_{jj} \neq 0$ for some i and j. Consider the submatrix

$$A[i,j] = \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}.$$

The symmetric part of this submatrix is

$$S(A[i,j]) = \begin{pmatrix} a_{ii} & \frac{a_{ij}+a_{ji}}{2} \\ \frac{a_{ij}+a_{ji}}{2} & a_{jj} \end{pmatrix}.$$

Since $rk(S(A)) = 1$, we must have $rk(S(A[i,j])) = 0$ or 1. But $a_{ii} \neq 0$ and $a_{jj} \neq 0$, and so $rk(S(A[i,j])) = 1$. Thus there exists a nonzero scalar k such that $a_{ij} + a_{ji} = 2ka_{ii}$ and $a_{jj} = 2k(a_{ij} + a_{ji})$. Hence $a_{ij} + a_{ji} = 2ka_{ii}$ where $k^2 = a_{jj}/a_{ii}$.

Conversely, suppose that $A = [a_{ij}]$ satisfies all the four properties as above. We shall show that $S(A)$ has rank 1. From (i), all the diagonal elements of A are nonnegative. From (ii), A has at least one positive diagonal element and if any diagonal element is zero then by (iii), all the entries of the corresponding row and column of $S(A)$ are zero. So these rows and columns of $S(A)$ could be deleted without affecting the rank of $S(A)$. So, without loss of generality, we may assume that all the diagonal elements are positive. Also from (iv), we have:

$$a_{22} = k_1^2 a_{11} \text{ and } \frac{a_{12} + a_{21}}{2} = k_1 a_{11},$$
$$a_{33} = k_2^2 a_{11} \text{ and } \frac{a_{13} + a_{31}}{2} = k_2 a_{11},$$
$$a_{33} = (\frac{k_2}{k_1})^2 a_{22} \text{ and } \frac{a_{23} + a_{32}}{2} = \frac{k_2}{k_1} a_{22} = k_1 k_2 a_{11}.$$

By induction we then have, for $j = 2, 3, 4, \dots, n$,

$$a_{jj} = k_{j-1}^2 a_{11} \text{ and } \frac{a_{1j} + a_{j1}}{2} = k_{j-1} a_{11},$$

$$a_{jj} = (\frac{k_{j-1}}{k_1})^2 a_{22} \text{ and } \frac{a_{j2}+a_{2j}}{2} = k_1 k_{j-1} a_{11},$$
$$a_{jj} = (\frac{k_{j-1}}{k_2})^2 a_{33} \text{ and } \frac{a_{3j}+a_{j3}}{2} = k_2 k_{j-1} a_{11}$$

and so on. So,

$$a_{jj} = (\frac{k_{j-1}}{k_{j-2}})^2 a_{j-1j-1} \text{ and } \frac{a_{j-1j}+a_{jj-1}}{2} = k_{j-2} k_{j-1} a_{11}.$$

Thus, $S(A)$ is given by:

$$S(A) = \begin{pmatrix} a_{11} & k_1 a_{11} & k_2 a_{11} & \dots & k_{n-1} a_{11} \\ k_1 a_{11} & k_1^2 a_{11} & k_1 k_2 a_{11} & \dots & k_1 k_{n-1} a_{11} \\ k_2 a_{11} & k_2 k_1 a_{11} & k_2^2 a_{11} & \dots & k_2 k_{n-1} a_{11} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ k_{n-1} a_{11} & k_{n-1} k_1 a_{11} & k_{n-1} k_2 a_{11} & \dots & k_{n-1}^2 a_{11} \end{pmatrix}. \tag{1}$$

Thus $rk(S(A)) = 1$. Hence A is almost skew-symmetric. $\square$

Remark 3.2. Let $a_{11}, a_{22}, \dots, a_{nn}$ be nonnegative real numbers, with at least one of these being nonzero. Using the theorem above, we can construct an almost skew-symmetric matrix A with $a_{11}, a_{22}, \dots, a_{nn}$ as its diagonal elements. If $a_{ii} = 0$, we choose any a_{ik} and a_{ki} such that $a_{ik} + a_{ki} = 0$ for $k = 1, 2, \dots, n$. If $a_{ii} \neq 0$ and $a_{jj} \neq 0$ then we take any a_{ij} and a_{ji} such that $a_{ij} + a_{ji} = k a_{ii}$, where $k^2 = \frac{a_{jj}}{a_{ii}}$. Thus $A = (a_{ij})$ is an almost skew-symmetric matrix. If all a_{ii} are positive then $S(A)$ is of the form (1), and each k_i has two possible values and so $S(A)$ has 2^{n-1} possible choices.

