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Tachyon Condensation on the Elliptic Curve

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Abstract

We use the framework of matrix factorizations to study topological B-type D-branes on the cubic curve. Specifically, we elucidate how the brane RR charges are encoded in the matrix factors, by analyzing their structure in terms of sections of vector bundles in conjunction with equivariant R-symmetry. One particular advantage of matrix factorizations is that explicit moduli dependence is built in, thus giving us full control over the open-string moduli space. It allows one to study phenomena like discontinuous jumps of the cohomology over the moduli space, as well as formation of bound states at threshold. One interesting aspect is that certain gauge symmetries inherent to the matrix formulation lead to a non-trivial global structure of the moduli space. We also investigate topological tachyon condensation, which enables us to construct, in a systematic fashion, higher-dimensional matrix factorizations out of smaller ones; this amounts to obtaining branes with higher RR charges as composites of ones with minimal charges. As an application, we explicitly construct all rank two matrix factorizations.

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1. Introduction

1.1. Motivation and overview

D-branes play a very important rôle in our understanding non-perturbative properties of string and field theories, as well as in building semi-realistic models. However, the naive geometrical notion of a D-brane, in which it is thought of as wrapping some p-dimensional cycle of a Calabi-Yau manifold, is a classical concept that is valid only in certain limiting situations, such as the large radius limit. When distances are small or curvatures large, quantum corrections tend to blur notions of classical geometry, such as the dimension of a wrapped submanifold. Moreover, branes can become unstable and decay in ways that are not visible classically.

Therefore one needs to adopt a more suitable language for describing general D-brane configurations. For topological B-type D-branes, the proper mathematical framework is a certain enhanced, bounded derived category of coherent sheaves [1,2,3] (and via homological mirror symmetry [4,5], this maps to the Fukaya category of A-type branes wrapping special Lagrangian cycles). This framework retains more data than the more familiar characterization just in terms of K-theory (*i.e.*, RR charges) and thus provides a much sharper description of D-branes. That is, the category also contains the information about the brane locations, and other possible (bundle or sheaf) moduli. For instance, a configuration consisting of an anti-$D0$-brane located at some point ζ_1 of the compactification manifold, plus a $D0$-brane located at some other point ζ_2, is trivial from the K-theory point of view, but is a non-trivial object in the categorical description as long as $\zeta_1 \neq \zeta_2$. Obviously, this extra information is crucial for understanding questions such as whether, in a given D-configuration, deformations are obstructed or not (*i.e.*, what is the effective superpotential and the moduli space of its flat directions). Moreover, the language of categories is tailor-made for addressing questions about stability and bound state formation, which can be described more physically by tachyon condensation. Excellent reviews of these matters may be found in refs. [6,7].

Often physicists associate derived categories with just an abstract collection of objects (the D-branes) and maps (open strings) between them, and wonder what concrete physical benefit such a picture might provide. Indeed, by merely tracing arrows around a quiver diagram, all one obtains is a list of possible terms in the effective superpotential and these terms are merely added up with unit coefficients. However, there is more to these maps than just being pointers between objects: in general they depend on the various parameters like brane-location moduli, and thus encode valuable extra information beyond mere combinatorics. Thus, superpotential terms derived from quiver diagrams will, in general, have pre-factors depending on the various moduli of the geometry, a fact that is often neglected in the physics literature.

The abstract notions of objects and morphisms (maps) can, in fact, be easily translated into a language more familiar to physicists via the following two logical steps. First, as has been proven recently [8] in quite some generality (see also [9]), the relevant category of topological B-type D-branes is isomorphic to a certain category of matrix factorizations [10,11], which encodes the specific D-geometry in question. Second, such matrix factorizations have a direct interpretation [12,13] in terms of two-dimensional topological (twisted $N = 2$ supersymmetric) boundary Landau-Ginzburg theory [14,15]. Specifically, the maps alluded to above feature (partly) as boundary superpotentials. Thus, via this chain of arguments, boundary Landau-Ginzburg theory provides a very explicit realization of the topological field theory of B-type D-branes. A sample computation was presented in ref. [16] demonstrating how it can be used, in conjunction with mirror symmetry, to explicitly determine moduli-dependent, instanton-corrected contributions to superpotentials on intersecting branes.

