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Toeplitz Corona and the Douglas property for free functions

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ABSTRACT

The well known Douglas Lemma says that for operators A, B on Hilbert space that $AA^* - BB^* \succeq 0$ implies $B = AC$ for some contraction operator C. The result carries over directly to classical operator-valued Toeplitz operators simply by replacement of operator by Toeplitz operator throughout. Free functions generalize the notion of free polynomials and formal power series and trace back to the work of J. Taylor in the 1970s. They are of current interest, in part because of their connections with free probability and engineering systems theory. In this article, for given free functions a and b with noncommutative domain $\mathcal{K}$ defined by free polynomial inequalities, we obtain a sufficient condition in terms of the difference $aa^* - bb^*$ for the existence of a free function c taking contractive values on $\mathcal{K}$ such that $b = ac$. The connection to recent work of Agler and McCarthy and their free Toeplitz Corona Theorem is expounded.

1. Introduction

Free functions can be traced back to the work of Taylor [27,28] and generalize formal power series which appear in the study of finite automata [26]. More recently they have been of interest for their connections with free probability and engineering systems theory, see for instance, [34,33,32,7,14,16,1,22,23,21,24,20,2–4,6].

This article provides a conceptually different proof of a result in [2] of a sufficient condition for the existence of a factorization $b = ac$, for free functions a, b and a free contractive-valued function c on a free domain determined by free polynomials. In the classical context, this is the problem of Leech. See [17]. As a consquence of our main result, the Toeplitz Corona Theorem of [2] is obtained. For more on the Corona and the Toeplitz-Corona problems, see [2,9,10,18,19,25,29,31,12,5,8,30].

All Hilbert spaces considered in this article are complex and separable. Let $\mathbb{C}^{n\times n}$ denote the set of $n \times n$ complex matrices and $\mathbb{C}^d_\infty$ denote graded set $((\mathbb{C}^{n\times n})^d)_n$, where $(\mathbb{C}^{n\times n})^d$ is the set of d-tuples $X = (X_1, \dots, X_d)$ of $n \times n$ matrices. Observe that the graded set $\mathbb{C}^d_\infty$ is closed with respect to direct sums and unitary conjugations.

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More generally, a *non-commutative set* $\mathcal{L} = (\mathcal{L}(n))_n$ is a graded set where $\mathcal{L}(n) \subset (\mathbb{C}^{n\times n})^d$ such that for $X \in \mathcal{L}(m)$, $Y \in \mathcal{L}(n)$ and a unitary matrix $U \in \mathbb{C}^{m\times m}$,

(i) $X \oplus Y = (X_1 \oplus Y_1, \dots, X_d \oplus Y_d) \in \mathcal{L}(m+n)$; and
(ii) $U^*XU = (U^*X_1U, \dots, U^*X_dU) \in \mathcal{L}(m)$.

It is to be noted that a non-commutative set is defined using property (i) only, in [16].

Let $B(\mathcal{H}, \mathcal{E})$ denote the set of bounded operators from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{E}$. We will use the notation $B(\mathcal{H})$ for $B(\mathcal{H}, \mathcal{H})$.

A $B(\mathcal{H}, \mathcal{E})$*-valued non-commutative function* defined on the non-commutative set $\mathcal{L}$ is a function such that for $X \in \mathcal{L}(m)$, $Y \in \mathcal{L}(n)$,

(i) $f(X) \in B(\mathcal{H} \otimes \mathbb{C}^m, \mathcal{E} \otimes \mathbb{C}^m)$.
(ii) $f(X \oplus Y) = f(X) \oplus f(Y)$.
(iii) $f(S^{-1}XS) = (I_{\mathcal{E}} \otimes S^{-1})f(X)(I_{\mathcal{H}} \otimes S)$ whenever $S \in \mathbb{C}^{m\times m}$ is invertible and $S^{-1}XS \in \mathcal{L}(m)$.

We will say that such a function is *bounded* if $\sup_{n\in\mathbb{N}} E_n < \infty$, where $E_n = \sup_{X\in\mathcal{L}(n)} \|f(X)\|$. Henceforth we will use the abbreviation "nc" for "non-commutative".

A typical example of an nc function is a *free polynomial in the d non-commuting variables $x_1, \dots, x_d$*, which is defined as follows.

Let $\mathcal{F}_d$ be the semigroup of words formed using the d-symbols $x_1, \dots, x_d$ and the empty word $\emptyset$ denote the identity element of $\mathcal{F}_d$. A $B(\mathbb{C}^k)$*-valued free polynomial in the non-commuting variables $x_1, \dots, x_d$* is a finite formal sum of the form $\sum_{w\in\mathcal{F}_d} p_w w$, where $p_w \in B(\mathbb{C}^k)$. For $w = x_{j_1}x_{j_2}\dots x_{j_m}$, the evaluation of p at $X \in (\mathbb{C}^{n\times n})^d$, is given by $p(X) = \sum_{w\in\mathcal{F}_d} p_w \otimes X^w \in B(\mathbb{C}^k \otimes \mathbb{C}^n)$, where $X^w = X_{j_1}X_{j_2}\dots X_{j_m}$. For $0 \in (\mathbb{C}^{n\times n})^d$, $p(0) := p_\emptyset \otimes I_n$. It is easy to see that p is a $B(\mathbb{C}^k)$-valued nc function defined on the nc set $\mathbb{C}^d_\infty$.