Remark 3.3. The set of all skew-symmetric matrices is topologically closed. However, the set of all almost skew-symmetric matrices does not share this property. Let $A_n = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & 0 \end{pmatrix}$, $n \in \mathbb{N}$. Then A_n is an almost skew-symmetric matrix for each n and converges to the zero matrix, which is not almost skew-symmetric. The sum of two almost skew-symmetric matrices may not be an almost skew-symmetric matrix. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Both A and B are almost skew-symmetric matrices but $A + B$ is not. In the next result, we characterize those almost skew-symmetric matrices A and B having the property that $A + B$ is also almost skew-symmetric.

Theorem 3.4. *Let $A, B \in \mathbb{R}^{n \times n}$ be two almost skew-symmetric matrices with $\delta(A) > 0$ and $\delta(B) > 0$. Suppose that $S(A) = uu^t$ and $S(B) = ww^t$ for some $u, w \in \mathbb{R}^n$. Then $A + B$ is an almost skew-symmetric matrix if and only if u and w are linearly dependent.*

Proof. Let $w = \alpha u$ where $\alpha \in \mathbb{R}$. Then $S(A+B) = S(A) + S(B) = (1+\alpha^2)uu^t$. Thus $rk(S(A+B)) = 1$. Hence $A+B$ is an almost skew-symmetric matrix. Conversely, let $A+B$ be an almost skew-symmetric matrix. Then for some $v \in \mathbb{R}^n$,

$$vv^t = S(A+B) = S(A) + S(B) = uu^t + ww^t.$$

It now follows that u and w are multiples of v. $\square$

Corollary 3.1. *Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be two almost skew-symmetric matrices with $\delta(A) > 0$ and $\delta(B) > 0$. Then $A+B$ is an almost skew-symmetric matrix if and only if $diag(A)$ and $diag(B)$ are linearly dependent.*

Proof. Let $diag(B) = \alpha diag(A)$ where $\alpha \in \mathbb{R}$. From Theorem 3.3, $diag(A)$ and $diag(B)$ are nonnegative and have at least one positive coordinate. So $\alpha > 0$. By the same result, it follows that, $S(B) = \alpha S(A)$. Thus $S(A+B) = (1+\alpha)S(A)$ and so $rk(S(A+B)) = 1$. Hence $A+B$ is an almost skew-symmetric matrix. Conversely, let $A+B$ be an almost skew-symmetric matrix and let $S(A) = uu^t$ and $S(B) = ww^t$ for some $u, w \in \mathbb{R}^n$. By Theorem 3.4, $w = \alpha u$ where $\alpha \in \mathbb{R}$. Thus $diag(B) = (w_1^2, w_2^2, \ldots, w_n^2)^t = \alpha^2(u_1^2, u_2^2, \ldots, u_n^2)^t = \alpha^2 diag(A)$, showing that $diag(A)$ and $diag(B)$ are linearly dependent. $\square$

Corollary 3.2. *Let $A, B \in \mathbb{R}^{n\times n}$ be two almost skew-symmetric matrices with $\delta(A) > 0$ and $\delta(B) > 0$. Suppose that $S(A) = uu^t$ and $S(B) = ww^t$ for some $u, w \in \mathbb{R}^n$. Then $A-B$ is an almost skew-symmetric matrix if and only if u and w are linearly dependent with $u \neq \pm w$.*

Proof. Let $w = \alpha u$ where $\alpha \in \mathbb{R}$ and $\alpha \neq \pm 1$. Then $S(A-B) = S(A) - S(B) = (1-\alpha^2)uu^t$. Thus $rk(S(A-B)) = 1$. Hence $A-B$ is almost skew-symmetric. Conversely, let $A-B$ be an almost skew-symmetric matrix. If $\delta(A-B) > 0$ then for some $0 \neq v \in \mathbb{R}^n$, $vv^t = S(A-B) = S(A) - S(B) = uu^t - ww^t$. Again, it follows that u and w are linearly dependent. If $\delta(A-B) < 0$ we consider $B-A$ and a similar argument shows that u and w are linearly dependent. $\square$

Remark 3.4. It is easy to see that the subdirect sum of two skew-symmetric matrices is skew-symmetric. However, the subdirect sum of two almost skew-symmetric matrices may not be an almost skew-symmetric. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Both A and B are almost skew-symmetric matrices but $A \oplus_1 B$ is not. In the next result, we characterize those almost skew-symmetric matrices A and B having the property that $A \oplus_k B$ is also almost skew-symmetric.

