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Truncated spectral regularization for an ill-posed nonhomogeneous parabolic problem

Ajoy Jana, M. Thamban Nair *

Department of Mathematics, IIT Madras, India

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ABSTRACT

The non homogeneous backward Cauchy problem $u_t + Au = f(t)$, $u(\tau) = \phi$ for $0 \le t < \tau$ is considered, where A is a densely defined positive self-adjoint unbounded operator on a Hilbert space H with $f \in L^1([0,\tau], H)$ and $\phi \in H$ is known to be an ill-posed problem. A truncated spectral representation of the mild solution of the above problem is shown to be a regularized approximation, and error analysis is considered when both ϕ and f are noisy. Error estimates are derived under appropriate choice of the regularization parameter. The results obtained unify and generalize many of the results available in the literature.

1. Introduction

Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a densely defined positive self-adjoint unbounded operator. For $\tau > 0$, $\phi \in H$ and $f \in L^1([0,\tau], H)$, consider the problem of solving the *final value problem*, denoted briefly as FVP,

$$u_t + Au = f(t), \qquad 0 \le t < \tau \tag{1.1}$$

$$u(\tau) = \phi. \tag{1.2}$$

Here, $L^1([0,\tau], H)$ denotes the space of all H-valued integrable functions on $[0,\tau]$, i.e., $g \in L^1([0,\tau], H)$ if and only if $g : [0,\tau] \to H$ is measurable and

$$\int_0^\tau \|g(t)\| dt < \infty.$$

* Corresponding author.
E-mail addresses: janaajoy340@gmail.com (A. Jana), mtnair@iitm.ac.in (M.T. Nair).

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The problem is to find a function $u : [0, \tau] \to H$ which is differentiable and satisfies the equations (1.1) and (1.2). It is well known that the above FVP is ill-posed (cf. Goldstein [6]). Therefore, in order to obtain stable approximate solutions for (1.1)–(1.2), some regularization method has to be employed. A particular case of the above FVP which has got wide applications in science and engineering is the backward heat conduction problem (BHCP) in which the Hilbert space H is the space $L^2(\Omega)$, where Ω is a domain in $\mathbb{R}^k$ for some $k \in \mathbb{N}$, and $-A = \Delta$, the Laplacian operator in $L^2(\Omega)$ (see Isakov [7], Nair [10]).

The homogeneous FVP, that is, when $f = 0$, has been studied by many authors using different approaches. Many of them have used the *quasi-reversibility method*, introduced by Lattes and Lions [8]. The main idea of this method is to consider a perturbed form of the operator A (see e.g., Miller [9], Showalter [12] and Boussetila and Rebbani [2]). Another approach to study the homogeneous FVP considered by some authors is by perturbing the final value; such method is called *quasi-boundary value method* (see, e.g. Clark and Oppenheimer [3], Denche and Bessila [4], Denche and Djezzar [5]). Clark and Oppenheimer, Denche and Bessila have restricted their study of quasi-boundary value method when operator A is having discrete spectrum. In [1], Boussetila and Rebbani have studied homogeneous FVB by perturbing the final value as well as the operator A.

We may recall from semigroup theory (cf. [11]) that if $u(\cdot)$ is a solution of the equation

$$u_t + Au = f(t), \qquad 0 < t \leq \tau,$$

then it has the representation

$$u(t) = S(t)\phi_0 + \int_0^t S(t-s)f(s)ds$$

where $\phi_0 = u(0)$ and $\{S(t) : t \geq 0\}$ is the C_0 semigroup generated by $-A$. In fact,

$$S(t) = e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda,$$

where $\{E_\lambda : \lambda \geq 0\}$ is the resolution of identity of A, and $\{e^{-tA} : t \geq 0\}$ is a differentiable semigroup (cf. [11]). With the above notation,

$$u(t) = \int_0^\infty e^{-t\lambda} dE_\lambda \phi_0 + \int_0^t \left(\int_0^\infty e^{-(t-s)\lambda} dE_\lambda f(s) \right) ds.$$

Note that the above representation is meaningful whenever $f \in L^1([0, \tau], H)$, and in that case $u : [0, \tau] \to H$ defined by

$$u(t) = \int_0^\infty e^{-t\lambda} dE_\lambda \phi_0 + \int_0^t \left(\int_0^\infty e^{-(t-s)\lambda} dE_\lambda f(s) \right) ds \tag{1.3}$$

is called the *mild solution* of the initial value problem (IVP)

$$u_t + Au = f(t), \qquad 0 < t \leq \tau, \tag{1.4}$$

$$u(0) = \phi_0 \tag{1.5}$$

for $\phi_0 \in H$. If the final value $\phi := u(\tau)$ is known instead of the initial value $\phi_0 := u(0)$, then $u(\cdot)$ in (1.3) can be written as

$$u(t) = \int_0^\infty e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \left(\int_0^\infty e^{(s-t)\lambda} dE_\lambda f(s) \right) ds, \quad t \in [0, \tau], \tag{1.6}$$

provided

$$\int_0^\infty e^{2\tau\lambda} d\|E_\lambda \phi\|^2 < \infty \quad \text{and} \quad \int_0^\tau \left\| \int_0^\infty e^{\lambda s} dE_\lambda f(s) \right\| ds < \infty. \tag{1.7}$$

We shall justify this statement in Section 2.

In particular, for the homogeneous FVP, that is, for $f = 0$, we have

$$u(t) = \int_0^\infty e^{(\tau-t)\lambda} dE_\lambda \phi, \quad t \in [0, \tau]$$

whenever

$$\int_0^\infty e^{2\tau\lambda} d\|E_\lambda \phi\|^2 < \infty.$$

Since the operator $e^{(\tau-t)A}$ and $e^{(s-t)A}$ are unbounded, the representation of $u(\cdot)$ in (1.6) shows small perturbations in ϕ or f or in both can lead to large deviations in the solution. Therefore, if the data ϕ and f are noisy, then to get a stable approximate solution, some regularization method has to be employed.

In [13], Tuan considered a regularized solution for the homogeneous problem by using the truncation of the above representation, namely

$$u_\beta(t) = \int_0^\beta e^{(\tau-t)\lambda} dE_\lambda \phi, \quad t \in [0, \tau], \quad \beta > 0.$$

Analogously, in [14], the authors Tuan and Trong considered the regularized solution of the non-homogeneous FVP, as

$$u_\beta(t) = \int_0^\beta e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \left(\int_0^\beta e^{(s-t)\lambda} dE_\lambda f(s) \right) ds, \quad t \in [0, \tau], \quad \beta > 0 \tag{1.8}$$

and obtained the error estimate as

$$\|u_\beta(t) - u(t)\| = O(e^{-t\beta}), \quad t \in [0, \tau], \quad \beta > 0 \tag{1.9}$$

under certain assumptions on ϕ and f so that the representation of $u(\cdot)$ in (1.6) is meaningful. Note that the estimate (1.9) is not useful if $t = 0$. For obtaining an estimate which also includes the case of $t = 0$, the conditions considered in [14] are

$$\int_0^\infty \lambda^{2p} d\|E_\lambda u(t)\| < \infty \quad \text{and} \quad \int_0^\infty e^{2\lambda q} d\|E_\lambda u(t)\| < \infty$$

for some $p > 0$, $q > 0$, which lead to the estimates

$$\|u_\beta(t) - u(t)\| = O(e^{-t\beta}\beta^{-p})$$

and

$$\|u_\beta(t) - u(t)\| = O(e^{-\beta(t+q)}),$$

respectively. They restricted their study for inexact data in ϕ only.