Let ϵ and δ be $B(\mathbb{C}^k)$-valued free polynomials in $x_1, \dots, x_d$ and let $\mathcal{K}$ denote the graded set $(\mathcal{K}(n))_n$, where

$$\mathcal{K}(n) = \{X \in (\mathbb{C}^{n\times n})^d : \exists\, c > 0 \text{ such that } \epsilon(X)\epsilon(X)^* - \delta(X)\delta(X)^* \succ c(I_k \otimes I_n)\}. \tag{1}$$

Observe that the graded set $\mathcal{K} = (\mathcal{K}(n))_n$ is an nc set. Throughout this article, we will consider this nc set with the additional assumption that $0 \in \mathcal{K}(1)$. Our main result is the following.

Proposition 1. *Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ be Hilbert spaces and suppose that a and b are bounded $B(\mathcal{E}_2, \mathcal{E}_3)$ and $B(\mathcal{E}_1, \mathcal{E}_3)$ valued nc-functions on $\mathcal{K}$. There exists a $B(\mathcal{E}_1, \mathcal{E}_2)$ valued nc-function f such that, for all n and $X \in \mathcal{K}(n)$,*

(i) $\|f(X)\| \le 1$; and
(ii) $a(X)f(X) = b(X)$,

if there exists a $B(\ell^2 \otimes \mathbb{C}^k, \mathcal{E}_3)$-valued nc function h defined on $\mathcal{K}$ such that

$$a(T)a(R)^* - b(T)b(R)^* = h(T)[I_{\ell^2} \otimes (\epsilon(T)\epsilon(R)^* - \delta(T)\delta(R)^*)]h(R)^* \tag{2}$$

for all $n \in \mathbb{N}$ and $R, T \in \mathcal{K}(n)$.

A key ingredient in the proof is the existence of a left-invariant Haar probability measure on the compact group of unitary matrices in $\mathbb{C}^{n\times n}$.

Observe that if $\epsilon = I_k\emptyset$, where $\emptyset \in \mathcal{F}_d$ is the empty word, then $\mathcal{K}$ is the domain $G_\delta = (G_\delta(n))$ considered in [2], where

$$G_\delta(n) = \{X = (X_1, \ldots, X_d) : \|\delta(X)\| < 1\} \subset (\mathbb{C}^{n\times n})^d, \tag{3}$$

with the additional assumption that $0 \in G_\delta(1)$. The following theorem for the domain G_δ has been proved in [2].

Theorem 1. *Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ be finite-dimensional Hilbert spaces and suppose that a and b are bounded $B(\mathcal{E}_2, \mathcal{E}_3)$ and $B(\mathcal{E}_1, \mathcal{E}_3)$ valued nc-functions on $\mathcal{K} = G_\delta$. The following are equivalent.*

(i) There exists a $B(\ell^2 \otimes \mathbb{C}^k, \mathcal{E}_3)$ valued nc-function h defined on $\mathcal{K}$ such that

$$a(T)a(R)^* - b(T)b(R)^* = h(T)[I_{\ell^2} \otimes ((I_k \otimes I_n) - \delta(T)\delta(R)^*)]h(R)^*$$

for all $n \in \mathbb{N}$ and $R, T \in \mathcal{K}(n)$.

(ii) There exists a bounded $B(\mathcal{E}_1, \mathcal{E}_2)$ valued nc-function f such that $\|f(X)\| \leq 1$ and $a(X)f(X) = b(X)$, for all $n \in \mathbb{N}$ and $X \in \mathcal{K}(n)$.

(iii) $a(X)a(X)^* - b(X)b(X)^* \succeq 0$ for all $n \in \mathbb{N}$ and $X \in \mathcal{K}(n)$.

It is immediate that a proof of the implication $(i) \implies (ii)$ of Theorem 1, follows from Proposition 1 by taking $\epsilon = I_k\emptyset$. Thus the proof given here of Proposition 1, exploiting the Haar measure, provides an alternate and conceptually different proof of $(i) \implies (ii)$ than the one given in [2].

The article is organized as follows. Section 2 contains some preliminary lemmas that will be used in the sequel. Section 3 contains the proofs of Proposition 1 (the main result of this article) and Theorem 1. The article ends with the Toeplitz-Corona theorem of [2] for the nc domain $\mathcal{K} = G_\delta$ with $0 \in \mathcal{K}(1)$.

2. Preliminaries

Lemma 1. *Let $\mathcal{X}$, $\mathcal{Y}$ be separable Hilbert spaces and $W \in B(\mathcal{X} \otimes \mathbb{C}^n, \mathcal{Y} \otimes \mathbb{C}^n)$. If $W = (I_\mathcal{Y} \otimes V)W(I_\mathcal{X} \otimes V^*)$ for all unitaries $V \in \mathbb{C}^{n\times n}$, then there exists an operator $\mathcal{W} \in B(\mathcal{X}, \mathcal{Y})$ such that $W = \mathcal{W} \otimes I_n$.*