Theorem 3.5. *Let $0 \leq k \leq m, n$ and $A_{22}, B_{11} \in \mathbb{R}^{k\times k}$. Suppose that $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}^{m\times m}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathbb{R}^{n\times n}$ are two almost skew-symmetric matrices with $\delta(A) > 0$ and $\delta(B) > 0$. Then $A \oplus_k B$ is an almost skew-symmetric matrix if and only if the following hold:*

(i) $diag(A_{11})$ and $diag(B_{22})$ are zero vectors.
(ii) $diag(A_{22})$ and $diag(B_{11})$ are linearly dependent.

Proof. Suppose that $A \oplus_k B$ is almost skew-symmetric. If $diag(A_{11})$ and $diag(B_{22})$ both are nonzero vectors then $rk(S(A \oplus_k B)) \geq 2$, a contradiction. So, at least one of these is zero. Without loss of generality, let $diag(A_{11}) \neq 0$ and $diag(B_{22}) = 0$. Since B is almost skew-symmetric matrix and $diag(B_{22}) = 0$ so, B_{11} is almost skew-symmetric matrix with $\delta(B_{11}) > 0$. If $diag(A_{22}) = 0$ then, by Theorem 3.3,

$$S(A \oplus_k B) = \begin{pmatrix} S(A_{11}) & 0 & 0 \\ 0 & S(B_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $diag(A_{22}) \neq 0$ then

$$S(A \oplus_k B) = \begin{pmatrix} S(A_{11}) & \frac{1}{2}(A_{12} + A_{21}) & 0 \\ \frac{1}{2}(A_{12} + A_{21}) & S(A_{22} + B_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From Eq. (1), in both cases $rk(S(A \oplus_k B)) \geq 2$, a contradiction. Thus, both $diag(A_{11})$ and $diag(B_{22})$ are zero vectors. Since A is almost skew-symmetric matrix and $diag(A_{11}) = 0$ so, A_{22} is almost skew-symmetric matrix with $\delta(A_{22}) > 0$. Since $A \oplus_k B$ is almost skew-symmetric matrix it follows that $A_{22} + B_{11}$ is also almost skew-symmetric matrix. Hence by Corollary 3.1, $diag(A_{22})$ and $diag(B_{11})$ are linearly dependent.

Conversely, suppose that A and B satisfy both the properties. Then, by Theorem 3.3,

$$S(A \oplus_k B) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & S(A_{22} + B_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since A and B are almost skew-symmetric matrices with $\delta(A) > 0$, $\delta(B) > 0$ and $diag(A_{11}) = 0$, $diag(B_{22}) = 0$ it follows that A_{22} and B_{11} are almost skew-symmetric matrices with $\delta(A_{22}) > 0$, $\delta(B_{11}) > 0$. Thus by Corollary 3.1, we have $rk(S(A_{22} + B_{11})) = 1$. Hence $rk(S(A \oplus_k B)) = 1$. $\square$

In the next result, we study the converse question.

Theorem 3.6. *Let* $C = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{pmatrix}$ *be an almost skew-symmetric matrix with* $\delta(C) > 0$ *where* C_{11}, C_{22}, *and* C_{33} *are square matrices. Then* C *can be written as either a subdirect sum of an almost skew-symmetric matrix* A *and a skew-symmetric matrix* B *or as a subdirect sum of two almost skew-symmetric matrices* A *and* B.

