The Classical Capacity of a Quantum Erasure Queue-Channel

Prabha Mandayam
Department of Physics
IIT Madras, Chennai, India

Krishna Jagannathan, Avhishek Chatterjee
Department of Electrical Engineering
IIT Madras, Chennai, India

***Abstract*—We consider a setting where a stream of qubits is processed sequentially. We derive fundamental limits on the rate at which classical information can be transmitted using qubits that decohere as they wait to be processed. Specifically, we model the sequential processing of qubits using a single server queue, and derive expressions for the classical capacity of such a quantum 'queue-channel.' Focusing on quantum erasures, we obtain an explicit single-letter capacity formula in terms of the stationary waiting time of qubits in the queue. Our capacity proof also implies that a 'classical' coding/decoding strategy is optimal, i.e., an encoder which uses only orthogonal product states, and a decoder which measures in a fixed product basis, are sufficient to achieve the classical capacity of the quantum erasure queue-channel. More broadly, our work begins to quantitatively address the impact of decoherence on the performance limits of quantum information processing systems.**

I. Introduction

Unlike classical bits, quantum bits (or qubits) undergo rapid *decoherence* in time, due to certain unavoidable physical phenomena. Information stored in a qubit may be completely or partially lost as the qubit decoheres. The nature and the rate of the decoherence process depend on the particular physical implementation of the quantum state, as well as other factors such as the environment and temperature. For example, superconducting Josephson junction based qubits have average coherence times that are typically of the order of a few tens of microseconds [1, Table 2].

Decoherence of qubits poses a major challenge to scalability of quantum information processing systems — therefore, it is imperative to obtain a quantitative understanding of the impact of decoherence on the performance limits of quantum information processing systems.

In this paper, we consider a setting where a stream of qubits is processed *sequentially* – for example, this 'processing' of qubits could involve transmitting them through an optical medium, or performing logical gate operations on them. We derive fundamental limits on the rate at which classical information can be transmitted using qubits that *decohere as they wait to be processed.*

To be more precise, we model the sequential processing of a stream of qubits using a single server queue. The qubits arrive at the queue according to some stationary point process, and the qubits are processed at a fixed average rate. The qubits undergo decoherence (leading to errors or erasures) as they wait to be processed, and the probability of error/erasure of each qubit is modeled as a function of the time spent in the queue by *that* qubit. We call this system a 'queue-channel' (a term borrowed from [2]), and characterise the information capacity of the queue-channel, for the case when decoherence leads to *erasures.*

A. Related Work

An information theoretic notion of reliability of a queuing system with state-dependent errors was introduced and studied in [2], where the authors considered queue-length dependent errors motivated mainly by human computation and crowd-sourcing. The classic paper of Anantharam and Verdú considered timing channels where information is encoded in the times between consecutive information packets, and these packets are subsequently processed according to some queueing discipline [3]. Due to randomness in the sojourn times of packets through servers, the encoded timing information is distorted, which the receiver must decode. In contrast to [3], *we are not concerned with information encoded in the timing between packets — in our work, all the information is in the qubits (i.e., symbols/packets, in the setting of [3]).*

In a recent paper [4], we considered the queue-channel problem described above, and derived the capacity of such a queue-channel under certain technically restrictive conditions. Specifically, [4] restricts the encoder to using only orthogonal product states, and the decoder measures in a *fixed* product basis. In this restricted setting, qubits are essentially made to behave like 'classical bits that decohere,' and the underlying quantum channel effectively simulates a classical channel known as the *induced classical* channel. In general, the classical capacity of the underlying quantum channel could be *larger* than the capacity of the induced classical channel, because the former allows for entangled channel uses and more general (joint) measurements at the decoder.

As an aside, we remark that the erasure queue-channel treated in [4] can be used to model a multimedia-streaming scenario, where information packets become useless (erased) after a certain time.

B. Our Contributions

In this paper, we completely characterise the classical capacity of the quantum erasure queue-channel, without the restrictions in [4]. Specifically, we allow for possibly entangled channel uses by the encoder, and arbitrary measurements at the decoder.

Notably, we show that the erasure queue-channel capacity remains the *same* as the capacity expression derived in [4, Theorem 1] for the induced classical channel. That is, the capacity does not increase by allowing for entangled channel uses by the encoder, and arbitrary measurements at the decoder. In other words, the classical coding/decoding strategy in [4] (proposed for the induced classical channel), is indeed sufficient to realise the classical capacity of the underlying quantum erasure queue-channel.