Proof. The result is an embodiment of the fact that the only $n \times n$ matrices which commute with all $n \times n$ matrices are multiples of the identity. Since $(I_\mathcal{Y} \otimes V)W = W(I_\mathcal{X} \otimes V)$ for every unitary $V \in \mathbb{C}^{n\times n}$, it follows that

$$(I_\mathcal{Y} \otimes X)W = W(I_\mathcal{X} \otimes X) \tag{4}$$

for every $X \in \mathbb{C}^{n\times n}$. Let $\{e_1, \ldots, e_n\}$ denote an orthonormal basis for $\mathbb{C}^n$ and let $E_{j,k} = e_j e_k^*$ denote the resulting matrix units. Write $W = \sum W_{j,k} \otimes E_{j,k}$ for operators $W_{j,k} : \mathcal{X} \to \mathcal{Y}$. Choosing, for $1 \leq \alpha, \beta \leq n$, the matrix $X = e_\alpha e_\beta^*$, from Eq. (4) it follows that

$$\sum_k W_{\beta,k} \otimes e_\alpha e_k^* = \sum_j W_{j,\alpha} \otimes e_j e_\beta^*.$$

Hence, $W_{\beta,k} = 0$ for $k \neq \beta$, $W_{j,\alpha} = 0$ for $j \neq \alpha$ and $W_{\alpha,\alpha} = W_{\beta,\beta}$ and the result follows by taking $\mathcal{W} = W_{\alpha,\alpha}$. □

Lemma 2. *Let $\mathcal{H}$ be a Hilbert space and suppose $A, B \in B(\mathcal{H})$. If $AA^* - BB^* \succ cI$ for some $c > 0$, then there exists a unique $E \in B(\mathcal{H})$ such that $B^* = E^*A^*$ and $\|E^*\| \leq 1$. Moreover, if $\mathcal{H}$ is finite dimensional, then E is unique and $\|E^*\| < 1$.*

Proof. The Douglas lemma [11] implies the existence of a contraction E such that $B = AE$ assuming only that $AA^* - BB^* \succeq 0$. Since the hypotheses imply that $AA^* \succeq cI$ is invertible, in the case that $\mathcal{H}$ is finite dimensional, it follows that A is invertible and $E = A^{-1}B$ is uniquely determined. Moreover, since $A(I - EE^*)A^* \succeq cI$ and A is invertible, E is a strict contraction. □

3. The proofs

Let $G^{(n)} = \{U \in \mathbb{C}^{n\times n} : U^*U = I\}$. It is well known that $G^{(n)}$ is a compact group with respect to multiplication. Hence there exists a unique left-invariant Haar measure $h^{(n)}$ on $G^{(n)}$ such that $h^{(n)}(G) = 1$ and

$$\int_{G^{(n)}} f(U)dh^{(n)}(U) = \int_{G^{(n)}} f(VU)dh^{(n)}(U), \tag{5}$$

for all continuous functions $f : G^{(n)} \to \mathbb{C}$ and $U, V \in G^{(n)}$. For more details see [9].

Recall the nc set $\mathcal{K}$ defined in (1) and the assumption that $0 \in \mathcal{K}(1)$.

Proof of Proposition 1. Fix $n \in \mathbb{N}$. For all $R, T \in \mathcal{K}(n)$, rearranging (2) yields,

$$\begin{aligned} &a(T)a(R)^* + h(T)[I_{\ell^2} \otimes \delta(T)\delta(R)^*]h(R)^* \\ &\quad = h(T)[I_{\ell^2} \otimes \epsilon(T)\epsilon(R)^*]h(R)^* + b(T)b(R)^*. \end{aligned} \tag{6}$$

Consider the closed subspaces:

$$\mathcal{D}^{(n)} = \overline{span}\left\{\begin{bmatrix} (I_{\ell^2} \otimes \delta(R)^*)h(R)^* \\ a(R)^* \end{bmatrix} x \,:\, x \in \mathcal{E}_3 \otimes \mathbb{C}^n, R \in \mathcal{K}(n)\right\},$$
$$\mathcal{R}^{(n)} = \overline{span}\left\{\begin{bmatrix} (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^* \\ b(R)^* \end{bmatrix} x \,:\, x \in \mathcal{E}_3 \otimes \mathbb{C}^n, R \in \mathcal{K}(n)\right\}$$

of $(\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n) \oplus (\mathcal{E}_2 \otimes \mathbb{C}^n)$ and $(\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n) \oplus (\mathcal{E}_1 \otimes \mathbb{C}^n)$ respectively.

Let $W^{(n)} : \mathcal{D}^{(n)} \to \mathcal{R}^{(n)}$ be the linear map obtained by extending the map

$$\begin{bmatrix} (I_{\ell^2} \otimes \delta(R)^*)h(R)^* \\ a(R)^* \end{bmatrix} x \to \begin{bmatrix} (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^* \\ b(R)^* \end{bmatrix} x$$

linearly to all of $\mathcal{D}^{(n)}$. It follows from Eq. (6) that $W_n : \mathcal{D}^{(n)} \to \mathcal{R}^{(n)}$ is an isometry (and hence the map is indeed well defined). Since the dimensions of ${\mathcal{D}^{(n)}}^\perp$ and ${\mathcal{R}^{(n)}}^\perp$ are equal (to infinity), it follows that $W^{(n)} : \mathcal{D}^{(n)} \to \mathcal{R}^{(n)}$ can be extended to a unitary $V^{(n)}$. Thus

$$V^{(n)} := \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} : (\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n) \oplus (\mathcal{E}_2 \otimes \mathbb{C}^n) \to (\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n) \oplus (\mathcal{E}_1 \otimes \mathbb{C}^n)$$

and satisfies

$$\begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \begin{pmatrix} (I_{\ell^2} \otimes \delta(R)^*)h(R)^* \\ a(R)^* \end{pmatrix} = \begin{pmatrix} (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^* \\ b(R)^* \end{pmatrix} \tag{7}$$

i.e.