Proof. We observe that at least one of $diag(C_{11})$, $diag(C_{33})$ is the zero vector. Without loss of generality, suppose that $diag(C_{11}) \neq 0$ and $diag(C_{33}) = 0$. Since C is almost skew-symmetric and $diag(C_{33}) = 0$, it follows that

$$A = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & c_{22} \end{pmatrix}$$

is almost skew-symmetric. Thus C can be written as a subdirect sum of an almost skew-symmetric matrix A and a skew-symmetric matrix B, where

$$B = \begin{pmatrix} 0 & C_{23} \\ C_{32} & C_{33} \end{pmatrix}.$$

If both $diag(C_{11})$ and $diag(C_{33})$ are zero vectors then C_{22} is an almost skew-symmetric matrix, since C is almost skew-symmetric. In this case, $C = A \oplus_k B$ where

$$A = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & \frac{1}{2}C_{22} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{1}{2}C_{22} & C_{23} \\ C_{32} & C_{33} \end{pmatrix}. \qquad \square$$

Remark 3.5. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}^{n\times n}$ be an almost skew-symmetric matrix with $\delta(A) > 0$, $diag(A_{11}) = 0$ and $A_{22} \in \mathbb{R}^{k\times k}$. Suppose that $B = uu^t$ where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^{n\times n}$. Further, let $diag(u_1u_1^t)$ and $diag(A_{22})$ be linearly dependent. Then B is also an almost skew-symmetric matrix with $\delta(B) > 0$. By Theorem 3.5, $A \oplus_k B$ is an almost skew-symmetric matrix if and only if $B = \begin{pmatrix} u_1u_1^t & 0 \\ 0 & 0 \end{pmatrix}$, since $diag(B_{22}) = 0$ and so $u_2 = 0$.

4. Considerations involving the Moore–Penrose inverse

In what follows, we show that results hitherto known for invertible almost skew-symmetric matrices have extensions to the Moore–Penrose inverse case. The first result (Theorem 4.1) extends the statement that if A is an invertible almost skew-symmetric matrix then A^{-1} is also almost skew-symmetric matrix. This generalization to the Moore–Penrose inverse was also proved in [9, Theorem 4.1] with an additional assumption that A is range-symmetric. The next result (Theorem 4.2) proves an analogue of the statement: If A is an invertible almost skew-symmetric matrix then the skew-symmetric part $K(A)$ satisfies $K(A^{-1}) = K(A^{-t}AA^{-t})$ [8, Theorem 4.1]. In the third assertion (Theorem 4.3), we extends formulas for $K(A^{-1})$, $S(A^{-1})$ and $V(A^{-1})$ (proved in [8, Theorem 4.2]) to the case of the Moore–Penrose inverse.

Theorem 4.1. *Let $A \in \mathbb{R}^{n\times n}$. Then A is almost skew-symmetric if and only if $A^\dagger$ is almost skew-symmetric.*

Proof. Since the operation of Moore–Penrose inversion is involutory, it suffices to prove the implication in one direction. Suppose A is an almost skew-symmetric matrix. Then A is range-symmetric. So, $AA^\dagger A^t = A^t$ and $A(A^\dagger + (A^\dagger)^t)A^t = A^t + A$. Thus $rk(S(A)) \leq rk(S(A^\dagger))$. Since $A^\dagger$ is also range-symmetric by reversing the roles of A and $A^\dagger$, we have $A^\dagger A(A^\dagger)^t = (A^\dagger)^t$ and $A^\dagger(A + A^t(A^\dagger)^t) = (A^\dagger)^t + A^\dagger$. Thus $rk(S(A^\dagger)) \leq rk(S(A))$. Hence $rk(S(A)) = rk(S(A^\dagger))$ and so $A^\dagger$ is almost skew-symmetric. □

Theorem 4.2. *Let $A \in \mathbb{R}^{n\times n}$ be an almost skew-symmetric matrix. Then $K(A^\dagger) = K((A^\dagger)^t A(A^\dagger)^t)$.*