In hindsight, this result may not be altogether surprising, considering that the classical capacity of the *memoryless* quantum erasure channel is the same as the capacity of the classical erasure channel. In other words, a classical coding strategy is sufficient to realise the classical capacity of the (memoryless) quantum erasure channel – see [5].

Proving the erasure queue-channel capacity result involves overcoming some technical challenges. First, the erasure queue-channel is non-stationary. Second, the erasure events corresponding to consecutive qubits are correlated through their waiting times, which are in turn governed by the queuing process. This leads to memory across consecutive channel uses.

Interestingly, we note that the capacity result in [4, Theorem 1] for the induced classical channel readily offers an 'achievable rate' for the quantum erasure queue-channel – after all, any rate that is achievable with the restrictions in [4] can be achieved without those restrictions. Much of the technical challenge therefore lies in proving a 'converse theorem,' i.e., in showing a capacity upper bound that matches the expression in [4, Theorem 1].

Our upper bound proof proceeds via the following key steps. The first step involves showing a certain conditional independence of n consecutive channel uses, *conditioned* on the sequence of qubit waiting times $(W_1, W_2, \ldots, W_n)$. Specifically, we show that the n-qubit queue-channel factors into a *tensor product* of single-use erasure channels, for any given sequence of the waiting times. Next, we use a general capacity upper bound proved in [6, Lemma 5], which we then simplify using the conditional independence result from the first step. Finally, we invoke the celebrated *additivity* result of Holevo [7] for the quantum erasure channel, and obtain a single-letter capacity upper bound. This upper bound matches the achievable rate using classical coding/decoding strategies, completing the proof.

II. System Model

The model we study is similar to the one considered in [4]. Specifically, a source generates a classical bit stream, which is encoded into qubits. These qubits are sent sequentially to a continuous-time single server queue according to a stationary point process of rate λ. The service times for each qubit are independent and identically distributed (i.i.d.). After getting

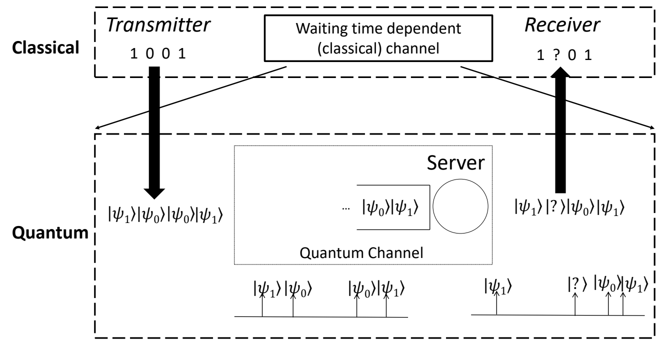

Fig. 1. Schematic of the queue-channel depicting the case of quantum erasure.

processed by the server, each qubit is measured and interpreted as a classical bit.

The service time of the jth qubit is denoted by S_j and has a cumulative distribution F_S. The average service rate of each qubit is μ, i.e., $\mathbf{E}_{F_S}[S] = 1/\mu$. In the interest of simplicity and tractability, we assume Poisson arrivals, i.e., the time between two consecutive arrivals is i.i.d. with an exponential distribution with parameter λ. For stability of the queue, we assume $\lambda < \mu$. For ease of notation let us assume $\mu = 1$. (Our results easily extend to general μ). For the service discipline, we consider First-Come-First-Served (FCFS).

Let A_j and D_j be the arrival and the departure epochs of jth qubit, respectively and $W_j = D_j - A_j$ be the total time that jth qubit spends in the queue.