The purpose of the present paper is to use the language of matrix factorizations to develop, from a physicist's point of view, a better understanding of tachyon condensation and the process of composite formation of B-type D-branes. Specifically, we will analyze, in some detail, D-branes on the cubic curve, Σ, which is the simplest situation with both bulk and boundary (brane) moduli and can be studied fairly explicitly.

Mathematically, the classification of bundles on the elliptic curve is a completely solved problem [17]: An indecomposable bundle is uniquely determined by rank and first Chern class of the bundle $\mathcal{E}$, plus a continuous parameter ζ: $(r(\mathcal{E}), c_1(\mathcal{E}), \zeta) \equiv (N_2, N_0, \zeta)$. In physical terms, this corresponds to the number of $D2$ and $D0$-branes plus, essentially, the location of the $D0$-brane on Σ. The mirror map to the Fukaya category of A-type branes is understood as well [18]. Matrix factorizations describing bundles on Σ have been described in ref. [19], but only for fixed moduli and not via tachyon condensation.[1] We will make use of these mathematical results to construct and analyze matrix factorizations explicitly depending on moduli, extending prior work [21,16] in a systematic fashion. Specifically we will show how the bundle data $(r(\mathcal{E}), c_1(\mathcal{E}), \zeta)$ are explicitly encoded by certain properties of the matrices. This leads to an algorithm that allows one to recover the brane data encoded in a given matrix factorization.

As an application of this we study (topological) tachyon condensation [22] of pairs of branes. This problem has two parts: first, determining the open-string spectrum by solving the relevant cohomology problem, and second, identifying the bundle data (N_2, N_0, ζ) of the matrix factorization that results from perturbing the direct product of matrix factorizations with an open string cohomology element. While we will encounter a few minor subtleties (such as discontinuous jumps of the cohomology upon varying moduli, and the formation of composites at threshold), the physical results are entirely as expected: brane composites can be formed according to the vector addition of brane charges, provided one properly chooses the perturbing tachyonic operators and appropriately tunes the moduli.

We will show explicitly how all configurations with rank $r = N_2 \leq 2$ can be built out of a minimal generating set of two-dimensional factorizations, which correspond to D-branes whose RR charges generate the full charge lattice. It is pretty clear that by iteratively applying the same logic, brane composites corresponding to arbitrary points (N_2, N_0) on the charge lattice can be generated. Of course, this does not come as a surprise, but this isn't the point of the present paper – the point is to understand how rather abstract mathematical concepts can be realized in a concrete physical framework, namely boundary Landau-Ginzburg theory and how to understand condensation within that framework. We expect that the insight we gain will be useful for attacking more complicated geometries, like branes on threefolds.

The plan of the paper is as follows: In the remainder of this section, we will review the description of B-branes by matrix factorizations in very simple terms; this is aimed at non-experts. Moreover, we will outline the main points of tachyon condensation in such models and present a few examples. Section 2 is then devoted

[1] While writing up this paper, we received a paper [20] that also deals with matrix factorizations pertaining to the elliptic curve, and which has some overlap with our work.

to a general discussion of how the bundle data of a given brane configuration are encoded in the corresponding matrix factorization. An important rôle is played by the holomorphic sections of bundles on the elliptic curve, which are given by Riemann theta functions and Appell functions, for rank one line bundles and rank two vector bundles, respectively.

In Section 3 we will reconsider the known 2×2 and 3×3 dimensional factorizations, and discuss in detail their structure in terms of transition functions of the relevant bundles. Section 4 deals with the open-string moduli space of the 3×3 factorization, which has a non-trivial global structure due to gauge symmetries inherent to the factorization; also, we will find how its moduli space can be compactified by adding an exceptional 4×4 factorization at the boundary, which appears to describe a pure, rigid anti-$D2$-brane.

In Section 5 we address how to properly formulate tachyon condensation in term of equivariant R-symmetry; this is necessary for disentangling the various different branes that are described by a given matrix factorization, and for subsequently identifying the various tachyon channels between pairs of them. Section 6 provides some more tools for obtaining new matrix factorizations from known ones; in particular we introduce certain bound states at threshold, which resolve a certain singularity in the multi-brane moduli space. Finally, in Section 7 we apply the techniques developed in the preceding sections, and show how the factorizations describing certain rank two vector bundles can be systematically generated by condensation of lower rank branes. We have also relegated some more technical material to the appendices.

