Thermodynamic structure of gravitational field equations

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Thermodynamic structure of Gravitational field equations

Dawood Kothawala [1]

Inter-University Centre for Astronomy and Astrophysics, Post Bag 04, Ganeshkhind, Pune 411007, INDIA

E-mail: tdawood@iucaa.ernet.in

Abstract. There is an intriguing analogy between the gravitational dynamics of horizons and thermodynamics, which is not yet understood at a deeper level. In fact, it has been shown for several cases that the on-horizon structure of the gravitational field equations in Einstein theory have the structure of *first law of thermodynamics*. In this talk, we discuss how such a structure arises and show that the field equations near any static horizon can be written as: $T\mathrm{d}S - \mathrm{d}E = P\mathrm{d}V$. Moreover, the result extends beyond Einstein theory and holds for Lanczos-Lovelock lagrangians as well. The entropy S we obtain is precisely the Noether charge entropy of Wald, and E provides a natural generalization of *quasi-local energy* of the horizon. We comment on several implications of this result, particularly the notion of gravitational entropy [treated as the Noether charge of diffeomorphism invariance] associated with horizons and it's role in gravitational dynamics arising out of virtual displacements of the horizon.

1. Introduction

There exists a strong mathematical resemblance between the on-horizon structure of the gravitational field equations and the *first law of thermodynamics*. This intriguing structure was first noted by Padmanabhan [1] in the context static spherically symmetric horizon in Einstein gravity. In [1] it was shown that the differential form of the on-horizon Einstein equation [more specifically, the G_0^0 and G_r^r components] can be interpreted as virtual displacement of the horizon, and is, in fact, equivalent to the first law fo thermodynamics: $T\mathrm{d}S = \mathrm{d}E + P_{\perp}\mathrm{d}V$. In this talk, we describe how such a result arises out of near-horizon symmetries of the gravitational field tensor and holds not only for Einstein theory, but arbitrary Lanczos-Lovelock theory as well. In particular, our discussion brings out the key ingredients responsible for the thermodynamic structure of field equations as well as the generality of the result. Moreover, one can identify the expression for horizon entropy with the Noether charge entropy defined by Wald. Therefore, the question of how "gravity knows about thermodynamics at all", finds a natural explanation in the analysis we present. [2]

Notation: The metric signature is $(-,+,+,+)$; latin indices go from 0 to 3, and greek indices from 1 to 3. Capitalized latin indices go over the $(D-2)$ transverse coordinates.

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[1] Work done in collaboration with T. Padmanabhan, IUCAA, India.
[2] Only the references most directly related to the present talk have been cited here; for further details and an exhaustive list of references, see [2].

2. Gravitational field equations as a thermodynamic identity

2.1. Background

The coordinate system best suited for our discussion is given by the metric:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \mathrm{d}n^2 + \sigma_{AB}\mathrm{d}y^A\mathrm{d}y^B \tag{1}$$

where $\sigma_{AB}(n, y^A)$ is the transverse metric, and the Killing horizon, generated by the timelike Killing vector field $\boldsymbol{\xi} = \boldsymbol{\partial}_t$, is approached as $N^2 \to 0$. Demanding the finiteness of curvature invariants on the horizon leads to the following Taylor series expansion for the lapse $N(n, x, y)$ and the transverse metric σ_{AB} [3]:

$$N(n, y) = \kappa n \left[1 - \frac{1}{2}R_\perp(y; n = 0)n^2 + O(n^3) \right] \tag{2}$$

$$\sigma_{AB} = [\sigma_H(y)]_{AB} + \frac{1}{2}[\sigma_2(y)]_{AB}\, n^2 + \mathcal{O}(n^3) \tag{3}$$

where we have collectively called the transverse coordinates as y, and $R_\perp$ is the Ricciscalar corresponding to $y = $ constant surface. We shall be interested in a small region in the neighbourhood of the spacelike $(D-2)$-surface; more precisely, we shall assume $n \ll R_\perp^{-1/2}$, and would be interested in the $\kappa \to \infty$. That is, we shall require the length scale set by κ to be the smallest of all length scales. The $t = $ constant part of the metric is written by employing *Gaussian normal coordinates* for the spatial part of the metric spanned by (n, y^A), n being the normal distance to the horizon. Consider the $(y = $ constant$)$ null vectors given by:

$$l_i = \left(-1, +N^{-1}, 0, 0\right)$$
$$k_i = \left(-1, -N^{-1}, 0, 0\right) \tag{4}$$