In this paper, we carry out a unified error analysis by suggesting conditions on ϕ and f as

$$\int_0^\infty [h(\lambda)]^2 e^{2\lambda\tau} d\|E_\lambda \phi\|^2 \le \rho^2 \quad \text{and} \quad \int_0^\tau \Big\| \int_0^\infty [h(\lambda)] e^{\lambda s} dE_\lambda f(s) \Big\| ds \le \eta$$

for some $\rho > 0$ and $\eta \ge 0$, where $h : (0, \infty) \to (0, \infty)$ is a monotonically increasing piecewise continuous function. In fact, we prove that

$$\|u_\beta(t) - u(t)\| \le O\left(\frac{e^{-t\beta}}{h(\beta)}\right), \quad t \in [0, \tau].$$

When the data ϕ and f are noisy, that is, if we have ϕ_ε and f_δ in place of ϕ and f respectively with

$$\|\phi_\varepsilon - \phi\| \le \varepsilon \quad \text{and} \quad \|f - f_\delta\| \le \delta$$

for some $\varepsilon, \delta > 0$, then we suggest a strategy of choosing the regularization parameter $\beta := \beta(\varepsilon, \delta)$ which yields an error estimate in terms of a function of ε, δ as well as the convergence

$$u_{\beta,\varepsilon,\delta}(t) \to u(t) \text{ as } \varepsilon \to 0 \text{ and } \delta \to 0,$$

for each $t \in [0, \tau]$.

In Section 2 we present the preliminary results required for our analysis, where we also define the concept of a mild solution of the FVP using the spectral representation of functions of the unbounded operator A and also prove some properties of the mild solution. In Section 3, a regularized approximation of the mild solution is defined using *spectral cut-off*, and prove our main theorems of the paper with exact data ϕ, f and also with noisy data ϕ_ε, f_δ, and deduce as special cases many of the known results. Analysis for separate noisy data in ϕ and f is obtained as particular cases of what we have discussed in Section 3.

2. Preliminaries

2.1. Some consequences of spectral theorem

Let $A : D(A) \subset H \to H$ be a densely defined positive self-adjoint unbounded operator on the Hilbert space H. Recall from spectral theorem (cf. Yosida [15]) that

$$Au := \int_0^\infty \lambda dE_\lambda u, \quad u \in D(A)$$

where

$$D(A) := \{u \in H : \int_0^\infty \lambda^2 d\|E_\lambda u\|^2 < \infty\},$$

and for any continuous or piecewise continuous function $g : [0, \infty) \to [0, \infty)$, the operator $g(A)$ is defined by

$$g(A)u := \int_0^\infty g(\lambda) dE_\lambda u, \quad u \in D(g(A)),$$

where

$$D(g(A)) := \{u \in H : \int_0^\infty g(\lambda)^2 d\|E_\lambda u\|^2 < \infty\}.$$

In particular, we define the operator e^{tA} as

$$e^{tA}u := \int_0^\infty e^{\lambda t} dE_\lambda u, \quad \forall u \in D(e^{tA})$$

where

$$D(e^{tA}) := \{u \in H : \int_0^\infty e^{2\lambda t} d\|E_\lambda u\|^2 < \infty\}.$$

We may also observe that, with

$$S(t) = e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda, \quad t \geq 0,$$

the family $\{S(t) : t \geq 0\}$ of bounded linear operators on H is a differentiable strongly continuous (or C_0) semigroup with $\|S(t)\| \leq 1$ for all $t \geq 0$, $-A$ is its infinitesimal generator (cf. [11]). With these notations, the following lemmas can be proved.

Lemma 2.1. *For $t > 0$, let us denote the range of the operator e^{-tA} by $R(e^{-tA})$ and domain of the operator e^{tA} by $D(e^{tA})$. Then*

$$R(e^{-tA}) \subseteq D(e^{tA}) \subseteq D(A^n) \quad \forall n \in \mathbb{N}.$$

Lemma 2.2. *For $t \geq 0$,*

$$e^{-tA}e^{tA} = I \quad on \quad D(e^{tA}),$$
$$e^{tA}e^{-tA} = I \quad on \quad H.$$

In particular, for $t > 0$, the operator e^{tA} is a closed operator and $S(t) := e^{-tA}$ is injective with its range $D(e^{tA})$.

2.2. Spectral representation of the solution

Now we derive the expression (1.6). More precisely, we prove the following theorem.

Theorem 2.3. *Suppose the equation (1.1) has a solution $u(\cdot)$. Suppose $\phi := u(\tau)$ and $f \in L^1([0, \tau], H)$ satisfy the following conditions:*

(1) $\phi \in D(e^{\tau A})$,
(2) $f(s) \in D(e^{sA})$ *for all* $s \in [0, \tau]$ *and*
(3) *the function* $s \mapsto e^{sA} f(s)$, $s \in [0, \tau]$ *belongs to* $L^1([0, \tau], H)$.

Then

$$u(t) = \int_0^\infty e^{(\tau - t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\infty e^{(s-t)\lambda} dE_\lambda f(s) ds.$$

Proof. Since $f \in L^1([0, \tau], H)$, by using results from semigroup theory, the solution $u(t)$ of the initial value problem (1.1) is given by (cf. [11])

$$u(t) = S(t)u(0) + \int_0^t S(t-s) f(s) ds. \tag{2.1}$$

Using the notation e^{-tA} for $S(t)$, (2.1) takes the form

$$u(t) = e^{-tA} u(0) + \int_0^t e^{-(t-s)A} f(s) ds. \tag{2.2}$$

In particular,

$$\phi := u(\tau) = e^{-\tau A} u(0) + \int_0^\tau e^{-(\tau - s)A} f(s) ds.$$

Since $\phi \in D(e^{\tau A})$, $f(s) \in D(e^{sA})$ for all $s \in [0, \tau]$ and the function $s \mapsto e^{sA} f(s)$ belongs to $L^1([0, \tau], H)$, we have

$$\begin{aligned} e^{\tau A}\phi &= u(0) + e^{\tau A}\Big[\int_0^\tau e^{-(\tau - s)A} f(s) ds\Big] \\ &= u(0) + \int_0^\tau e^{sA} f(s) ds. \end{aligned}$$

Therefore,

$$u(0) = e^{\tau A}\phi - \int_0^\tau e^{sA} f(s) ds.$$

Substituting the above representation of $u(0)$ in (2.2), we get

$$\begin{aligned} u(t) &= e^{-tA}\Big[e^{\tau A}\phi - \int_0^\tau e^{sA} f(s) ds\Big] + \int_0^t e^{-(t-s)A} f(s) ds \\ &= e^{(\tau - t)A}\phi - \int_0^\tau e^{(s-t)A} f(s) ds + \int_0^t e^{-(t-s)A} f(s) ds \end{aligned}$$

$$= e^{(\tau-t)A}\phi - \int_t^\tau e^{(s-t)A} f(s)ds.$$

Thus,

$$u(t) = \int_0^\infty e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\infty e^{(s-t)\lambda} dE_\lambda f(s)ds. \qquad \square$$

Remark 2.4. We observe that a sufficient condition on f, in order to satisfy the condition (3) in Theorem 2.3, is

$$\int_0^\tau \int_0^\infty e^{2s\lambda} d\|E_\lambda f(s)\|^2 ds < \infty, \tag{2.3}$$

or equivalently, the function $s \mapsto e^{sA} f(s)$ belongs to $L^2([0,\tau], H)$, since, by Cauchy–Schwarz inequality,

$$\int_0^\tau \Big\| \int_0^\infty e^{s\lambda} dE_\lambda f(s) \Big\| ds \leq \sqrt{\tau} \left(\int_0^\tau \Big\| \int_0^\infty e^{s\lambda} dE_\lambda f(s) \Big\|^2 ds \right)^{1/2}$$

$$= \sqrt{\tau} \left(\int_0^\tau \int_0^\infty e^{2s\lambda} d\|E_\lambda (f(s))\|^2 ds \right)^{1/2}.$$

The conditions in (2.3) and $\phi \in D(e^{\tau A})$ are exactly the assumptions of Tuan and Trong [14] for obtaining the estimate (1.9). ♢

In view of Theorem 2.3, we introduce the following definition.