A. Noise Model

As the qubits wait to be served, they undergo decoherence, leading to errors at the receiver. This decoherence is modeled in general as a completely positive trace preserving (CPTP) map on the the qubit states [8]. In general, a given symbol $X_j \in \mathcal{X}$, is encoded as a positive trace-class operator (called the *density operator*) ρ_j on a Hilbert space $\mathcal{H}^A$ of dimension $|\mathcal{X}|$. The noise channel is denoted as a map $\mathcal{N} : \mathcal{S}(\mathcal{H}^A) \to \mathcal{S}(\mathcal{H}^B)$, where $\mathcal{S}(\mathcal{H}^A)$ $(\mathcal{S}(\mathcal{H}^B))$ denotes the set of density operators on the *input* Hilbert space $\mathcal{H}^A$ (*output* Hilbert space $\mathcal{H}^B$). The probability that a given state ρ_j undergoes decoherence is indeed a function of the waiting time W_j and the noise map is accordingly parameterized in terms of the waiting time as $\mathcal{N}_{W_j}$. The noisy output state after the action of the map $\mathcal{N}_{W_j}$ is denoted as $\sigma_j \equiv \mathcal{N}_{W_j}(\rho_j)$. This noisy state is measured by the receiver by performing a general quantum measurement and decoded as the output symbol $Y_j \in \mathcal{Y}$.

An n-length transmission over the above channel is denoted as follows. Inputs are drawn from the set $\mathcal{X}^{(n)}$ of length n symbols $\{X^n = (X_1 X_2 \ldots X_n)\}$, and encoded into density operators $\rho_{X^n} \in \mathcal{S}((\mathcal{H}^A)^{\otimes n})$ on an n-fold tensor product of the input Hilbert space $\mathcal{H}^A$. The n-qubit channel is denoted $\mathcal{N}^{(n)}_{W^n}$ and parameterized by the sequence of waiting times $W^n = (W_1, W_2, \ldots, W_n)$. Note that the encoded state ρ_{X^n}

could be entangled across multiple channel uses. Furthermore, in general, the n-qubit queue-channel is not a stationary, memoryless channel and does not automatically factor into an n-fold tensor product of single qubit channels.

We refer to this system as a quantum queue-channel, and characterize the classical capacity of this system (in bits/sec).

In this paper, we restrict attention to a quantum erasure channel [9] which acts on the jth state ρ_j as follows: ρ_j remains unaffected with probability $1 - p(W_j)$, and is erased to a (fixed) erasure state $|e\rangle\langle e|$ with probability $p(W_j)$, where $p : [0, \infty) \rightarrow [0, 1]$ is typically increasing.

Note that we have modelled the erasure probability of the jth qubit as an explicit function of the time W_j spent in the queue. For instance, in several physical scenarios, the decoherence time of a single qubit maybe modelled as an exponential random variable. In other words, the probability of a qubit erasure after waiting for a time W is given by $p(W) = 1 - e^{-\kappa W}$, where $1/\kappa$ is a characteristic time constant of the physical system under consideration [8, Section 8.3].

III. Quantum Queue-Channel Capacity

We are interested in defining and computing the information capacity of the waiting time dependent queue-channel.

A. Definitions

Let M be the message transmitted from a set $\mathcal{M}$ and $\hat{M} \in \mathcal{M}$ be its estimate at the receiver.

Definition 1: An (n, R, ϵ, T) quantum code consists of an encoding function $X^n = f(M)$, leading to an encoded n-qubit quantum state ρ_{X^n} corresponding to message M, and the decoding function $\hat{M} = g(\Lambda, \mathcal{N}^{(n)}(\rho_{X^n}), A^n, D^n)$ corresponding to a measurement Λ at the receiver's end, where the cardinality of the message set $|\mathcal{M}| = 2^{nR}$, and for each codeword, the expected total time for all the symbols to reach the receiver is less than T.

Definition 2: If the decoder chooses $\hat{M}$ with average probability of error less than ϵ, the code is said to be ϵ-achievable. For any $0 < \epsilon < 1$, if there exists an ϵ-achievable code (n, R, ϵ, T), the rate $\frac{R}{T}$ is said to be achievable.

Definition 3: The information capacity of the queue-channel is the supremum of all achievable rates for a given (Poisson) arrival process with arrival rate λ and is denoted by C bits per unit time.

Note that the information capacity of the queue-channel depends on the arrival rate, the service process, and the noise model. We assume that the receiver knows the realization of the arrival and the departure time of each symbol.

In order to evaluate the classical capacity of the quantum queue-channel, we invoke the capacity formula obtained in [6], for the classical capacity of general quantum channels which are neither stationary nor memoryless. The following proposition is a direct consequence of the general channel capacity expression in [6].