$$\sum_{\ell=1}^{k} A^{(n)}(I_{\ell^2} \otimes \delta(R)^*)h(R)^* + B^{(n)}a(R)^* = (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^*, \tag{8}$$

$$C^{(n)}(I_{\ell^2} \otimes \delta(R)^*)h(R)^* + D^{(n)}a(R)^* = b(R)^*. \tag{9}$$

The rest of the proof will only use the fact that $V^{(n)}$ is a contraction (although it is in fact unitary). Let $U \in G^{(n)}$. Observe that $U^*RU \in \mathcal{K}(n)$. Moreover,

$$\begin{aligned}(I_{\ell^2} \otimes \delta(U^*RU)^*)h(U^*RU)^* &= (I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(R)^*)h(R)^*(I_{\mathcal{E}_3} \otimes U),\\ a(U^*RU)^* &= (I_{\mathcal{E}_2} \otimes U^*)a(R)^*(I_{\mathcal{E}_3} \otimes U) \text{ and}\\ b(U^*RU)^* &= (I_{\mathcal{E}_1} \otimes U^*)b(R)^*(I_{\mathcal{E}_3} \otimes U).\end{aligned}$$

Thus replacing R in Eqs. (8) and (9) by U^*RU yields,

$$\begin{aligned}&A^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(R)^*)h(R)^*(I_{\mathcal{E}_3} \otimes U)\\ &\quad + B^{(n)}(I_{\mathcal{E}_2} \otimes U^*)a(R)^*(I_{\mathcal{E}_3} \otimes U) = (I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \epsilon(R)^*)h(R)^*(I_{\mathcal{E}_3} \otimes U),\end{aligned} \tag{10}$$

and

$$\begin{aligned}&C^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(R)^*)h(R)^*(I_{\mathcal{E}_3} \otimes U)\\ &\quad + D^{(n)}(I_{\mathcal{E}_2} \otimes U^*)a(R)^*(I_{\mathcal{E}_3} \otimes U) = (I_{\mathcal{E}_1} \otimes U^*)b(R)^*(I_{\mathcal{E}_3} \otimes U).\end{aligned} \tag{11}$$

Multiplying Eq. (10) on the left by $(I_{\ell^2} \otimes I_k \otimes U)$ and on the right by $(I_{\mathcal{E}_3} \otimes U^*)$ and Eq. (11) on the left by $(I_{\mathcal{E}_1} \otimes U)$ and on the left by $(I_{\mathcal{E}_3} \otimes U^*)$ yields,

$$\begin{aligned}&(I_{\ell^2} \otimes I_k \otimes U)A^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(R)^*)h(R)^*\\ &\quad + (I_{\ell^2} \times I_k \otimes U)B^{(n)}(I_{\mathcal{E}_2} \otimes U^*)a(R)^* = (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^*,\end{aligned} \tag{12}$$

and

$$\begin{aligned}&(I_{\mathcal{E}_1} \otimes U)C^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(R)^*)h(R)^*\\ &\quad + (I_{\mathcal{E}_1} \otimes U)D^{(n)}(I_{\mathcal{E}_2} \otimes U^*)a(R)^* = b(R)^*.\end{aligned} \tag{13}$$

Let $\tilde{A}^{(n)}$, $\tilde{B}^{(n)}$, $\tilde{C}^{(n)}$ and $\tilde{D}^{(n)}$ denote the bounded (in fact, contractive) operators that satisfy

$$\begin{aligned}\langle \tilde{A}^{(n)}x, y\rangle &= \int_{G^(n)} \langle A^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)x, (I_{\ell^2} \otimes I_k \otimes U^*)y\rangle \, dh^{(n)}(U)\\ \langle \tilde{B}^{(n)}a, b\rangle &= \int_{G^(n)} \langle B^{(n)}(I_{\mathcal{E}_2} \otimes U^*)a, (I_{\ell^2} \otimes I_k \otimes U^*)b\rangle \, dh^{(n)}(U)\\ \langle \tilde{C}^{(n)}z, w\rangle &= \int_{G^(n)} \langle C^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)z, (I_{\mathcal{E}_1} \otimes U^*)w\rangle \, dh^{(n)}(U)\\ \langle \tilde{D}^{(n)}g, h\rangle &= \int_{G^(n)} \langle D^{(n)}(I_{\mathcal{E}_2} \otimes U^*)g, (I_{\mathcal{E}_1} \otimes U^*)h\rangle \, dh^{(n)}(U)\end{aligned} \tag{14}$$

for all $x, y, b, z \in \ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n$; $a, g \in \mathcal{E}_2 \otimes \mathbb{C}^n$; $w, h \in \mathcal{E}_1 \otimes \mathbb{C}^n$. Moreover, for $x \in \mathcal{E}_3 \otimes \mathbb{C}^n$ and $y \in \ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n$, $u \in \mathcal{E}_3 \otimes \mathbb{C}^n$ and $v \in \mathcal{E}_1 \otimes \mathbb{C}^n$, it follows from Eqs. (14), (12) and (13) that