Definition 2.5. If $\phi \in H$ and $f \in L^1([0,\tau], H)$ satisfy the conditions (1)–(3) in Theorem 2.3, that is,

(1) $\phi \in D(e^{\tau A})$,
(2) $f(s) \in D(e^{sA})$ for all $s \in [0,\tau]$ and
(3) the function $s \mapsto e^{sA} f(s)$, $s \in [0,\tau]$ belongs to $L^1([0,\tau], H)$,

then the function $u : [0,\tau] \to H$ defined by

$$u(t) = \int_0^\infty e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\infty e^{(s-t)\lambda} dE_\lambda f(s)ds, \tag{2.4}$$

is called the **mild solution** of the FVP given by (1.1) and (1.2). ♢

Theorem 2.6. *Suppose $\phi \in H$ and $f \in L^1([0,\tau], H)$ satisfy the conditions (1)–(3) in Definition 2.5. Let $u(\cdot)$ be the mild solution of the FVP given by (1.1) and (1.2). Then*

(i) *$u(t) \in D(A^n)$ for all $t \in (0,\tau)$, $n \in \mathbb{N}$ and*
(ii) *$u(\cdot)$ is continuous on $[0,\tau]$.*

Proof. We observe that the mild solution $u(\cdot)$ of the FVP given by (1.1) and (1.2) is

$$u(t) = e^{(\tau-t)A}\phi - \int_t^\tau e^{(s-t)A} f(s)ds, \tag{2.5}$$

which can be rewritten as

$$u(t) = e^{-tA}\Big(e^{\tau A}\phi - \int_t^\tau e^{sA} f(s)ds\Big)$$

so that

$$u(t) \in R(e^{-tA}), \forall t \in (0, \tau).$$

Hence, by Lemma 2.1, we obtain (i).

Now, we prove (ii). Since for $x \in H$, the function $t \mapsto e^{-tA}x$ is continuous on $[0, \infty)$ (cf. Pazy [11]). Hence, the function $t \mapsto e^{(\tau-t)A}\phi = e^{-tA}(e^{\tau A}\phi)$ is continuous on $[0, \tau]$. Hence, it is enough to show that the function $t \mapsto v(t) := -\int_t^\tau e^{(s-t)A} f(s)ds$ is continuous on $[0, \tau]$. Let $t_0 \in [0, \tau]$ and $t \in (0, \tau)$, then

$$\begin{aligned} v(t) - v(t_0) &= -\int_t^\tau e^{(s-t)A} f(s)ds + \int_{t_0}^\tau e^{(s-t_0)A} f(s)ds \\ &= -e^{-tA}\Big(\int_{t_0}^\tau e^{sA} f(s)ds + \int_t^{t_0} e^{sA} f(s)ds\Big) + e^{-t_0A}\Big(\int_{t_0}^\tau e^{sA} f(s)ds\Big) \\ &= \Big(e^{-tA} - e^{-t_0A}\Big)y - e^{-tA}\Big(\int_t^{t_0} e^{sA} f(s)ds\Big) \end{aligned}$$

where $y = -\int_{t_0}^\tau e^{sA} f(s)ds$. By Cauchy–Schwarz inequality,

$$\begin{aligned} \Big\|e^{-tA}\Big(\int_t^{t_0} e^{sA} f(s)ds\Big)\Big\| &\le \Big\|\int_t^{t_0} e^{sA} f(s)ds\Big\| \\ &\le \Big(\int_0^\tau \|e^{sA} f(s)\|^2 ds\Big)^{\frac{1}{2}} (|t - t_0|)^{\frac{1}{2}}. \end{aligned}$$

From the above, we have

$$\lim_{t\to t_0} e^{-tA}\Big(\int_t^{t_0} e^{sA} f(s)ds\Big) = 0.$$

Also, by the continuity of $t \mapsto e^{-tA}y$,

$$\lim_{t\to t_0} \Big(e^{-tA} - e^{-t_0A}\Big)y = 0.$$

Hence $\lim_{t\to t_0} v(t) = v(t_0)$. Thus, $t \mapsto v(t)$ is continuous on $[0, \tau]$, and (ii) is proved. □

3. Regularization and error analysis

3.1. Error estimate with exact data

Let $\phi \in H$ and $f \in L^1([0,\tau], H)$ satisfy the conditions (1)–(3) in Definition 2.5 and let $u(\cdot)$ be the mild solution of the FVP given by (1.1) and (1.2), that is,

$$u(t) = \int_0^\infty e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\infty e^{(s-t)\lambda} dE_\lambda f(s) ds. \tag{3.1}$$

As we have already remarked in Section 1, the dependence of $u(\cdot)$ on ϕ and f is not continuous due to the terms $e^{(\tau-t)\lambda}$ and $e^{(s-t)\lambda}$ in the first integral and last integral in (3.1). Therefore, following [14] for noisy data in ϕ, we consider the regularized solution for $\beta > 0$ as

$$u_\beta(\phi, f, t) = \int_0^\beta e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\beta e^{(s-t)\lambda} dE_\lambda f(s) ds. \tag{3.2}$$

The following theorem shows the continuous dependence of $u_\beta(\phi, f, t)$ on ϕ and f.

Theorem 3.1. *Let $\phi_1, \phi_2 \in H$ and $f_1, f_2 \in L^1([0,\tau], H)$. Then*

$$\|u_\beta(\phi_1, f_1, t) - u_\beta(\phi_2, f_2, t)\| \le e^{(\tau-t)\beta}\Big(\|\phi_1 - \phi_2\| + \|f_1 - f_2\|\Big), \quad 0 \le t \le \tau.$$

Proof. For $t \in [0,\tau]$,

$$u_\beta(\phi_1, f_1, t) - u_\beta(\phi_2, f_2, t) = \int_0^\beta e^{\lambda(\tau-t)} dE_\lambda(\phi_1 - \phi_2) - \int_t^\tau \int_0^\beta e^{(s-t)\lambda} dE_\lambda (f_1 - f_2)(s) ds.$$

Therefore,

$$\|u_\beta(\phi_1, f_1, t) - u_\beta(\phi_2, f_2, t)\| \le \Big\| \int_0^\beta e^{\lambda(\tau-t)} dE_\lambda(\phi_1 - \phi_2) \Big\| + \int_t^\tau \Big\| \int_0^\beta e^{(s-t)\lambda} dE_\lambda (f_1 - f_2)(s) \Big\| ds. \tag{3.3}$$

Now, using the fact that $e^{2\lambda(\tau-t)} \le e^{2\beta(\tau-t)}$ for $0 \le \lambda \le \beta$,

$$\begin{aligned}\Big\| \int_0^\beta e^{\lambda(\tau-t)} dE_\lambda(\phi_1 - \phi_2) \Big\|^2 &= \int_0^\beta e^{2\lambda(\tau-t)} d\|E_\lambda(\phi_1 - \phi_2)\|^2 \\ &\le e^{2\beta(\tau-t)} \|\phi_1 - \phi_2\|^2.\end{aligned}$$

Thus,

$$\Big\| \int_0^\beta e^{\lambda(\tau-t)} dE_\lambda(\phi_1 - \phi_2) \Big\| \le e^{\beta(\tau-t)} \|\phi_1 - \phi_2\|. \tag{3.4}$$

Again, using similar inequation as in (3.4) and the fact $e^{(s-t)\lambda} \leq e^{(\tau-t)\beta}$, for all $s \in [t, \tau]$, we get

$$\int_t^\tau \left\| \int_0^\beta e^{(s-t)\lambda} dE_\lambda (f_1 - f_2)(s) \right\| ds \leq \int_t^\tau e^{(s-t)\beta} \|(f_1 - f_2)(s)\| ds \leq e^{(\tau-t)\beta} \|f_1 - f_2\|. \tag{3.5}$$

From (3.3), using (3.4) and (3.5), we get

$$\|u_\beta(\phi_1, f_1, t) - u_\beta(\phi_2, f_2, t)\| \leq e^{(\tau-t)\beta} \Big(\|\phi_1 - \phi_2\| + \|f_1 - f_2\| \Big). \qquad \square$$