Proposition 1: The capacity of the queue-channel (in bits/sec) described in Sec. II is given by

$$C = \lambda \sup_{\{\vec{P}, \vec{\rho}\}} \underline{\mathbf{I}}(\{\vec{P}, \vec{\rho}\}, \vec{\mathcal{N}}_{\vec{W}}), \quad (1)$$

where, $\underline{\mathbf{I}}(\{\vec{P}, \vec{\rho}\}, \vec{\mathcal{N}}_{\vec{W}})$ is the quantum analog of the spectral inf-information rate defined in [10]. Here, $\vec{P}$ is the totality of sequences $\{P^n(X^n)\}_{n=1}^{\infty}$ of probability distributions (with finite support) over input sequences X^n, and $\{\vec{\rho}\}$ denotes the sequences of states $\{\rho_{X^n}\}$ corresponding to the encoding $X^n \rightarrow \rho_{X^n}$. Finally, $\vec{\mathcal{N}}_{\vec{W}}$ denotes the sequence of channels $\{\mathcal{N}_{W^{(n)}}^{(n)}\}_{n=1}^{\infty}$, which are parameterised by the corresponding waiting time sequences $\{W^{(n)}\}_{n=1}^{\infty}$.

B. Erasure Queue-Channels

Erasure channels are ubiquitous in classical as well as quantum information theory. Our model captures a quantum information system where qubits decohere with time into erased (or non-informative) quantum states.

A single-use quantum erasure channel, for a qubit with waiting time W_1 is characterized by a pair of noise operators (or *Kraus* operators), namely the operator $E : \rho_1 \rightarrow |e\rangle\langle e|$ which maps any input density operator ρ_1 to a fixed erasure state $|e\rangle\langle e|$, and the identity operator $I : \rho_1 \rightarrow \rho_1$. If the input state has a waiting time W_1, the erasure and identity operations occur with probabilities $q_E(W_1) = p(W_1)$ and $q_I(W_1) = 1 - p(W_1)$ respectively. Thus, the final state after the action of the erasure queue-channel for a given input state ρ_1, with waiting time W_1 is,

$$\mathcal{N}_{W_1}(\rho_1) = q_E(W_1)\, E\rho_1 E^{\dagger} + q_I(W_1)\, \rho_1.$$

Theorem 1: For the erasure queue-channel defined above, the capacity $C = \lambda\ \mathbf{E}_{\pi}\left[1 - p(W)\right]$ bits/sec, irrespective of the receiver's knowledge of the arrival and the departure times, where π is the stationary distribution of the waiting time in the queue.

Proof Outline: Obtaining the queue-channel capacity of the quantum erasure channel poses certain technical challenges, since the erasure probabilities are correlated across different channel uses. In other words, the probability that the i^{th} qubit gets erased is a function of its waiting time in the queue, which in turn depends on the waiting time of the previous $(i-1)^{\text{th}}$ qubit and so on. Furthermore, the channel is non-stationary. However, for the queue-channel model considered here, the n-qubit queue-channel does factor into a *tensor product* of single-use channels, *conditioned* on the sequence of waiting times $(W_1, W_2, \ldots, W_n)$. This is formally shown in Lemma 1 below.

In order to obtain an *upper bound on capacity*, we proceed via the following key steps. First, we invoke an upper bound proved in [6, Lemma 5], to bound the capacity as the limit inferior of the *Holevo capacity* of a sequence of erasure channels. Next, we use the tensor product from of the channel obtained in Lemma 1 in conjunction with the celebrated *additivity* result of Holevo [7] for the quantum erasure channel. This results

$$
\begin{aligned}
\mathcal{N}^{(n)}_{W^{(n)}}(\rho_{12\ldots n}) &= \sum_{k_1,k_2,\ldots,k_n} q_{k_1k_2\ldots k_n}(W_1,W_2,\ldots,W_n)\, A_{k_1}\otimes A_{k_2}\ldots\otimes A_{k_n}\,(\rho_{12\ldots n})\, A^\dagger_{k_1}\otimes A^\dagger_{k_2}\ldots\otimes A^\dagger_{k_n} \\
&= \sum_{k_1,k_2,\ldots,k_n} q_{k_1}(W_1)q_{k_2}(W_2)\ldots q_{k_n}(W_n)A_{k_1}\otimes A_{k_2}\ldots\otimes A_{k_n}\,(\rho_{12\ldots n})\, A^\dagger_{k_1}\otimes A^\dagger_{k_2}\ldots\otimes A_{k_n} \\
&= (\mathcal{N}_{W_1}\otimes\mathcal{N}_{W_2}\ldots\otimes\mathcal{N}_{W_n})\,(\rho_{12\ldots n}). \qquad (2)
\end{aligned}
$$

in a single-letter expression for the queue-channel capacity of the quantum erasure channel, as shown in Proposition 2.