Proof. Suppose that A is an almost skew-symmetric matrix such that $S(A) = ww^t$, $w \in \mathbb{R}^n$. First we shall show that $w \in R(A)$. Now $A + A^t = 2ww^t$ so that for any $x \in \mathbb{R}^n$, $Ax + A^t x = 2w^t xw$. Thus $w \in R(A) + R(A^t)$. Since $R(A) = R(A^t)$, we have $w \in R(A)$. Now, we shall show that $\lambda = 1 - 2w^t A^\dagger w \neq 0$. If $\lambda = 0$, then $w^t A^\dagger w = \frac{1}{2}$. Now, $\frac{1}{2}(A + A^t) = ww^t$ and so $\frac{1}{2}(A + A^t)A^\dagger w = ww^t A^\dagger w = \frac{1}{2}w$. Since $w \in R(A)$, we have $AA^\dagger w = w$. Thus $A^t A^\dagger w = 0$. Thus $A^\dagger w \in N(A^t) \cap R(A^\dagger)$. Since $R(A^\dagger) = R(A)$ and $N(A^t) = R(A)^\perp$, we have $A^\dagger w = 0$, contradicting $w^t A^\dagger w = \frac{1}{2}$. So $\lambda \neq 0$. Since $w \in R(A)$ and $\lambda \neq 0$, applying the generalized Sherman–Morrison–Woodbury formula to $A^t = -A + 2ww^t$, we obtain:

$$(A^t)^\dagger = (-A)^\dagger - 2\lambda^{-1}A^\dagger ww^t A^\dagger.$$

Thus $S(A^\dagger) = -\lambda^{-1}A^\dagger ww^t A^\dagger$. Since $S(A^\dagger)$ is symmetric, we have $A^\dagger ww^t A^\dagger = (A^\dagger ww^t A^\dagger)^t = (A^\dagger)^t ww^t (A^\dagger)^t$. Then

$$\frac{1}{2}(A^\dagger(A + A^t)A^\dagger) = \frac{1}{2}((A^\dagger)^t(A + A^t)(A^\dagger)^t).$$

Upon premultiplying by A and postmultiplying by A^t, we have

$$\frac{1}{2}(AA^\dagger(A + A^t)A^\dagger A^t) = \frac{1}{2}(A(A^t)^\dagger(A + A^t)(A^\dagger)^t A^t).$$

This simplifies to

$$\frac{1}{2}(A^t + A^t A^\dagger A^t) = \frac{1}{2}(A(A^\dagger)^t A + A).$$

Thus $\frac{1}{2}(A - A^t) = \frac{1}{2}(A^t A^\dagger A^t - A(A^\dagger)^t A)$ so that $K(A) = K(A^t A^\dagger A^t)$. Interchanging the roles of A and $A^\dagger$ yields the desired conclusion. □

Theorem 4.3. *Let $A \in \mathbb{R}^{n\times n}$ be an almost skew-symmetric matrix. Then*

(i) $K(A^\dagger) = K(A)^\dagger$.

Let $S(A) = ww^t$ and $u = K(A)^\dagger w$. Then the following hold:

(ii) $S(A^\dagger) = uu^t$ *and*
(iii) $V(A^\dagger) = \frac{\|K(A)^\dagger u\|^2}{\|u\|^2}$.

Proof. As before, we have $w \in R(A)$ and $\mu = 1 + w^t K(A)^\dagger w = 1$. Note that $(K(A^\dagger))^t = (K(A)^t)^\dagger = -(K(A))^\dagger$, i.e., $K(A)^\dagger$ is skew-symmetric. Applying the generalized Sherman–Morrison–Woodbury formula to $A = K(A) + ww^t$, we obtain

$$\begin{aligned} A^\dagger &= K(A)^\dagger - K(A)^\dagger ww^t K(A)^\dagger \\ &= K(A)^\dagger + K(A)^\dagger ww^t (K(A)^\dagger)^t \\ &= uu^t + K(A)^\dagger. \end{aligned}$$

Since $K(A)^\dagger$ is skew-symmetric and uu^t is symmetric, the uniqueness of the decomposition of a matrix into symmetric and skew-symmetric summands yields (i) and (ii).

(iii) Since A is almost skew-symmetric, $A^\dagger$ is also almost skew-symmetric, by Theorem 4.1. Thus $rk(S(A^\dagger)) = 1$ and hence $u \neq 0$. Using (i) and (ii) and from the definition of variance, it follows that