Recall that the conditions (1)–(3) in Definition 2.5 imply

$$\int_0^\infty e^{2\lambda\tau} d\|E_\lambda \phi\|^2 \leq \rho^2 \quad \text{and} \quad \int_0^\tau \left\| \int_0^\infty e^{\lambda s} dE_\lambda f(s) \right\| ds \leq \eta \tag{3.6}$$

for some $\rho > 0$, $\eta \geq 0$. Now, we prove one of the main theorems of this paper under the general conditions on ϕ and f, namely,

$$\int_0^\infty [h(\lambda)]^2 e^{2\lambda\tau} d\|E_\lambda \phi\|^2 \leq \rho^2 \quad \text{and} \quad \int_0^\tau \left\| \int_0^\infty [h(\lambda)] e^{\lambda s} dE_\lambda f(s) \right\| ds \leq \eta \tag{3.7}$$

for some $\rho > 0$, $\eta \geq 0$ (depending on the function h), where $h : (0, \infty) \to (0, \infty)$ is a monotonically increasing piecewise continuous function, which can lead to better error estimates than those possible under (3.6).

Theorem 3.2. *Suppose $\phi \in H$ and $f \in L^1([0, \tau], H)$ satisfy the conditions in (3.7). Let $u(t)$ and $u_\beta(t) := u_\beta(\phi, f, t)$ be as in (3.1) and (3.2), respectively. Then*

$$\|u(t) - u_\beta(t)\| \leq \frac{(\rho + \eta)e^{-t\beta}}{h(\beta)}, \qquad 0 \leq t \leq \tau.$$

In particular the following hold.

(i) *For $0 < t \leq \tau$,*

$$\|u(t) - u_\beta(t)\| \to 0 \quad \text{as} \quad \beta \to \infty.$$

(ii) *If $\lim_{\lambda \to \infty} h(\lambda) = \infty$, then for $0 \leq t \leq \tau$,*

$$\|u(t) - u_\beta(t)\| \to 0 \quad \text{as} \quad \beta \to \infty.$$

Proof. Let $t \in [0, \tau]$. From (3.1) and (3.2), we obtain

$$u(t) - u_\beta(t) = \int_\beta^\infty e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_\beta^\infty e^{(s-t)\lambda} dE_\lambda f(s) ds.$$

Therefore

$$\|u(t) - u_\beta(t)\| \leq \left\| \int_\beta^\infty e^{(\tau-t)\lambda} dE_\lambda \phi \right\| + \int_t^\tau \left\| \int_\beta^\infty e^{(s-t)\lambda} dE_\lambda f(s) \right\| ds. \tag{3.8}$$

Now, using the fact that $\frac{e^{-2\lambda t}}{[h(\lambda)]^2} \leq \frac{e^{-2\beta t}}{[h(\beta)]^2}$ for $\lambda \geq \beta$,

$$\Big\| \int_\beta^\infty e^{(\tau-t)\lambda} dE_\lambda \phi \Big\|^2 = \int_\beta^\infty e^{2(\tau-t)\lambda} d\|E_\lambda \phi\|^2$$
$$= \int_\beta^\infty \frac{e^{-2\lambda t}}{[h(\lambda)]^2} [h(\lambda)]^2 e^{2\tau\lambda} d\|E_\lambda \phi\|^2$$
$$\leq \frac{e^{-2t\beta}}{[h(\beta)]^2} \int_0^\infty [h(\lambda)]^2 e^{2\tau\lambda} d\|E_\lambda \phi\|^2$$
$$\leq \frac{e^{-2t\beta}}{[h(\beta)]^2} \rho^2.$$

Thus,

$$\Big\| \int_\beta^\infty e^{(\tau-t)\lambda} dE_\lambda \phi \Big\| \leq \rho \frac{e^{-t\beta}}{h(\beta)}. \tag{3.9}$$

Also,

$$\Big\| \int_\beta^\infty e^{(s-t)\lambda} dE_\lambda f(s) \Big\|^2 = \int_\beta^\infty e^{2(s-t)\lambda} d\|E_\lambda f(s)\|^2$$
$$= \int_\beta^\infty \frac{e^{-2t\lambda}}{[h(\lambda)]^2} [h(\lambda)]^2 e^{2s\lambda} d\|E_\lambda f(s)\|^2$$
$$\leq \frac{e^{-2t\beta}}{[h(\beta)]^2} \int_0^\infty [h(\lambda)]^2 e^{2s\lambda} d\|E_\lambda f(s)\|^2$$
$$= \frac{e^{-2\beta t}}{[h(\beta)]^2} \Big\| \int_0^\infty [h(\lambda)] e^{\lambda s} dE_\lambda f(s) \Big\|^2.$$

Thus,

$$\int_t^\tau \Big\| \int_\beta^\infty e^{(s-t)\lambda} dE_\lambda f(s) \Big\| ds \leq \eta \frac{e^{-t\beta}}{h(\beta)}. \tag{3.10}$$

From (3.8), using (3.9) and (3.10), we get

$$\|u(t) - u_\beta(t)\| \leq (\rho + \eta) \frac{e^{-t\beta}}{h(\beta)}.$$

Note that for any give $\beta_0 > 0$,

$$\frac{e^{-t\beta}}{h(\beta)} \leq \frac{e^{-t\beta_0}}{h(\beta_0)} \quad \forall \beta \geq \beta_0,$$

so that for $0 < t \leq \tau$,

$$\|u(t) - u_\beta(t)\| \to 0 \quad \text{as} \quad \beta \to \infty.$$

In case $h(\beta) \to \infty$ as $\beta \to \infty$, we obtain

$$\|u(t) - u_\beta(t)\| \to 0 \quad \text{as} \quad \beta \to \infty$$

for every $t \in [0, \tau]$. $\square$

Remark 3.3. We remark that if $h : (0, \infty) \to (0, \infty)$ in Theorem 3.2 is a bounded function then we cannot infer convergence of $u_\beta(0)$ to $u(0)$ as $\beta \to \infty$.

Let us consider a few special cases of the function h in Theorem 3.2:

(i) Suppose $h(\lambda) = \lambda^p$, $\lambda > 0$ for some $p > 0$. Then the conditions on ϕ and f in (3.7) take the forms

$$\int_0^\infty \lambda^{2p} e^{2\lambda\tau} d\|E_\lambda \phi\|^2 \leq \rho^2 \quad \text{and} \quad \int_0^\tau \Big\| \int_0^\infty \lambda^p e^{\lambda s} dE_\lambda f(s) \Big\| ds \leq \eta$$

and we obtain

$$\|u(t) - u_\beta(t)\| \leq (\rho + \eta)\beta^{-p} e^{-t\beta} \quad \text{for all} \quad t \in [0, \tau].$$

In particular,

$$\|u(0) - u_\beta(0)\| \leq (\rho + \eta)\beta^{-p}.$$

(ii) Suppose $h(\lambda) = e^{q\lambda}$, $\lambda > 0$ for some $q > 0$. Then the conditions on ϕ and f in (3.7) take the forms

$$\int_0^\infty e^{2\lambda(\tau+q)} d\|E_\lambda \phi\|^2 \leq \rho^2 \quad \text{and} \quad \int_0^\tau \Big\| \int_0^\infty e^{\lambda(s+q)} dE_\lambda f(s) \Big\| ds \leq \eta$$

and we obtain

$$\|u(t) - u_\beta(t)\| \leq (\rho + \eta) e^{-\beta(t+q)} \quad \text{for all} \quad t \in [0, \tau].$$