$$\begin{aligned}
&\left\langle [\tilde{A}^{(n)}(I_{\ell^2} \otimes \delta(R)^*)h(R)^* + \tilde{B}^{(n)}a(R)^*]x, y \right\rangle \\
&\quad = \int\limits_{G^{(n)}} \Big\langle [(I_{\ell^2} \otimes I_k \otimes U)A^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(R)^*)h(R)^* \\
&\qquad + (I_{\ell^2} \otimes I_k \otimes U)B^{(n)}(I_{\mathcal{E}_2} \otimes U^*)a(R)^*]x, y \Big\rangle \, dh^{(n)}(U) \\
&\quad = \int\limits_{G^{(n)}} \langle (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^* x, y \rangle \, dh^{(n)}(U) \\
&\quad = \langle (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^* x, y \rangle
\end{aligned} \tag{15}$$

as well as

$$\begin{aligned}
&\left\langle [\tilde{C}^{(n)}(I_{\ell^2} \otimes \delta(R)^*)h(R)^* + \tilde{D}^{(n)}a(R)^*]u, v \right\rangle \\
&\quad = \int\limits_{G^{(n)}} \Big\langle [(I_{\mathcal{E}_1} \otimes U)C^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(R)^*)h(R)^* \\
&\qquad + (I_{\mathcal{E}_1} \otimes U)D^{(n)}(I_{\mathcal{E}_2} \otimes U^*)a(R)^*]u, v \Big\rangle \, dh^{(n)}(U) \\
&\quad = \int\limits_{G^{(n)}} \langle b(R)^* u, v \rangle \, dh^{(n)}(U). \\
&\quad = \langle b(R)^* u, v \rangle
\end{aligned} \tag{16}$$

Eqs. (15) and (16) together imply that

$$\begin{pmatrix} \tilde{A}^{(n)} & \tilde{B}^{(n)} \\ \tilde{C}^{(n)} & \tilde{D}^{(n)} \end{pmatrix} \begin{pmatrix} (I_{\ell^2} \otimes \delta(R)^*)h(R)^* \\ a(R)^* \end{pmatrix} = \begin{pmatrix} (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^* \\ b(R)^* \end{pmatrix}.$$

Also, observe that $\begin{pmatrix} \tilde{A}^{(n)} & \tilde{B}^{(n)} \\ \tilde{C}^{(n)} & \tilde{D}^{(n)} \end{pmatrix}$ is a contraction. Lastly, for $V \in G^{(n)}$, the left invariance property of the Haar measure h implies that $\tilde{A}^{(n)}$, $\tilde{B}^{(n)}$, $\tilde{C}^{(n)}$ and $\tilde{D}^{(n)}$ are invariant under conjugation by $I \otimes V$ and hence

$$\begin{aligned}
\tilde{A}^{(n)} &= (I_{\ell^2} \otimes I_k \otimes V)\tilde{A}^{(n)}(I_{\ell^2} \otimes I_k \otimes V^*) \\
\tilde{B}^{(n)} &= (I_{\ell^2} \otimes I_k \otimes V)\tilde{B}^{(n)}(I_{\mathcal{E}_2} \otimes V^*) \\
\tilde{C}^{(n)} &= (I_{\mathcal{E}_1} \otimes V)\tilde{C}^{(n)}(I_{\ell^2} \otimes I_k \otimes V^*) \\
\tilde{D}^{(n)} &= (I_{\mathcal{E}_1} \otimes V)\tilde{D}^{(n)}(I_{\mathcal{E}_2} \otimes V^*).
\end{aligned}$$

It follows from Lemma 1 that there exist bounded operators $\mathcal{A}^{(n)}$, $\mathcal{B}^{(n)}$, $\mathcal{C}^{(n)}$, and $\mathcal{D}^{(n)}$ such that $\tilde{A}^{(n)} = \mathcal{A}^{(n)} \otimes I_n$, $\tilde{B}^{(n)} = \mathcal{B}^{(n)} \otimes I_n$, $\tilde{C}^{(n)} = \mathcal{C}^{(n)} \otimes I_n$ and $\tilde{D}^{(n)} = \mathcal{D}^{(n)} \otimes I_n$, where $\mathcal{A}^{(n)} \in B(\ell^2 \otimes \mathbb{C}^k)$, $\mathcal{B}^{(n)} \in B(\mathcal{E}_2, \ell^2 \otimes \mathbb{C}^k)$, $\mathcal{C}^{(n)} \in B(\ell^2 \otimes \mathbb{C}^k, \mathcal{E}_1)$ and $\mathcal{D}^{(n)} \in B(\mathcal{E}_2, \mathcal{E}_1)$. Moreover,

$$\begin{pmatrix} \mathcal{A}^{(n)} & \mathcal{B}^{(n)} \\ \mathcal{C}^{(n)} & \mathcal{D}^{(n)} \end{pmatrix} : (\ell^2 \otimes \mathbb{C}^k) \oplus \mathcal{E}_2 \to (\ell^2 \otimes \mathbb{C}^k) \oplus \mathcal{E}_1$$

is a contraction.