$$V(A^\dagger) = \frac{\|K(A)^\dagger u\|^2}{\|u\|^2}. \qquad \square$$

5. Tucker's theorem for almost skew-symmetric matrices

It is well known that, among the theorems of the alternative, the Farkas lemma, proved in the early part of the twentieth century, stands out. This is perhaps due to the all important application of the proof of the duality theorem in linear programming. A proof of the existence of a stationary probability vector of a Markov matrix also has been shown to follow from Farkas' lemma. For more details of these statements and proofs we refer to the excellent book [6]. Over the period of eleven decades, Farkas' lemma has undergone a plethora of generalizations. Several new proofs have also been given. Let us recall a proof of Farkas' lemma, given by Broyden [4]. He shows that (Theorem 1.3 in [4]) if Q is an orthogonal matrix, then there exists a positive vector x such that $Qx = Sx$, where S is a diagonal matrix whose diagonal elements are equal to either plus or minus one (such a matrix being called a sign matrix). Using this result he proves Tucker's theorem from which Farkas' lemma is an easy corollary. For a proof of the other theorems of the alternative using Tucker's theorem, we refer to [5]. This section presents the outcome of an attempt to explore a version of Tucker's theorem for almost skew-symmetric matrices. This is presented in Theorem 5.2. Surprisingly, the latter is obtained from the original Tucker's theorem. More intriguingly, this leads to yet another proof of Farkas' lemma. It is perplexing for the reason that one would expect a result resembling Farkas' lemma and not the result itself. This is presented in Theorem 5.3. On the other hand, Tucker's theorem also has been proved using Farkas' lemma [5,10] where in the latter a version called Motzkin transportation theorem is utilized. In the light of these results, it follows that Tucker's theorem and Farkas' lemma are equivalent. With this perspective, let us add that our result of a version of Tucker's theorem is sandwiched between Tucker's theorem for skew-symmetric matrices and Farkas' lemma and hence it is also equivalent to these two. This perhaps is the most interesting aspect of Theorem 5.2.

In the proof of Tucker's theorem [4], the following fact is used: If A is a skew-symmetric matrix, then the Cayley transformation of A given by $Q = (I + A)^{-1}(I - A)$ is well defined and is an orthogonal matrix. The curiosity of whether the Cayley transformation exists for an almost skew-symmetric matrix and the question as to what extent it differs from an orthogonal matrix is answered in the following theorem. This is presented as a stand alone result. However, we believe that this could be the starting point of other investigations on the Cayley transformation of almost skew-symmetric matrices. Hence, this is interesting in its own right.

Theorem 5.1. *Let $A \in \mathbb{R}^{n\times n}$ be an almost skew-symmetric matrix with $\delta(A) > 0$. Then $(I+A)$ is invertible. Further, if $Q = (I+A)^{-1}(I-A)$, then $rk(I-Q^tQ) = rk(I-QQ^t) = 1$.*

Proof. Let $(I + A)x = 0$. Then $0 \le x^tAx = -x^tx = \|x\|$. So, $x = 0$. Hence $(I + A)$ is invertible. Let

$$Q = (I + A)^{-1}(I - A) = (I - A)(I + A)^{-1}.$$

Then

$$\begin{aligned} I - Q^tQ &= (I + A^t)^{-1}(I + A^t)(I + A)(I + A)^{-1} - (I + A^t)^{-1}(I - A^t)(I - A)(I + A)^{-1} \\ &= (I + A^t)^{-1}(2(A + A^t))(I + A)^{-1} \\ &= 4(I + A^t)^{-1}S(A)(I + A)^{-1}. \end{aligned}$$

Thus $rk(I - Q^tQ) = 1$. Similarly we get

$$I - QQ^t = 4(I + A)^{-1}S(A)(I + A^t)^{-1}.$$

Hence $rk(I - QQ^t) = 1$. $\square$

As was mentioned in the introduction, Tucker's theorem states that if A is a skew-symmetric matrix then there exists a vector u satisfying: $u \geq 0$, $Au \geq 0$ and $Au + u > 0$. In what follows, we prove a version for almost skew-symmetric matrices.

Theorem 5.2. *Let $A \in \mathbb{R}^{n\times n}$ be an almost skew-symmetric matrix with $\delta(A) > 0$. Then there exist vectors $u \geq 0$, $v \geq 0$ and $v \in N(S(A))$ such that $Au \geq 0$, $Au + v > 0$ and $u - A^tv > 0$.*