In particular,

$$\|u(0) - u_\beta(0)\| \leq (\rho + \eta) e^{-q\beta}.$$

(iii) Suppose $h(\lambda) = 1$, $\lambda > 0$. Then the conditions on ϕ and f in (3.7) take the forms

$$\int_0^\infty e^{2\lambda\tau} d\|E_\lambda \phi\|^2 \leq \rho^2 \quad \text{and} \quad \int_0^\tau \Big\| \int_0^\infty e^{\lambda s} dE_\lambda f(s) \Big\| ds \leq \eta$$

and we obtain

$$\|u(t) - u_\beta(t)\| \leq (\rho + \eta) e^{-\beta t} \quad \text{for all} \quad t \in (0, \tau].$$

In this case, we cannot infer $u_\beta(0) \to u(0)$ as $\beta \to \infty$. $\diamondsuit$

Corollary 3.4. *Suppose*

$$\int_0^\tau \int_0^\infty [h(\lambda)]^2 e^{2\lambda s} d\|E_\lambda f(s)\|^2 ds \le \eta^2$$

holds in place of the the second integral in (3.7). Then

$$\|u(t) - u_\beta(t)\| \le \frac{(\rho + \eta\sqrt{\tau})e^{-t\beta}}{h(\beta)}, \quad 0 \le t \le \tau.$$

Proof. By Cauchy–Schwarz inequality, we have

$$\begin{aligned}\Big(\int_0^\tau \Big\| \int_0^\infty [h(\lambda)] e^{\lambda s} dE_\lambda f(s) \Big\| ds\Big)^2 &\le \tau \int_0^\tau \Big\| \int_0^\infty [h(\lambda)] e^{\lambda s} dE_\lambda f(s) \Big\|^2 ds \\ &\le \tau \int_0^\tau \int_0^\infty [h(\lambda)]^2 e^{2\lambda s} d\|E_\lambda f(s)\|^2 ds \\ &\le \tau\eta^2.\end{aligned}$$

Hence, the result follows immediately from Theorem 3.2. □

Remark 3.5. Suppose that $h(A)e^{sA}f(s)$ is well defined for all $s \in [0, \tau]$. Then the assumptions on f in Theorem 3.2 and Corollary 3.4 correspond to the assumptions that the function $s \mapsto h(A)e^{sA}f(s)$ belongs to $L^1([0, \tau], H)$ and $L^2([0, \tau], H)$, respectively. Therefore, the assumption on f in Theorem 3.2 is weaker than that in Corollary 3.4. ♢

3.2. Error estimate under noisy data

Now, let us assume that the data ϕ and f are noisy, that is, if we have ϕ_ε and f_δ in place of ϕ and f respectively with

$$\|\phi_\varepsilon - \phi\| \le \varepsilon \quad \text{and} \quad \|f - f_\delta\| \le \delta.$$

Let $u(t)$ be as in (3.1) and let $u_{\beta,\epsilon,\delta}(t) := u_\beta(\phi_\epsilon, f_\delta, t)$ be defined as in (3.2), that is,

$$u_{\beta,\varepsilon,\delta}(t) = \int_0^\beta e^{(\tau-t)\lambda} dE_\lambda \phi_\varepsilon - \int_t^\tau \int_0^\beta e^{(s-t)\lambda} dE_\lambda f_\delta(s) ds$$

for each $\beta > 0$.

Theorem 3.6. *Suppose $\phi \in H$ and $f \in L^1([0, \tau], H)$ satisfy the conditions in (3.7). Then*

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le e^{-t\beta}\left((\epsilon + \delta)e^{\tau\beta} + \frac{(\rho + \eta)}{h(\beta)}\right), \quad 0 \le t \le \tau.$$

Proof. Let $0 \le t \le \tau$ and let $u_\beta(t) := u_\beta(\phi, f, t)$ be defined as in (3.2). Using Theorem 3.1, we have

$$\|u_\beta(t) - u_{\beta,\varepsilon,\delta}(t)\| \le e^{(\tau-t)\beta}\Big(\|\phi - \phi_\epsilon\| + \|f - f_\delta\|\Big)$$

so that

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le e^{(\tau-t)\beta}\Big(\|\phi - \phi_\epsilon\| + \|f - f_\delta\|\Big) + \|u(t) - u_\beta(t)\|.$$

Since $\|\phi - \phi_\varepsilon\| \le \varepsilon$ and $\|f - f_\delta\| \le \delta$,

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le (\epsilon + \delta)e^{(\tau-t)\beta} + \|u(t) - u_\beta(t)\|. \tag{3.11}$$

Now (3.11), using Theorem 3.2, implies

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le e^{-t\beta}\left((\epsilon + \delta)e^{\tau\beta} + \frac{(\rho + \eta)}{h(\beta)}\right), \quad 0 \le t \le \tau. \qquad \square$$

Corollary 3.7. *The following results hold.*

(i) *Suppose* $\phi \in H$, $f \in L^1([0,\tau], H)$ *satisfy the conditions*

$$\int_0^\infty e^{2\lambda\tau} d\|E_\lambda\phi\|^2 \le \rho^2 \quad \text{and} \quad \int_0^\tau \Big\| \int_0^\infty e^{\lambda s} dE_\lambda f(s) \Big\| ds \le \eta$$

for some $\rho > 0$ *and* $\eta \ge 0$. *Then*

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le e^{-t\beta}\left((\epsilon + \delta)e^{\tau\beta} + \rho + \eta\right), \quad 0 \le t \le \tau.$$

(ii) *Suppose* $\phi \in H$, $f \in L^1([0,\tau], H)$ *satisfy the conditions*

$$\int_0^\infty \lambda^{2p} e^{2\lambda\tau} d\|E_\lambda\phi\|^2 \le \rho^2 \quad \text{and} \quad \int_0^\tau \Big\| \int_0^\infty \lambda^p e^{\lambda s} dE_\lambda f(s) \Big\| ds \le \eta$$

for some $\rho, p > 0$ *and* $\eta \ge 0$. *Then*

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le (\epsilon + \delta)e^{(\tau-t)\beta} + (\rho + \eta)e^{-t\beta}\beta^{-p}, \quad 0 \le t \le \tau.$$

(iii) *Suppose* $\phi \in H$, $f \in L^1([0,\tau], H)$ *satisfy the conditions*

$$\int_0^\infty e^{2\lambda(\tau+q)} d\|E_\lambda\phi\|^2 \le \rho^2 \quad \text{and} \quad \int_0^\tau \Big\| \int_0^\infty e^{\lambda(s+q)} dE_\lambda f(s) \Big\| ds \le \eta$$

for some $\rho, q > 0$ *and* $\eta \ge 0$. *Then*

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le (\epsilon + \delta)e^{(\tau-t)\beta} + (\rho + \eta)e^{-(t+q)\beta}, \quad 0 \le t \le \tau.$$

Proof. The results in (i), (ii) and (iii) follow from Theorem 3.6 by taking $h(\lambda) = 1$, $h(\lambda) = \lambda^p$ and $h(\lambda) = e^{q\lambda}$, respectively. $\square$

Remark 3.8. Suppose $h(\beta) \to \infty$ as $\beta \to \infty$. Then, there exists $\beta_0 > 0$ such that $\frac{1}{h(\beta)} \le 1$, $\forall \beta > \beta_0$. Therefore,

$$e^{-t\beta}\left((\epsilon + \delta)e^{\tau\beta} + \frac{\rho + \eta}{h(\beta)}\right) \le e^{-t\beta}\left((\epsilon + \delta)e^{\tau\beta} + \rho + \eta\right) \quad \forall \beta > \beta_0.$$

Thus, in this case, the estimate in Theorem 3.6 leads to the estimate in Corollary 3.7(i). However, the estimate in Corollary 3.7(i) is not useful for $t = 0$. $\diamondsuit$

Remark 3.9. Let us consider the homogeneous FVP, that is, $f = 0$. In this case we have $\eta = 0$ and

$$u(t) = \int_0^\infty e^{(\tau-t)\lambda} dE_\lambda \phi, \quad t \in [0, \tau]$$

so that

$$\|u(0)\|^2 = \int_0^\infty e^{2\lambda\tau} d\|E_\lambda \phi\|^2.$$

In this case, we have negated δ for the following expressions.