In the limit we are interested in, it is easy to check that these vectors satisfy the geodesic equation in affinely parametrized form, that is; $\nabla_{\boldsymbol{l}}\boldsymbol{l} = 0 = \nabla_{\boldsymbol{k}}\boldsymbol{k}$. [Explicit calculation shows that $\nabla_{\boldsymbol{l}}\boldsymbol{l}$ has only "y"-components, given by:

$$\nabla_{\boldsymbol{l}}\boldsymbol{l} = -\left(2\kappa^{-2}\right)\left[\sigma^{AB}\,\partial_B R_\perp\right]_{n=0}\,\boldsymbol{\partial}_A + O(n^2) \tag{5}$$

which go to zero in the limit $\partial R_\perp/\kappa^2 \to 0$, which is the limit we are interested in.] The affine parameter λ, defined by $\boldsymbol{l} \cdot \nabla\lambda = 1$ can be found by using the above form of $N(n, y)$; to the leading order, we find that, $\lambda \sim \lambda_H + (1/2)\kappa n^2$ where $\lambda = \lambda_H$ is the location of the horizon. Note that, $N^2\boldsymbol{l} \to \boldsymbol{\xi}|_H$, which implies, $2\kappa\left(\lambda - \lambda_H\right)\boldsymbol{l} \to \boldsymbol{\xi}|_H$. In subsequent analysis, the differentials of various geometric quantities (such as entropy) defined on the horizon, which are directly involved in the statement of the first law of thermodynamics, are to be interpreted as variations with respect to the affine parameter along the outgoing null geodesics, i.e., λ. This, of course, is the most natural variation that can be chosen on a *null surface*. All throughout, we shall take the *on the horizon* limit by considering a foliation defined by $n = $ constant surfaces and then taking the limit $n \to 0$.

Aside: Before proceeding to the calculation, we should mention the connection of above metric with local coordinate system of an accelerated observer. The most common form in which the latter is written are the so called Fermi normal coordinates, which takes the form (in self-evident notation):

$$g_{00} = -\left[(1 + a_\mu y^\mu)^2 + R_{0\mu0\nu}y^\mu y^\nu\right] + \mathcal{O}(y^3)$$

$$g_{0\mu} = -\frac{2}{3}R_{0\rho\mu\sigma}y^\rho y^\sigma + \mathcal{O}(y^3)$$

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}\,y^\rho\,y^\sigma + \mathcal{O}(y^3) \tag{6}$$

It is worth noting that the metric we considered above has a very similar form. Of course, it is simpler than the above metric since while constructing Fermi coordinates, no restriction such as staticity or constant acceleration is imposed. However, it is worth pursuing this point further since one ultimately wants to connect the dynamics of gravity with local horizons of accelerated observers in curved spacetime [see section 3], and so it is important to make precise notions such as local boost Killing vector field, local Rindler temperature etc. Work towards this is currently in progress.