Let $\mathcal{H} = (\ell^2 \otimes \mathbb{C}^k) \oplus \mathcal{E}_2$ and $\mathcal{E} = (\ell^2 \otimes \mathbb{C}^k) \oplus \mathcal{E}_1$. Observe that $\mathcal{H} \oplus \mathcal{E}$ is separable. At this point, it has been proved that there exists an operator $\mathcal{V} \in B(\mathcal{H}, \mathcal{E})$ such that $\|\mathcal{V}\| \leq 1$ and

$$\mathcal{V} \otimes I_n \begin{pmatrix} (I \otimes \delta(R)^*)h(R)^* \\ a(R)^* \end{pmatrix} = \begin{pmatrix} (I \otimes \epsilon(R)^*)h(R)^* \\ b(R)^* \end{pmatrix}. \tag{17}$$

Let

$$L_n = \left\{ \begin{pmatrix} 0 & 0 \\ \mathcal{V} & 0 \end{pmatrix} : \|\mathcal{V}\| \leq 1 \text{ and } (\mathcal{V} \otimes I_n) \text{ solves } (17) \right\} \subset B(\mathcal{H} \oplus \mathcal{E}).$$

The argument above implies that $L_n \neq \emptyset$ for each $n \in \mathbb{N}$. It is also the case that L_n is a WOT-closed subset of the WOT-compact unit ball of $B(\mathcal{H} \oplus \mathcal{E})$. Thus L_n is WOT-compact for each $n \in \mathbb{N}$. Moreover since $0 \in \mathcal{K}(1)$, it follows that $L_n \supset L_{n+1}$. By the nested intersection property of compact sets, $\bigcap_{n\in\mathbb{N}} L_n$ is non-empty. Say $\begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \in \bigcap_{n\in\mathbb{N}} L_n$, where $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in B(\ell^2 \otimes \mathbb{C}^k)$, $B \in B(\mathcal{E}_2, \ell^2 \otimes \mathbb{C}^k)$, $C \in B(\ell^2 \otimes \mathbb{C}^k, \mathcal{E}_1)$ and $D \in B(\mathcal{E}_2, \mathcal{E}_1)$.

For all $n \in \mathbb{N}$ and $R \in \mathcal{K}(n)$, we have,

$$(A \otimes I_n)(I_{\ell^2} \otimes \delta(R)^*)h(R)^* + (B \otimes I_n)a(R)^* = (I_{\ell^2} \otimes \epsilon(R)^*)h(R)^* \tag{18}$$

$$(C \otimes I_n)(I_{\ell^2} \otimes \delta(R)^*)h(R)^* + (D \otimes I_n)a(R)^* = b(R)^*. \tag{19}$$

By Lemma 2, for each $n \in \mathbb{N}$ and $R \in \mathcal{K}(n)$ there exists a uniquely determined strict contraction $\gamma(R) \in B(\mathbb{C}^k \otimes \mathbb{C}^n)$ such that

$$\delta(R)^* = \gamma(R)^* \epsilon(R)^*. \tag{20}$$

Since $\|A \otimes I_n\| \leq 1$ and $\|\gamma(R)^*\| < 1$, rearranging Eq. (18) and using (20) yields,

$$(I_{\ell^2} \otimes \epsilon(R)^*)h(R)^* = \{I_{\ell^2} \otimes I_k \otimes I_n - (A \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\}^{-1}(B \otimes I_n)a(R)^*. \tag{21}$$

Using (21) and (20) in (19) yields,

$$\begin{aligned}&[(C \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\{I_{\ell^2} \otimes I_k \otimes I_n \\ &\quad - (A \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\}^{-1}(B \otimes I_n) + (D \otimes I_n)]a(R)^* = b(R)^*.\end{aligned} \tag{22}$$

For $n \in \mathbb{N}$, $R \in \mathcal{K}(n)$, define the function f on $\mathcal{K}$ by

$$\begin{aligned}f(R) = {} & [(C \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\{I_{\ell^2} \otimes I_k \otimes I_n \\ & - (A \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\}^{-1}(B \otimes I_n) + (D \otimes I_n)]^*\end{aligned} \tag{23}$$

Thus f is a $B(\mathcal{E}_1, \mathcal{E}_2)$-valued graded function which satisfies $a(R)f(R) = b(R)$. Moreover, for $R \in \mathcal{K}(n)$ and $S \in \mathcal{K}(m)$, since $\gamma(R \oplus S) = \gamma(R) \oplus \gamma(S)$, it follows that,

$$\begin{aligned}f(R \oplus S) &= [(C \otimes I_{n+m})(I_{\ell^2} \otimes \gamma(R \oplus S)^*)\{I_{\ell^2} \otimes I_k \otimes I_{n+m} \\ &\qquad - (A \otimes I_{n+m})(I_{\ell^2} \otimes \gamma(R \oplus S)^*)\}^{-1}(B \otimes I_{n+m}) + (D \otimes I_{n+m})]^* \\ &= f(R) \oplus f(S).\end{aligned}$$

i.e. f preserves direct sums.