(i) Suppose $h \equiv 1$. Then the condition on ϕ in Corollary 3.7(i) can be replaced by $\|u(0)\| \le \rho$ and the error estimate in Theorem 3.6 becomes

$$\|u(t) - u_{\beta,\epsilon}(t)\| \le e^{-t\beta} \left(\epsilon e^{\tau\beta} + \rho\right).$$

This is the estimate obtained by Tuan ([13], Theorem 2.3) for homogeneous final value parabolic problem.

(ii) Suppose $h(\lambda) = \lambda^p$ for some $p > 0$. Then the condition on ϕ in Theorem 3.6, which is same as the condition in Corollary 3.7(ii), is

$$\int_0^\infty \lambda^{2p} e^{2\lambda\tau} d\|E_\lambda \phi\|^2 \le \rho^2, \tag{3.12}$$

and the error estimate in Theorem 3.6 becomes

$$\|u(t) - u_{\beta,\epsilon}(t)\| \le \epsilon e^{(\tau-t)\beta} + \rho\beta^{-p} e^{-t\beta}.$$

If we choose $\beta = \frac{a}{\tau} \ln(\frac{1}{\varepsilon})$, $(0 < a < 1)$, then

$$\|u(t) - u_{\beta,\epsilon}(t)\| \le \varepsilon^{\frac{at}{\tau}+1-a} + \left(\frac{\tau}{a}\right)^p \rho \Big(\ln(\frac{1}{\varepsilon})\Big)^{-p}, \quad \forall t \in [0, \tau].$$

This is the estimate obtained by Tuan ([13], Theorem 2.4(a)) for homogeneous final value parabolic problem under the assumption that

$$\int_0^\infty \lambda^{2p} d\|E_\lambda u(t)\|^2 < \infty \quad \forall\, t \in [0, \tau]. \tag{3.13}$$

We show below that (3.12) and (3.13) are equivalent, so that result of ([13], Theorem 2.4(a)) is a particular case of our Theorem 3.6.

Using spectral theorem for the operator A, we have

$$u(t) = \int_0^\infty e^{(\tau-t)\lambda} dE_\lambda \phi = e^{(\tau-t)A} \phi$$

so that

$$E_\mu u(t) = \chi_{[0,\,\mu]}(A)u(t) = \chi_{[0,\,\mu]}(A)e^{(\tau-t)A}\phi = \int_0^\mu e^{(\tau-t)\lambda} dE_\lambda \phi$$

and

$$\|E_\mu u(t)\|^2 = \int_0^\mu e^{2(\tau-t)\lambda} d\|E_\lambda \phi\|^2.$$

Therefore (cf. Yosida [15]),

$$\int_0^\infty \mu^{2p} d\|E_\mu u(t)\|^2 = \int_0^\infty \mu^{2p} e^{2(\tau-t)\mu} d\|E_\mu \phi\|^2.$$

Hence,

$$\int_0^\infty \mu^{2p} d\|E_\mu u(t)\|^2 < \infty \quad \forall\, t \in [0,\tau] \quad \text{if and only if} \quad \int_0^\infty \lambda^{2p} e^{2\tau\lambda} d\|E_\lambda \phi\|^2 < \infty,$$

that is, (3.12) and (3.13) are equivalent.

(iii) Suppose $h(\lambda) = e^{q\lambda}$ for some $q > 0$, corresponding to Theorem 3.6. Then, it is seen that

$$\int_0^\infty e^{2\lambda(q+\tau)} d\|E_\lambda \phi\|^2 \le \rho^2, \tag{3.14}$$

and the error estimate in Theorem 3.6 becomes

$$\|u(t) - u_{\beta,\epsilon}(t)\| \le \epsilon e^{(\tau-t)\beta} + \rho e^{-\beta(t+q)}.$$

If we choose $\beta = \frac{1}{\tau+q}\ln(\frac{1}{\varepsilon})$, then

$$\|u(t) - u_{\beta,\epsilon}(t)\| \le \varepsilon^{\frac{q}{\tau+q}} \left(\varepsilon^{\frac{t}{\tau+\rho}} + \rho \right).$$

This is the estimate obtained by Tuan ([13], Theorem 2.4(b)) for homogeneous FVP proved under the assumption that

$$\int_0^\infty e^{2q\mu} d\|E_\mu u(t)\|^2 < \infty \quad \forall\, t \in [0,\tau]. \tag{3.15}$$

As in (ii) above, it can be shown that (3.14) and (3.15) are equivalent, so that Theorem 2.4(b)) of Tuan [13] is a particular case of our Theorem 3.6. ♢

3.3. Error estimates under parameter choice strategies

The following three theorems follow from Theorem 3.6 by direct substitution.

Theorem 3.10. *Let $\phi \in H$ and $f \in L^1([0,\tau],H)$ satisfy the conditions in (3.7) with $h \equiv 1$ and let $c_{h,\rho,\eta} = \rho + \eta$. Then taking*

$$\beta := \frac{1}{\tau} \ln \Big[\frac{t c_{h,\rho,\eta}}{(\varepsilon+\delta)(\tau - t)} \Big],$$

we have

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le \Big[\frac{\tau}{t}\Big]^{\frac{t}{\tau}} \Big[\frac{\tau}{\tau - t}\Big]^{1-\frac{t}{\tau}} c_{h,\rho,\eta}^{1-\frac{t}{\tau}} (\varepsilon+\delta)^{t/\tau} \quad \text{for} \quad 0 < t < \tau. \tag{3.16}$$

Further,

$$\max_{0<t<\tau} \Big[\frac{\tau}{t}\Big]^{\frac{t}{\tau}} \Big[\frac{\tau}{\tau - t}\Big]^{1-\frac{t}{\tau}} = 2$$

and taking

$$\beta := \frac{1}{\tau} \ln \Big(\frac{c_{h,\rho,\eta}}{\varepsilon+\delta} \Big),$$
$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le 2 c_{h,\rho,\eta}^{1-\frac{t}{\tau}} (\varepsilon+\delta)^{\frac{t}{\tau}} \quad \text{for} \quad 0 \le t \le \tau. \tag{3.17}$$

Theorem 3.11. *Let $\phi \in H$ and $f \in L^1([0,\tau],H)$ satisfy the conditions in (3.7) with $h(\lambda) = \lambda^p$, $\lambda > 0$ for some $p > 0$, and let $c_{h,\rho,\eta} := \rho + \eta$. Then taking*

$$\beta := \frac{\gamma}{\tau - t} \ln \Big(\frac{1}{\varepsilon+\delta} \Big)$$

for some γ with $0 < \gamma < 1$,

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le (\varepsilon+\delta)^{1-\gamma} + c_{h,\rho,\eta} (\varepsilon+\delta)^{\frac{\gamma t}{\tau - t}} \Big(\frac{\tau - t}{\gamma}\Big)^p \Big[\ln\Big(\frac{1}{\varepsilon+\delta}\Big)\Big]^{-p}, \quad 0 \le t < \tau. \tag{3.18}$$

Further, if we choose

$$\beta := \frac{1}{\tau} \ln \Big(\frac{1}{\varepsilon+\delta} \Big),$$

then

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le (\varepsilon+\delta)^{t/\tau} \left(1 + c_{h,\rho,\eta} \tau^p \Big[\ln\Big(\frac{1}{\varepsilon+\delta}\Big)\Big]^{-p} \right), \quad 0 \le t \le \tau. \tag{3.19}$$