Proof. Let A be an almost skew-symmetric matrix with $\delta(A) > 0$. Define the block matrix $M \in \mathbb{R}^{2n\times 2n}$ by

$$M = \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}.$$

Then M is a skew-symmetric matrix. By Theorem 2.1, there exists $x \geq 0$ such that

$$Mx \geq 0 \text{ and } Mx + x > 0.$$

Let $x = (v^t, u^t)^t$. Then $u \geq 0$ and $v \geq 0$ such that

$$Au \geq 0, \ -A^tv \geq 0, \ Au + v > 0 \text{ and } u - A^tv > 0.$$

Now, we only require to show that $v \in N(S(A))$. We have $-A^tv \geq 0$ so that $-v^tA^tv \geq 0$ and so $-v^tAv \geq 0$. Thus $-v^t(A + A^t)v \geq 0$ so that $v^tS(A)v \leq 0$. But $v^tS(A)v \geq 0$, since $S(A)$ is positive semidefinite. Thus $v^tS(A)v = 0$. This means that $v^tww^tv = 0$ so that $w^tv = 0$. Thus $v \in N(S(A))$. $\square$

Using Theorem 5.2 we now prove Farkas' lemma.

Theorem 5.3. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then either*

(a) $Ax = b$ has solution $x \geq 0$

or (exclusive)

(b) there exists $z \in \mathbb{R}^m$ such that $A^tz \leq 0$ and $b^tz > 0$.

Proof. Define the block matrix $B \in \mathbb{R}^{(2m+n+1)\times(2m+n+1)}$ by

$$B = \begin{pmatrix} 0 & 0 & A & -b \\ 0 & 0 & -A & b \\ -A^t & A^t & 0 & 0 \\ b^t & -b^t & 0 & 1 \end{pmatrix}.$$

Then B is an almost skew-symmetric matrix with symmetric part $S(B) = diag(0, 0, \dots, 1)$. By Theorem 5.2, there exist vectors $u \geq 0$, $v \geq 0$ and $v \in N(S(B))$ such that

$$Bu \geq 0,\ \ Bu + v > 0 \text{ and } u - B^t v > 0.$$

Now

$$N(S(B)) = \{(v_1^t, v_2^t, v_3^t, 0)^t : v_1, v_2 \in \mathbb{R}^m \text{ and } v_3 \in \mathbb{R}^n\}.$$

If $Bu = 0$, then from $Bu + v > 0$ we obtain $v > 0$, a contradiction, since $v \in N(S(B))$. So $0 \neq Bu \geq 0$. Let $u = (z_1^t, z_2^t, x^t, p)^t$, $p \in \mathbb{R}$. Then

$$\begin{pmatrix} 0 & 0 & A & -b \\ 0 & 0 & -A & b \\ -A^t & A^t & 0 & 0 \\ b^t & -b^t & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ x \\ p \end{pmatrix} \geq 0.$$

Now, we consider the two cases $p > 0$ and $p = 0$. If $p > 0$ then the vector u can be normalized so that $p = 1$. This yields

$$Ax - b \geq 0 \text{ and } -Ax + b \geq 0.$$

Thus $Ax = b$, giving (a). If $p = 0$ then

$$-A^t z_1 + A^t z_2 \geq 0 \text{ and } b^t z_1 - b^t z_2 \geq 0.$$

Let $z = z_1 - z_2$. Then

$$A^t z \leq 0 \text{ and } b^t z \geq 0.$$

If $b^t z = 0$, then the last coordinate of $Bu+v$ will be 0, a contradiction, since $Bu+v > 0$. So $b^t z > 0$, yielding (b). $\square$

6. Preservers of almost skew-symmetry

In this short section, we present certain preservers of almost skew-symmetric matrices. First, consider the mapping $\phi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $\phi(A) = PAP^T$, where P is a fixed invertible matrix. Then ϕ is linear, bijective and ϕ^{-1} also has a similar form. Also, $\phi(A)$ is almost skew-symmetric if and only if so is A (with a similar statement being true for the inverse map ϕ^{-1}). In particular, if $P^T = P^{-1}$, then $\delta(\phi(A)) = \delta(A)$. The argument as above, is applicable for the mapping $A \mapsto PA^TP^T$. Next, define $\psi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $\psi(A) = PA^\dagger P^T$. Then ψ is bijective and not linear. By Theorem 4.1, it now follows that $\psi(A)$ is almost skew-symmetric if and only if A is almost skew-symmetric (with a similar statement being true for the inverse map ψ^{-1}). Let us conclude by pointing however, that we are not aware of the most general form of a linear preserver of almost skew-symmetry.

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