Finally, to show that f is an nc function, suppose $R \in \mathcal{K}(n)$ and S is an invertible $n \times n$ matrix such that $S^{-1}RS \in \mathcal{K}(n)$. We need to show that $f(S^{-1}RS) = (I_{\mathcal{E}_2} \otimes S^{-1})f(R)(I_{\mathcal{E}_1} \otimes S)$. Observe that $\gamma(R)^*$ is uniquely determined by (20), since $\epsilon(R)^*$ is invertible. From the form of f, it is enough to show $\gamma(S^{-1}RS) = (I_k \otimes S^{-1})\gamma(R)(I_k \otimes S)$. To this end, observe that,

$$
\begin{aligned}
(I_k \otimes S^*)\delta(R)^*(I_k \otimes (S^*)^{-1}) &= \delta(S^{-1}RS)^* \\
&= \gamma(S^{-1}RS)^*\epsilon(S^{-1}RS)^* \\
&= \gamma(S^{-1}RS)^*(I_k \otimes S^*)\epsilon(R)^*(I_k \otimes (S^*)^{-1}).
\end{aligned} \tag{24}
$$

Thus

$$
(I_k \otimes S^*)\gamma(R)^*\epsilon(R)^*(I_k \otimes (S^*)^{-1}) = \gamma(S^{-1}RS)^*(I_k \otimes S^*)\epsilon(R)^*(I_k \otimes (S^*)^{-1}).
$$

Since $\epsilon(R)^*(I_k \otimes (S^*)^{-1})$ is invertible, taking adjoints, it follows that

$$
(I_k \otimes S^{-1})\gamma(R)(I_k \otimes S) = \gamma(S^{-1}RS).
$$

The proof is complete if we show that $\|f(R)\| \leq 1$ for every $n \in \mathbb{N}$ and $R \in \mathcal{K}(n)$. Recall that for all $n \in \mathbb{N}$, $V \otimes I_n = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} A \otimes I_n & B \otimes I_n \\ C \otimes I_n & D \otimes I_n \end{pmatrix}$ is a contraction. Thus there exist bounded operators $\mathcal{P}$ and $\mathcal{Q}$ such that

$$
\begin{pmatrix} \mathcal{P}^*\mathcal{P} & \mathcal{P}^*\mathcal{Q} \\ \mathcal{Q}^*\mathcal{P} & \mathcal{Q}^*\mathcal{Q} \end{pmatrix} = \begin{pmatrix} I_{\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n} & 0 \\ 0 & I_{\mathcal{E}_2 \otimes \mathbb{C}^n} \end{pmatrix} - \begin{pmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & \mathcal{D}^* \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \succeq 0. \tag{25}
$$

For notational convenience, let $\Gamma(R) := (I_{\ell^2} \otimes \gamma(R)^*)$, $\Delta(R) := (I_{\ell^2} \otimes I_k \otimes I_n - \mathcal{A}\Gamma(R))$ and $\Phi(R) := \Delta(R)^{-1}$. We have $f(R)^* = \mathcal{D} + \mathcal{C}\Gamma(R)\Phi(R)\mathcal{B}$. Using Eq. (25), for $n \in \mathbb{N}$ and $X \in \mathcal{K}(n)$, we have, as in [13],

$$
\begin{aligned}
(I_{\mathcal{E}_2} \otimes I_n) - f(R)f(R)^* &= (I_{\mathcal{E}_2} \otimes I_n) - \mathcal{D}^*\mathcal{D} - \mathcal{B}^*\Phi(R)^*\Gamma(R)^*\mathcal{C}^*\mathcal{D} \\
&\quad - \mathcal{D}^*\mathcal{C}\Gamma(R)\Phi(R)\mathcal{B} - \mathcal{B}^*\Phi(R)^*\Gamma(R)^*\mathcal{C}^*\mathcal{C}\Gamma(R)\Phi(R)\mathcal{B} \\[1ex]
&= \mathcal{Q}^*\mathcal{Q} + B^*B + \mathcal{B}^*\Phi(R)^*\Gamma(R)^*(\mathcal{A}^*\mathcal{B} + \mathcal{P}^*\mathcal{Q}) \\
&\quad + (\mathcal{B}^*\mathcal{A} + \mathcal{Q}^*\mathcal{P})\Gamma(R)\Phi(R)\mathcal{B} \\
&\quad - \mathcal{B}^*\Phi(R)^*\Gamma(R)^*(I - \mathcal{A}^*\mathcal{A} - \mathcal{P}^*\mathcal{P})\Gamma(R)\Phi(R)\mathcal{B} \\[1ex]
&= \mathcal{B}^*\Phi(R)^*[\Delta(R)^*\Delta(R) + \Gamma(R)^*\mathcal{A}^*\Delta(R) + \Delta(R)^*\mathcal{A}\Gamma(R) \\
&\quad - \Gamma(R)^*(I - \mathcal{A}^*\mathcal{A})\Gamma(R)]\Phi(R)\mathcal{B} \\
&\quad + \mathcal{Q}^*\mathcal{Q} + \mathcal{B}^*\Phi(R)^*\Gamma(R)^*\mathcal{P}^*\mathcal{Q} + \mathcal{Q}^*\mathcal{P}\Gamma(R)\Phi(R)\mathcal{B} \\
&\quad + \mathcal{B}^*\Phi(R)^*\Gamma(R)^*\mathcal{P}^*\mathcal{P}\Gamma(R)\Phi(R)\mathcal{B} \\[1ex]
&= \mathcal{B}^*\Phi(R)^*[I - \Gamma(R)^*\Gamma(R)]\Phi(R)\mathcal{B} \\
&\quad + (\mathcal{Q} + \mathcal{P}\Gamma(R)\Phi(R)\mathcal{B})^*(\mathcal{Q} + \mathcal{P}\Gamma(R)\Phi(R)\mathcal{B}) \\[1ex]
&\succeq 0. \qquad \square
\end{aligned}
$$

Proof of Theorem 1. (i) implies (ii): Follows from Proposition 1, by letting $\epsilon = I_k\emptyset$.