2.2. Thermodynamic structure of Einstein tensor

We will now use the near horizon symmetries of the Einstein tensor to prove that the field equations near the horizon have a thermodynamic interpretation. We begin with the following expression for the on-horizon structure of the Einstein tensor (which can be found in [3];

$$G^{\widehat{\xi}}_{\widehat{\xi}}|_H = G^{\widehat{n}}_{\widehat{n}}|_H = \frac{1}{2}\text{tr}\left[\sigma_2\right] - \frac{1}{2}R_{\parallel} \tag{7}$$

where $R_{\parallel}$ is the Ricci scalar of the on-horizon transverse metric, $[\sigma_H]_{AB}$. The Einstein tensor components given above are evaluated in an orthonormal tetrad appropriate for a timelike observer moving along the orbit of the Killing vector field generating the Killing horizon. [3]

It is easy to show that, on the horizon: $\delta_\lambda\sqrt{\sigma} = (2\kappa)^{-1}\sqrt{\sigma}\,\text{tr}\left[\sigma_2\right]\,\delta\lambda$. We can use this to express $G^{\widehat{\xi}}_{\widehat{\xi}}$ in terms of variation of the transverse area with respect to affine parameter λ. Further, using the equality $G^{\widehat{\xi}}_{\widehat{\xi}}|_H = G^{\widehat{n}}_{\widehat{n}}|_H$ arising out of near-horizon symmetries alongwith the field equations, give

$$T\frac{\partial}{\partial\lambda}\left[\int\frac{1}{4}\sqrt{\sigma}\,\text{d}^2y\right]_H \delta\lambda - \left[\int_H\frac{1}{8\pi}R_{\parallel}\sqrt{\sigma}\,\text{d}^2y\right]\frac{\delta\lambda}{2} \;=\; \int_H P_\perp\sqrt{\sigma}\,\text{d}^2y\,\delta\lambda \;=\; \overline{F}\,\delta\lambda \tag{8}$$

where we have identified $T = \kappa/2\pi$ as the horizon temperature and used the interpretation of $T^{\widehat{n}}_{\widehat{n}}$ as normal pressure, $P_\perp$, *on the horizon*. The last equality defines $\overline{F}$ as the average normal force over the horizon "surface" (in the spirit of membrane paradigm) so that $\overline{F}\,\delta\lambda$ is the virtual work done in displacing the horizon by an affine distance $\delta\lambda$. The above equation can now be recognized as

$$T\delta_\lambda S - \delta_\lambda E = \overline{F}\,\delta\lambda \tag{9}$$

where

$$S \;=\; \frac{1}{4}\int_y\sqrt{\sigma}\,\text{d}^2y \tag{10}$$

$$E \;=\; \int_\lambda\frac{1}{16\pi}\left[\int_y R_{\parallel}\sqrt{\sigma}\,\text{d}^2y\right]\delta\lambda \tag{11}$$

give entropy and energy when evaluated at $\lambda = \lambda_H$. While the expression for entropy is familiar, the expression for energy is not, but can be shown to reduce to standard expressions in specific cases.

We have therefore shown that Einstein equations near a static horizon has a highly symmetric structure which has the form of the first law of thermodynamics. It is trivial to show that the above expression reproduces the original result of Padmanabhan for static spherical symmetric case.

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[3] The validity of expression (7), with the given Taylor series for $N(n, y^A)$ and $\sigma_{AB}(n, y^A)$, can be easily checked using a symbolic package such as `MAPLE`.

2.3. Thermodynamic structure of Lanczos-Lovelock tensor

A natural generalization of the above result would be to look at Lanczos-Lovelock lagrangians, which are the unique generalizations of Einstein tensor to higher dimensions, and yield equations of motion which are well behaved. We shall simply outline the derivation here; details can be found in [2].