Theorem 3.12. *Let $\phi \in H$ and $f \in L^1([0,\tau],H)$ satisfy the conditions in (3.7) with $h(\lambda) = e^{q\lambda}$, $t > 0$ for some $q > 0$, and let $c_{h,\rho,\eta} := \rho + \eta$. Then taking*

$$\beta = \frac{1}{\tau + q} \ln \Big[\frac{(t+q) c_{h,\rho,\eta}}{(\varepsilon+\delta)(\tau - t)} \Big],$$

we have

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le \Big(\frac{\tau + q}{\tau - t}\Big) \Big(\frac{\tau - t}{t + q}\Big)^{\frac{\tau+q}{t+q}} c_{h,\rho,\eta}^{1-\frac{t+q}{\tau+q}} (\varepsilon+\delta)^{\frac{t+q}{\tau+q}}, \quad 0 \le t < \tau. \tag{3.20}$$

Further,

$$\max_{0\le t<\tau}\Big(\frac{\tau+q}{\tau-t}\Big)\Big(\frac{\tau-t}{t+q}\Big)^{\frac{\tau+q}{t+q}}=2$$

and taking $\beta=\frac{1}{\tau+q}\ln(\frac{c_{h,\rho,\eta}}{\varepsilon+\delta})$,

$$\|u(t)-u_{\beta,\epsilon,\delta}(t)\|\le 2c_{h,\rho,\eta}^{1-\frac{t+q}{\tau+q}}(\varepsilon+\delta)^{\frac{t+q}{\tau+q}},\quad 0\le t\le\tau. \tag{3.21}$$

Remark 3.13. Let us explain the motivation for the choice of β in Theorems 3.10, 3.11, 3.12.

(i) Recall that, in Theorem 3.6, for $h\equiv 1$, that is, in Corollary 3.7(i), we obtained the estimate

$$\|u(t)-u_{\beta,\epsilon,\delta}(t)\|\le(\varepsilon+\delta)e^{(\tau-t)\beta}+c_{h,\rho,\eta}e^{-t\beta},\quad 0\le t\le\tau$$

where $c_{h,\rho,\eta}:=\rho+\eta$. Note that for fixed $\varepsilon,\delta>0$ and $0<t<\tau$,

$$e^{-t\beta}\to 0\quad\text{and}\quad(\varepsilon+\delta)e^{(\tau-t)\beta}\to\infty\quad\text{as}\quad\beta\to\infty.$$

So, a natural choice of β would be in such a way that the function

$$g(\beta)=(\varepsilon+\delta)e^{(\tau-t)\beta}+c_{h,\rho,\eta}e^{-t\beta},\quad 0<t<\tau$$

attains its minimum. It can be seen that for $\beta=\frac{1}{\tau}\ln\Big[\frac{tc_{h,\rho,\eta}}{(\varepsilon+\delta)(\tau-t)}\Big]=\beta_{\varepsilon,\delta}$ (say), the function g attains its minimum value and it is given by

$$g(\beta_{\varepsilon,\delta})=\Big[\frac{\tau}{t}\Big]^{\frac{t}{\tau}}\Big[\frac{\tau}{\tau-t}\Big]^{1-\frac{t}{\tau}}c_{h,\rho,\eta}^{1-\frac{t}{\tau}}(\varepsilon+\delta)^{t/\tau}.$$

Thus, we obtain (3.16).

Also, if we choose β such that

$$(\varepsilon+\delta)e^{(\tau-t)\beta}=c_{h,\rho,\eta}e^{-t\beta},$$

then we have $g(\beta)=2c_{h,\rho,\eta}e^{-t\beta}$. Note that

$$(\varepsilon+\delta)e^{(\tau-t)\beta}=c_{h,\rho,\eta}e^{-t\beta}\quad\text{iff}\quad\beta=\frac{1}{\tau}\ln\Big(\frac{c_{h,\rho,\eta}}{\varepsilon+\delta}\Big).$$

Thus, we obtain the estimate in (3.17).

Since $\lim_{\alpha\to 0^+}\alpha^\alpha=1$, we may also observe that

$$\lim_{t\to 0}\Big[\frac{\tau}{t}\Big]^{\frac{t}{\tau}}\Big[\frac{\tau}{\tau-t}\Big]^{1-\frac{t}{\tau}}=1=\lim_{t\to\tau}\Big[\frac{\tau}{t}\Big]^{\frac{t}{\tau}}\Big[\frac{\tau}{\tau-t}\Big]^{1-\frac{t}{\tau}}$$

and

$$\max_{0<t<\tau}\Big[\frac{\tau}{t}\Big]^{\frac{t}{\tau}}\Big[\frac{\tau}{\tau-t}\Big]^{1-\frac{t}{\tau}}=2.$$

(ii) In Theorem 3.6, for $h(\lambda)=\lambda^p$ for some $p>0$, i.e., in Corollary 3.7(ii), we obtained the estimate

$$\|u(t)-u_{\beta,\varepsilon,\delta}(t)\|\le(\varepsilon+\delta)e^{(\tau-t)\beta}+c_{h,\rho,\eta}e^{-t\beta}\beta^{-p},\quad 0\le t\le\tau$$

with $c_{h,\rho,\eta} := \rho + \eta$. We have to find $\beta := \beta(\varepsilon, \delta)$ in such a way that $\beta(\varepsilon, \delta) \to \infty$ as $\varepsilon \to 0$ and $\delta \to 0$. So, we may consider β in the form

$$\beta = \xi(t) \ln(\frac{1}{\varepsilon + \delta})$$

for some suitable positive function $\xi(t)$. Substituting this choice of β in the expression $(\varepsilon + \delta)e^{(\tau - t)\beta}$, we obtain

$$(\varepsilon + \delta)e^{(\tau-t)\beta} = (\varepsilon + \delta)^{1-(\tau-t)\xi(t)}.$$

Thus, it is necessary that $0 < \xi(t) < 1/(\tau - t)$. So, we may consider $\xi(t) = \gamma/(\tau - t)$ for some γ with $0 < \gamma < 1$, which leads to the choice

$$\beta = \frac{\gamma}{\tau - t} \ln\Big(\frac{1}{\varepsilon + \delta}\Big).$$

Thus, we obtain (3.18).

In (3.18), if we choose γ in such a way that

$$1 - \gamma = \frac{\gamma t}{\tau - t},$$

then (3.18) takes the form

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le (\varepsilon + \delta)^{1-\gamma}\Big\{1 + c_{h,\rho,\eta}\Big(\frac{\tau - t}{\gamma}\Big)^p \Big[\ln\Big(\frac{1}{\varepsilon + \delta}\Big)\Big]^{-p}\Big\}.$$

Note that

$$1 - \gamma = \frac{\gamma t}{\tau - t} \quad \text{iff} \quad \gamma = \frac{\tau - t}{\tau}.$$

Thus, the choice $\gamma = \frac{\tau - t}{\tau}$ leads to

$$\beta = \frac{\gamma}{\tau - t} \ln\Big(\frac{1}{\varepsilon + \delta}\Big) = \frac{1}{\tau} \ln\Big(\frac{1}{\varepsilon + \delta}\Big)$$

and this choice of β leads to (3.19).

(iii) In Theorem 3.6, for $h(\lambda) = e^{q\lambda}$, $\forall \lambda \in (0, \infty)$ for some $q > 0$ i.e., in Corollary 3.7(iii), we obtained the estimate

$$\|u(t) - u_{\beta,\epsilon,\delta}(t)\| \le (\varepsilon + \delta)e^{(\tau-t)\beta} + c_{h,\rho,\eta}e^{-(t+q)\beta}, \quad 0 \le t \le \tau,$$

where $c_{h,\rho,\eta} := \rho + \eta$.

So, we may choose the regularization parameter $\beta := \beta(\varepsilon, \delta)$ such that the function

$$g(\beta) = e^{-t\beta}\left((\varepsilon + \delta)e^{\tau\beta} + c_{h,\rho,\eta}e^{-q\beta}\right), \quad 0 \le t < \tau$$

attains its minimum. It can be seen that such a β is

$$\beta = \frac{1}{\tau + q} \ln\Big[\frac{(t + q)c_{h,\rho,\eta}}{(\varepsilon + \delta)(\tau - t)}\Big] = \beta_{\varepsilon,\delta} \ \text{(say)}$$

and the minimum value of g is

$$g(\beta_{\varepsilon,\delta}) = \left(\frac{\tau+q}{\tau-t}\right)\left(\frac{\tau-t}{t+q}\right)^{\frac{\tau+q}{t+q}} c_{h,\rho,\eta}^{1-\frac{t+q}{\tau+q}} (\varepsilon+\delta)^{\frac{t+q}{\tau+q}}, \quad 0 \le t < \tau.$$

Thus, we obtain the estimate (3.20).