Finally, the *achievability* proof follows by fixing a classical encoding and decoding strategy and showing that the capacity of the induced classical channel does indeed coincide with the upper bound. ∎

Lemma 1 (Conditional independence): The n-qubit erasure queue-channel factors into a tensor product of single-qubit erasure channels, conditioned on the sequence of waiting times $(W_1,W_2,\ldots,W_n)$.

Proof: Recall that the single-qubit erasure channel with associated waiting time W_1 can be described using the operators $\{E,I\}$ with associated probabilities of occurrence denoted as $q_E(W_1), q_I(W_1)$ respectively. Consider a sequence of n qubits transmitted via the erasure queue-channel with associated waiting times $W_1,W_2,\ldots,W_n$. The n-qubit erasure channel maybe described via n-fold tensor-product operators of the form $\{A_1\otimes A_2\otimes\ldots\otimes A_n\}$, where each single-qubit operator $A_i\in\{E,I\}$. There are 2^n such operators, occurring with probabilities $q_{k_1k_2\ldots k_n}(W_1,W_2,\ldots,W_n)$, with the indices $k_i\in\{E,I\}$, depending on whether A_i was an erasure or the identity operator.

This n-fold channel is a non-iid, correlated quantum channel in general, since the joint distribution $q_{k_1k_2\ldots k_n}(W_1,W_2,\ldots,W_n)$ does not factor into a product of the individual error probabilities for each qubit. However, for an FCFS queue, the probabilities $q_{k_1k_2\ldots k_n}$ satisfy the Markov property, $q_{k_1k_2\ldots k_n} = q_{k_n|k_{n-1}}q_{k_{n-1}|k_{n-2}}\ldots q_{k_2|k_1}q_{k_1}$. Furthermore, conditioned on the the waiting time sequence $W^{(n)} = (W_1,W_2,\ldots,W_n)$, the conditional probabilities simply reduce to $q_{k_i|k_{i-1}} = q_{k_i}(W_i)$.

Therefore, conditioned on the waiting times $(W_1,W_2,\ldots,W_n)$ we may represent the action of the n-qubit channel on any n-qubit state $\rho_{12\ldots n}$ as shown in (2). In other words, conditioned on the waiting time sequence W^n, the n-qubit channel factors into an n-fold tensor product of the form $\mathcal{N}_{W_1}\otimes\mathcal{N}_{W_2}\otimes\ldots\otimes\mathcal{N}_{W_n}$, as desired. ∎

We next state and prove an upper bound on the queue-channel capacity of the quantum erasure channel.

Proposition 2 (Upper bound on Capacity): The capacity of the quantum erasure queue-channel $C\le\lambda\,\mathbf{E}_\pi\,[1-p(W)]$.

Proof: We start with an upperbound on the quantum analog of the inf-information rate proved in [6, Lemma 5]:

$$
\underline{\mathbf{I}}(\{\vec{P},\vec{\rho}\}\vec{\mathcal{N}}) \le \liminf_{n\to\infty}\frac{1}{n}\chi(\{P^{(n)},\rho_{X^n}\},\mathcal{N}^{(n)}_{W^{(n)}}),
$$

where, $\chi(\{P^{(n)},\rho_{X^n}\},\mathcal{N}^{(n)}_{W^n})$ is the Holevo information of the ensemble $\{P^{(n)}(X^n),\mathcal{N}^{(n)}_{W^{(n)}}(\rho_{X^n})\}$.

Recall that the Holevo information of an ensemble $\mathcal{E}\equiv\{P_x,\mathcal{N}(\rho_x)\}$ is defined as the von Neumann entropy difference [9],

$$
\chi(\{P_x,\mathcal{N}(\rho_x)\}) = H\left(\sum_x P_x\mathcal{N}(\rho_x)\right) - \sum_x p_x H(\mathcal{N}(\rho_x)).
$$

The supremum of this entropy difference over all input ensembles $\{P(x),\rho_x\}$ defines the Holevo capacity of the channel $\mathcal{N}$:

$$
\chi(\mathcal{N}) := \sup_{\{P(x),\rho_x\}}\chi(\{P_x,\mathcal{N}(\rho_x)\}).
$$