The Lanczos-Lovelock (LL) lagrangians are given by

$$\mathcal{L}_m^{(D)} = \left(\frac{1}{16\pi}\frac{1}{2^m}\right) \boldsymbol{\delta}_{c_1 d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} \, \boldsymbol{R}_{a_1 b_1}^{c_1 d_1} \cdots \boldsymbol{R}_{a_m b_m}^{c_m d_m} \tag{12}$$

The corresponding equations of motion are $2E_j^i = T_j^i$, where

$$E_{j(m)}^i = \left(\frac{1}{16\pi}\frac{m}{2^m}\right) \boldsymbol{\delta}_{jd_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} \, \boldsymbol{R}_{a_1 b_1}^{id_1} \cdots \boldsymbol{R}_{a_m b_m}^{c_m d_m} - \frac{1}{2}\delta_j^i \mathcal{L}_m \tag{13}$$

where m is an integer, and Einstein theory corresponds to $m = 1$. These lagrangians have various special properties which have been discussed extensively in the literature. In particular, these lagrangians satisfy: $\nabla_a (\partial \mathcal{L}/\partial R_{abcd}) = 0$.

A detailed analysis using Gauss-Codazzi decomposition and Combinatorics leads to

$$E_{\hat{b}}^{\hat{a}}|_H = \left[\begin{array}{cc|c} E_\perp & 0 & 0 \\ 0 & E_\perp & 0 \\ \hline 0 & 0 & E_\parallel{}_{[(D-2)\times(D-2)]} \end{array} \right] \tag{14}$$

which generalizes the result of Einstein theory. The object of interest, viz. $E_\perp$, turns out to be

$$E_\perp = \underbrace{\left(\frac{1}{16\pi}\frac{m}{2^m}\right) \sigma^{CA} \, \mathcal{E}_A^B \, [\sigma_2]_{CB}}_{\text{corr to variation of a lower dim. action!}} - \left(\frac{1}{2}\right) \mathcal{L}_m^{(D-2)} \tag{15}$$

Once again, some simple algebraic manipulations similar to those done in the Einstein case puts this in the form (valid on the horizon):

$$2E_\perp \, \sqrt{\sigma} \, \delta\lambda = \frac{\kappa}{2\pi} \underbrace{\left(\frac{1}{8}\frac{m}{2^{m-1}}\right) \mathcal{E}^{BC} \, \delta_\lambda \sigma_{BC} \, \sqrt{\sigma}}_{\delta_\lambda S} - \underbrace{\mathcal{L}_m^{(D-2)} \sqrt{\sigma} \, \delta\lambda}_{\delta_\lambda E} \tag{16}$$

where

$$S = 4\pi m \int \mathrm{d}\Sigma \, \mathcal{L}_{m-1}^{(D-2)} \qquad \mapsto \frac{1}{4}\int \mathrm{d}\Sigma \tag{17}$$

$$E = \int^\lambda \delta\lambda \int \mathrm{d}\Sigma \, \mathcal{L}_m^{(D-2)} \qquad \mapsto \int_\lambda \frac{1}{16\pi} \left[\int_y R_\parallel \sqrt{\sigma} \, \mathrm{d}^2 y\right] \delta\lambda \tag{18}$$

where the arrows give corresponding expressions for $D = 4$ Einstein case. The thing to note is that both terms on RHS are expressible as "variations" of quantities locally defined on the horizon! In fact, S is precisely the Noether charge entropy as defined by Wald [4]. The quantity E as defined above gives an expression for energy which matches with known expressions for specific cases [for example, it gives $(1/2)r_H$ for spherically symmetric solutions in $D = 4$ Einstein theory, and reproduces the correct mass for spherically symmetric black hole solutions in Lovelock theory as calculated by others]. In fact, this expression for energy deserves a closer look, since it provides a very natural generalization of quasilocal energy for aspheric black holes in Einstein as well as Lovelock gravity.

Following exactly the same steps as in Einstein case, we conclude that the field equations can be written as: $T\delta_\lambda S - \delta_\lambda E = \overline{F}\delta\lambda$ on the horizon.