Also, if we choose β such that $(\varepsilon+\delta)e^{(\tau-t)\beta} = c_{h,\rho,\eta}e^{-(q+t)\beta}$, that is, if

$$\beta = \frac{1}{\tau+q}\ln\left(\frac{c_{h,\rho,\eta}}{\varepsilon+\delta}\right),$$

then we obtain the estimate (3.21). We may observe that

$$\max_{0\le t<\tau}\left(\frac{\tau+q}{\tau-t}\right)\left(\frac{\tau-t}{t+q}\right)^{\frac{\tau+q}{t+q}} = 2. \qquad \diamondsuit$$

General case: Let $h : (0,\infty) \to (0,\infty)$ be a continuous monotonically increasing function satisfying (3.7). Then

$$h(\beta)e^{\tau\beta} \to \infty \quad \text{as} \quad \beta \to \infty.$$

In view of the estimate in Theorem 3.6, using the arguments as in Nair [10], we would like to find $\beta = \beta_{\varepsilon,\delta}$ such that

$$\frac{c_{h,\rho,\eta}}{h(\beta)} = (\varepsilon+\delta)e^{\tau\beta}$$

i.e.,

$$\frac{1}{h(\beta)e^{\tau\beta}} = \frac{\varepsilon+\delta}{c_{h,\rho,\eta}},$$

where $c_{h,\rho,\eta} := \rho + \eta$.

We shall make use of the following lemma whose proof is obvious from the properties of continuous functions.

Lemma 3.14. *Let $h : (0,\infty) \to (0,\infty)$ be a continuous monotonically increasing function satisfying (3.7) and let*

$$\psi(\lambda) = \frac{1}{h(\lambda)e^{\tau\lambda}}, \quad \lambda > 0.$$

Then, $\psi : (0,\infty) \to (0,\infty)$ is continuous strictly monotonically decreasing function and for $0 < \varepsilon+\delta < \ell c_{h,\rho,\eta}$ with $c_{h,\rho,\eta} := \rho+\eta$ and $\ell := \lim_{\lambda\to 0}(1/h(\lambda))$, there exists a unique $\beta_{\varepsilon,\delta} > 0$ such that

$$\psi(\beta_{\varepsilon,\delta}) = \frac{\varepsilon+\delta}{c_{h,\rho,\eta}}$$

and $\beta_{\varepsilon,\delta} \to \infty$ as $(\varepsilon+\delta) \to 0$. i.e., $\beta_{\varepsilon,\delta} \to \infty$ as $\varepsilon \to 0$ and $\delta \to 0$.

Theorem 3.15. *Let $h : (0,\infty) \to (0,\infty)$ be a continuous monotonically increasing function and let $\phi \in H$ and $f \in L^1([0,\tau], H)$ satisfy the conditions in (3.7). For $0 < \varepsilon + \delta < \ell c_{h,\rho,\eta}$ with $c_{h,\rho,\eta} := \rho + \eta$ and $\ell := \lim_{\lambda\to 0}(1/h(\lambda))$, let $\beta_{\varepsilon,\delta} > 0$ be as in Lemma 3.14. Then*

$$\|u_{\beta_{\varepsilon,\delta},\varepsilon,\delta}(t) - u(t)\| \leq 2c_{h,\rho,\eta}\frac{e^{-t\beta_{\varepsilon,\delta}}}{h(\beta_{\varepsilon,\delta})}, \quad 0 \leq t \leq \tau. \tag{3.22}$$

In particular,

$$\|u_{\beta_{\varepsilon,\delta},\varepsilon,\delta}(t) - u(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0 \text{ and } \delta \to 0, \quad \forall t \in (0,\tau].$$

Further, if h is unbounded function, then

$$\|u_{\beta_{\varepsilon,\delta},\varepsilon,\delta}(t) - u(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0 \text{ and } \delta \to 0, \quad \forall t \in [0,\tau].$$

Proof. The estimate in (3.22) for the error $\|u_{\beta_{\varepsilon,\delta},\varepsilon,\delta}(t) - u(t)\|$ follows from Theorem 3.6. The remaining part of the theorem is a consequence of (3.22). □

Remark 3.16. Suppose $h : (0,\infty) \to (0,\infty)$ is monotonically increasing, but not continuous. Since ψ is a monotonically decreasing function on $(0,\infty)$ and $\psi(\lambda) \to 0$ as $\lambda \to \infty$, 0 is a limit point of $R(\psi)$, the range of ψ. Therefore, there exists a strictly monotonically decreasing sequence $\{\zeta_n\}$ in $R(\psi)$ such that $\zeta_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, let $\beta_n > 0$ be such that

$$\psi(\beta_n) = \zeta_n.$$

As ψ is strictly monotonically decreasing function, the sequence (β_n) is strictly increasing, and $\beta_n \to \infty$ as $n \to \infty$. Then, taking $\varepsilon_n + \delta_n = c_{h,\rho,\eta}\zeta_n$, we obtain

$$\|u_{\beta_n,\varepsilon_n,\delta_n}(t) - u(t)\| \leq 2c_{h,\rho,\eta}\frac{e^{-t\beta_n}}{h(\beta_n)}, \quad 0 \leq t \leq \tau.$$

In particular, for each $t \in (0,\tau]$,

$$\|u_{\beta_n,\varepsilon_n,\delta_n}(t) - u(t)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Further, if h is unbounded function, then for each $t \in [0,\tau]$,

$$\|u_{\beta_n,\varepsilon_n,\delta_n}(t) - u(t)\| \to 0 \quad \text{as} \quad n \to \infty. \qquad \diamondsuit$$

Corollary 3.17. *Let $\phi \in H$ and $f \in L^1([0,\tau], H)$ satisfy the conditions in (3.7). Then, with $\beta = \beta_{\varepsilon,\delta} = \frac{1}{\tau}\ln\left(\frac{c_{h,\rho,\eta}}{\varepsilon+\delta}\right)$,*

$$\|u(t) - u_{\beta,\varepsilon,\delta}(t)\| \leq 2c_{h,\rho,\eta}^{1-\frac{t}{\tau}}(\varepsilon+\delta)^{\frac{t}{\tau}}, \quad 0 \leq t \leq \tau.$$

Proof. Follows from Theorem 3.15 by taking $h(\lambda) \equiv 1$. □

Corollary 3.18. *Let $\phi \in H$ and $f \in L^1([0,\tau], H)$ satisfy the conditions in (3.7) and let $\beta = \beta_\varepsilon = \frac{1}{q+\tau}\ln(\frac{c_{h,\rho,\eta}}{\varepsilon+\delta})$ for some $q > 0$. Then*

$$\|u(t) - u_{\beta,\varepsilon,\delta}(t)\| \leq 2c_{h,\rho,\eta}^{\frac{\tau-t}{\tau+q}}(\varepsilon+\delta)^{\frac{t+q}{\tau+q}}, \quad 0 \leq t \leq \tau.$$

Proof. Follows from Theorem 3.15 by taking $h(\lambda) = e^{q\lambda}$, $\lambda > 0$. $\square$

Conclusion: We have defined mild solution for FVP for non-homogeneous parabolic problem equation, considered regularized approximation for the mild solution and derived error estimates under inexact data with appropriate a priori parameter choice strategies when the noise is not only in the final value, but also in the source term. The main theorems of this paper unify many of the results available in the literature, and they are presented in a simple and straight forward manner.

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