(ii) implies (iii): Observe that for each $n \in \mathbb{N}$ and $X \in \mathcal{K}(n)$,

$$\begin{aligned} a(X)a(X)^* - b(X)b(X)^* &= a(X)a(X)^* - a(X)f(X)f(X)^*a(X)^* \\ &= a(X)(I_{\mathcal{E}_2} \otimes I_n - f(X)f(X)^*)a(X)^* \\ &\succeq 0. \end{aligned} \tag{26}$$

(iii) implies (i): This is the content of Theorem 7.10 in [2]. □

Recall the non-commutative set $G_\delta = (G_\delta(n))_n$ from (3). The following is the Toeplitz-Corona theorem of [2] for the non-commutative domain $G_\delta = (G_\delta(n))$ with the assumption that $0 \in G_\delta(1)$. Observe that certain well-known non-commutative domains, for example, the non-commutative polydisc, can be realized as such G_δ, for suitable δ.

Theorem 2. *Let $a_1, \dots, a_\ell$ be bounded $\mathbb{C}$-valued nc-functions defined on G_δ and $\mu > 0$. If for all $n \in \mathbb{N}$ and $R \in G_\delta(n)$, $\sum_{i=1}^{\ell} a_i(R)a_i(R)^* \succeq \mu^2 I_n$, then there exist $\mathbb{C}$-valued nc functions $g_1, \dots, g_\ell$ defined on G_δ such that $\sum_{i=1}^{\ell} a_i(R)g_i(R) = I_n$ for each $n \in \mathbb{N}$ and $R \in G_\delta(n)$. Moreover the $B(\mathbb{C}, \mathbb{C}^\ell)$ valued nc function g satisfies $\|g(R)\| \le \frac{1}{\mu}$ for all $n \in \mathbb{N}$ and $R \in G_\delta(n)$, where $g(R) = e_1 \otimes g_1(R) + \cdots + e_\ell \otimes g_j(R)$ and $e_1, e_2, \dots, e_\ell$ are the standard unit (column) vectors in $\mathbb{C}^\ell$.*

Proof. Letting $\mathcal{E}_1 = \mathcal{E}_3 = \mathbb{C}$ and $\mathcal{E}_2 = \mathbb{C}^\ell$, $a(R) = e_1^* \otimes a_1(R) + \cdots + e_\ell^* \otimes a_\ell(R)$ and $b(R) = \mu I_n$ for $R \in G_\delta(n)$ in Theorem 1, the hypothesis becomes $a(R)a(R)^* - b(R)b(R)^* \succeq 0$. Theorem 1 now implies that there exists a $B(\mathbb{C}, \mathbb{C}^\ell)$ valued nc function f such that $\|f(R)\| \le 1$ and

$$[e_1^* \otimes a_1(R) + \cdots + e_\ell^* \otimes a_j(R)]f(R) = \mu I_n. \tag{27}$$

Choose $\mathbb{C}$-valued nc functions $f_1, \dots, f_\ell$ such that $f(R) = e_1 \otimes f_1(R) + \cdots e_\ell \otimes f_\ell(R)$. Using this in Eq. (27) yields,

$$\sum_{i=1}^{\ell} a_i(R)f_i(R) = \mu I_n.$$

Taking $g_i = \frac{1}{\mu} f_i$; $i = 1, 2, \dots, \ell$, completes the proof. □

4. Free spectrahedra

Let Λ denote a linear $k \times k$ matrix-valued nc polynomial,

$$\Lambda(x) = \sum_{j=1}^{g} A_j x_j,$$

where the A_j are $k \times k$ matrices. The corresponding *linear pencil* is the expression

$$L(x) = I_k - \Lambda(x) - \Lambda(x)^*.$$

A bit of algebra shows that

$$L(x) = (I_k - \Lambda)(x)\,(I_k - \Lambda)(x)^* - \Lambda(x)\Lambda(x)^*. \tag{28}$$

Given the linear pencil $L(x)$, define the *free (non-commutative) spectrahedron* (see [15]) associated with the linear pencil $L(x)$ by $\mathcal{R}_L = (\mathcal{R}_{L,n})_n$, where

$$\mathcal{R}_{L,n} = \{X \in (\mathbb{C}^{n\times n})^d \,:\, L(X) \succ 0\}.$$

If one associates with the linear pencil $L(x)$, the nc polynomials $\epsilon(x) = I_k - \Lambda(x)$ and $\delta(x) = \Lambda(x)$, then it follows from (28) that the spectrahedron $\mathcal{R}_L$ is the nc set $\mathcal{K} = (\mathcal{K}(n))_n$ constructed from nc polynomials ϵ and δ as in Eq. (1). The results of this article apply equally well to such spectrahedra.

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