Consider now the Holevo capacity $\chi(\mathcal{N}^{(n)}_{W^{(n)}})$ of the n-qubit erasure queue-channel, for a given sequence of waiting times $W^{(n)}$. Lemma 1 implies that

$$
\chi(\mathcal{N}^{(n)}_{W^n}) = \chi(\mathcal{N}_{W_1}\otimes\mathcal{N}_{W_2}\otimes\ldots\otimes\mathcal{N}_{W_n}).
$$

Next we invoke the well known additivity property of the Holevo capacity of the quantum erasure channel [7], to obtain, for any n,

$$
\chi(\mathcal{N}^{(n)}_{W^n}) = \sum_{i=1}^{n}\chi\left(\mathcal{N}_{W_i}\right).
$$

Recasting this in terms of the Holevo information of the encoding ensemble for the n-qubit channel, we get,

$$
\begin{aligned}
&\sup_{\{P^n(X^n),\rho_{X^n}\}}\chi(\{P^n_{X^n},\mathcal{N}^{(n)}_{W^{(n)}}(\rho_{X^n})\}) \\
&= \sum_{i=1}^{n}\sup_{\{P_i(X_i),\rho_i\}}\chi(\{P(X_i),\mathcal{N}_{W_i}(\rho_i)\}). \qquad (3)
\end{aligned}
$$

Finally, we use the fact that the Holevo information of a single use erasure channel corresponding to waiting time W_i is given by [5],

$$
\chi(\mathcal{N}_{W_i}) \equiv \sup_{\{P(x),\rho_x\}}\chi(\{P_x,\mathcal{N}_{W_i}(\rho_x)\}) = 1-p(W_i). \qquad (4)
$$

Combining the above sequence of steps, we thus get the following upper bound on the capacity of the erasure queue-channel:

$$\begin{aligned}
C &\overset{(a)}{=} \lambda \sup_{\{\vec{P},\vec{\rho}\}} \underline{\mathbf{I}}(\{\vec{P},\vec{\rho}\},\vec{\mathcal{N}}_{\vec{W}}) \\
&\overset{(b)}{\leq} \lambda \sup_{\{\vec{P},\vec{\rho}\}} \liminf_{n\to\infty} \frac{1}{n}\chi(\{P^{(n)},\rho_{X^n}\},\mathcal{N}^{(n)}_{W^{(n)}}) \\
&\overset{(c)}{\leq} \lambda \liminf_{n\to\infty} \frac{1}{n} \sup_{\{\vec{P},\vec{\rho}\}} \chi(\{P^{(n)},\rho_{X^n}\},\mathcal{N}^{(n)}_{W^{(n)}}) \\
&\overset{(d)}{=} \lambda \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sup_{\{P(X_i),\rho_i\}} \chi(\{P(X_i),\mathcal{N}_{W_i}(\rho_i)\}) \\
&\overset{(e)}{=} \lambda \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (1-p(W_i)) \overset{(f)}{=} \lambda\ \mathbf{E}_\pi\left[1-p(W)\right] \text{ a.s.}
\end{aligned}$$

Here, (a) is simply the definition of the queue-channel capacity as stated in Eq. (1), and, (b) is the upper bound from [6, Lemma 5]. The inequality in (c) follows from the fact that for each n, the Holevo information is upper bounded by the supremum over all input encodings. The equality in (d) follows from the conditional independence of the n-use channel and the additivity of the quantum erasure channel (see Eq. (3)) and (e) simply uses the Holevo capacity of the single use quantum erasure channel stated in Eq. (4). Finally, (f) follows from the ergodicity of the $\mathsf{M/GI/1}$ queue. ■

Proposition 3 (Lower bound on Capacity (Achievability)): The capacity of the quantum erasure queue-channel satisfies $C \geq \lambda\ \mathbf{E}_\pi\left[1-p(W)\right].$

Proof: We prove the lower bound by producing a particular encoding/decoding strategy that achieves the said capacity expression. In particular, we employ a classical strategy in which classical bits 0 and 1 are encoded into two fixed orthogonal states (say $|\psi_0\rangle$ and $|\psi_1\rangle$), and the decoder also measures in a fixed basis. The input codewords are unentangled across multiple channel uses and the decoder simply performs a product measurements. In this setting, the qubits essentially behave as classical bits, and the quantum erasure channel essentially simulates the *induced classical* channel. The capacity of the corresponding induced classical channel is equal to $\lambda\ \mathbf{E}_\pi\left[1-p(W)\right]$, as shown in [4, Theorem 1]. ■

We remark that the above capacity result does not depend on the specific functional form of $p(\cdot)$. Further, the capacity result holds for any stationary and ergodic queue – i.e., it does not assume any specific queueing model. If we assume the functional form $p(W) = 1-\exp(-\kappa W)$ for the erasure probability, the following corollary is immediate.