3. Comments and Discussion

To put the result in appropriate context [5], begin by noting that what we have shown is that the following relations

$$\underbrace{\boldsymbol{E}(\widehat{\boldsymbol{n}},\widehat{\boldsymbol{n}}) = -\boldsymbol{E}(\boldsymbol{u},\boldsymbol{u})}_{near-horizon\ symmetry} \ \& \ \underbrace{\boldsymbol{E}(\boldsymbol{u},\boldsymbol{u}) = (1/2)\,\boldsymbol{T}(\boldsymbol{u},\boldsymbol{u})}_{field\ eq}$$

have a thermodynamic structure. Democracy of all observers then implies that $\boldsymbol{E} = (1/2)\boldsymbol{T}$ are thermodynamic:

$$\boldsymbol{E}(\boldsymbol{u},\boldsymbol{u}) = \frac{1}{2}\,\boldsymbol{T}(\boldsymbol{u},\boldsymbol{u}) \ \underset{all\ \boldsymbol{u}}{\longmapsto} \ \boldsymbol{E} = \frac{1}{2}\,\boldsymbol{T}$$

Further, note that, in the limit $N \to 0$, we have $N^2\boldsymbol{l} \to \boldsymbol{\xi}$. Using this and $\boldsymbol{u} = \boldsymbol{\xi}/N$, we see that $\boldsymbol{T}(\boldsymbol{u},\boldsymbol{u}) \to \boldsymbol{T}(\boldsymbol{\xi},\boldsymbol{l})$, which is precisely the flow of energy when the horizon undergoes a virtual displacement along the outgoing null geodesics. Incidentally, there is another way to look at this which is worth mentioning: the object $g^{\perp}_{ab} = 2\xi_{(a}l_{b)}$ is the transverse part of the induced metric, and hence $T_{ab}\xi^a l^b = (1/2)\,\text{tr}_2[T_{ab}]$, where tr_2 is the trace with respect to the metric $2\xi_{(a}l_{b)}$. For spherically symmetric spacetimes, since $T_0^0 = T_r^r = P$, we have $(1/2)\,\text{tr}_2[T_{ab}] = P$, which is the result given in [1].

Finally, let us briefly comment on lagrangians which depend on metric and Riemann tensor (but not it's derivatives); i.e., $L = L\,[g_{ab}, R_{abcd}]$. Equations of motion are given by:

$$P_a{}^{cde}R_{bcde} - 2\nabla^c\nabla^d P_{acdb} - \frac{1}{2}Lg_{ab} = 8\pi T_{ab} \tag{19}$$

where P_{abcd} is defined as $\partial L/\partial R_{abcd}$, and inherit all the symmetries of the Riemann tensor. It is obvious that analyzing the near-horizon symmetries is not going to be easy, particularly due to the presence of the middle term on LHS; in fact, no such symmetries might exist at all for general lagrangians without imposing additional restrictions. One might concentrate [taking hint from the Lovelock case] on $T_{ab}\xi^a l^b$ without worrying about the symmetries of the field tensor, but then one risks loosing some crucial link between such symmetries and black hole entropy. Indeed, that such a link exists is most clearly evident from the work of Carlip [6], and it would in fact be worthwhile to connect Carlip's analysis with our result. Nonetheless, we observe that the thermodynamic structure of field equations are crucially linked to the near-horizon symmetries and select out a particular class of lagrangians, the Lanczos-Lovelock lagrangians. This seems reasonable since it is only for this class of lagrangians that one obtains second order equations of motion and the initial value problem is well defined. Since we do not have any criterion other than symmetry principles to analyze notions such as gravitational entropy in arbitrary theories of gravity, such a restriction to lagrangians is important since it gives us a handle on the sort of low-energy effective actions we may expect from a full theory of quantum gravity. It must be remembered that two apparently different types of symmetries [definitely connected] are evident in our result: the appearance of Wald entropy which arises as a Noether charge of diffeomorphism invariance (restricted to diffeomorphisms generating isometries) and the near-horizon symmetry of the field tensor, which in fact enforces, in the 2-dimensional subspace orthogonal to the horizon, conformal invariance of matter fields near the Killing horizon. Moreover, our result above already clearly provides a direct connection of entropy so obtained with gravitational dynamics.

4. Acknowledgments

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