Corollary 1: When the decoherence time of each qubit is exponentially distributed, i.e., $p(W) = 1-\exp(-\kappa W)$, the erasure queue-channel capacity is given by $\lambda \mathbf{E}_\pi\left[e^{-\kappa W}\right]$ bits/sec.

We remark that the capacity expression $\lambda \mathbf{E}_\pi\left[e^{-\kappa W}\right]$ is simply λ times the Laplace transform of the stationary waiting time W, evaluated at κ, which is the rate of decoherence. Using Pollaczek-Khinchin formula for an FCFS $\mathsf{M/GI/1}$ queue, we can obtain a closed-from expression for the capacity and the optimal arrival rate [4].

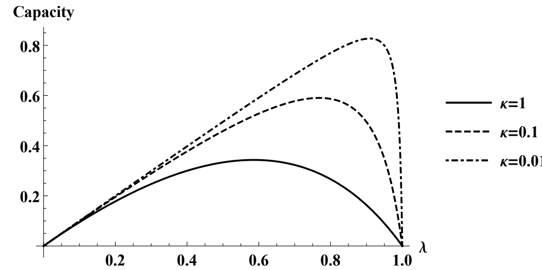

Fig. 2. The capacity of the $\mathsf{M/M/1}$ queue-channel (in bits/sec) plotted as a function of the arrival rate λ for different values of the decoherence parameter κ.

Proposition 4: For an FCFS $\mathsf{M/GI/1}$ erasure queue-channel (with $p(W) = 1-\exp(-\kappa W)$),

(i) the capacity is given by $\frac{\lambda(1-\lambda)}{1-\alpha\lambda}$ bits/sec, and
(ii) the capacity is maximised at

$$\lambda_{\mathsf{M/GI/1}} = \frac{1}{\alpha}\left(1-\sqrt{1-\alpha}\right) = \frac{1}{1+\sqrt{1-\alpha}},$$

where $\alpha = \frac{1-\tilde{F}_S(\kappa)}{\kappa}$, and $\tilde{F}_S(u) = \int \exp(-ux)dF_S(x)$ is the Laplace transform of the service time distribution.

This result offers interesting insights into the relation between the information capacity and the characteristic time-constant of the quantum states. Fig. 2 plots capacity versus arrival rate for an $\mathsf{M/M/1}$ queue of unit service rate. We note that $\kappa = 0.01$ corresponds to an average coherence time which is two orders of magnitude longer than the service time — a setting reminiscent of superconducting qubits [1]. We also notice from the shape of the capacity curve for $\kappa = 0.01$ that there is a drastic drop in the capacity, if the system is operated beyond the optimal arrival rate $\lambda_{\mathsf{M/M/1}}$. This is due to the drastic increase in delay induced decoherence as the arrival rate of qubits approaches the server capacity.

IV. Concluding Remarks and Future Work

In this paper, we considered a quantum erasure queue-channel, and derived a single-letter capacity formula in terms of the stationary waiting time in the queue. We also showed that a classical coding/decoding strategy is capacity achieving for this channel.

There is ample scope for further work along several directions. First, we can study waiting induced errors under other widely studied quantum channel models, such as the depolarising channel, phase damping and amplitude damping channels. The additivity (or otherwise) of such channels is likely to play a crucial role.

We have only considered uncoded quantum bits in this paper. We can also quantitatively evaluate the impact of using quantum codes to protect qubits from errors. Employing a code would enhance robustness to errors, but would also increase the waiting time due to the increased number of qubits to be processed. It would be interesting to characterise this tradeoff, and identify the regimes where using coded qubits would be beneficial or otherwise.

As we enter an era of quantum networks and noisy intermediate-scale quantum technologies [11], our work begins to quantitatively address the impact of decoherence on the performance limits of quantum information processing systems.

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