1.2. Recapitulation: B-type branes on the elliptic curve Σ

We will consider B-type branes on the cubic curve, Σ, defined by the following hypersurface in $\mathbb{P}^2$:

$$W(x) \ \equiv \ \left({x_1}^3 + {x_2}^3 + {x_3}^3\right) \ - \ 3\,a\,x_1 x_2 x_3 \ = \ 0\,. \tag{1.1}$$

Here, a is the complex structure modulus that is related to the standard modulus of the torus via:

$$\left(\frac{3\,a\,(a^3+8)}{a^3-1}\right)^3 \ = \ j(\tau)\,, \tag{1.2}$$

where $j(\tau) = q^{-1} + 744 + \ldots$ is the familiar modular invariant function in terms of $q = e^{2\pi i\tau}$, and τ is the flat coordinate of the complex structure moduli space, which coincides with the Kähler modulus of the mirror curve, $\hat{\Sigma}$. The coordinates on the cubic can be uniformized in terms of theta functions by writing $x_\ell = \mu_\ell(\xi)$, where

$$\mu_\ell(\xi) \;\equiv\; \mu_\ell(\xi|\tau) \;=\; \omega^{(\ell-1)}\,\Theta\left[\begin{matrix}\frac{1}{3}(1-\ell)-\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\,\middle|\,3\,\xi, 3\,\tau\right], \tag{1.3}$$

with $\omega \equiv e^{2\pi i/3}$. For further details we refer the reader to Appendix A. Here, ξ is an arbitrary point on the Jacobian of Σ, which coincides with Σ itself. One should also note that $\mu_\ell(\xi)$ is not a function on Σ, but is actually a section of the line bundle, $\mathcal{L}^3$, with first Chern number $c_1 = 3$. This will lead to a natural ambiguity in determining the bundle data associated with a given matrix factorization.

As has been discussed by now in many papers [12,13,23–36], topological B-type D-branes can be described by a two-dimensional, twisted $N = 2$ supersymmetric boundary Landau-Ginzburg model based on matrix factorizations of the form

$$J(x)\,E(x) \;=\; E(x)\,J(x) \;=\; W(x)\,\mathbb{1}_{n\times n}\,. \tag{1.4}$$

Here $J(x)$ and $E(x)$ are $n\times n$ polynomial matrices[2] with values in $\mathbb{C}[x_1, x_2, x_3]$, whose precise form depends on the D-brane configuration in question. It will be important later to make use of the fact that (1.4) is invariant under gauge transformations of the form:

$$\begin{aligned} J(x) \;&\to\; U_L(x)\,J(x)\,U_R(x)\;, \\ E(x) \;&\to\; U_R^{-1}(x)\,E(x)\,U_L^{-1}(x)\;, \end{aligned} \tag{1.5}$$

for polynomial matrices $U_{L,R}(x)$ that are invertible over $\mathbb{C}[x_1, x_2, x_3]$. In particular, this means we can do arbitrary row and column reduction operations on J (respectively, E) so long as one does the corresponding inverse operations on E (respectively, J). For further details of this procedure we refer the reader to Appendix B.

[2] We do not require n to be the dimension of a Clifford algebra, which would be needed if we introduced boundary fermions [14]. The implementation of more general boundary couplings not involving fermions, was discussed in refs. [25,29].

Mathematically, matrix factorizations form a category [10,11] with objects P of the form

$$P \equiv \left[\begin{tikzcd} P_1 \arrow[d, bend left, "E_P"] \\ P_0 \arrow[u, bend left, "J_P"] \end{tikzcd} \right] , \tag{1.6}$$

where P_0 and P_1 are certain projective modules over $\mathbb{C}[x_1 \ldots, x_m]$. This "composite" form of objects, P, has a simple physical interpretation. Recall that the usual bulk Landau-Ginzburg action is not conformally invariant, but represents an action that will flow, in the infra-red, to the conformal field theory of interest, and the superpotential will remain unrenormalized along the flow. By the same token, in the boundary theory, the "constituents" $P_{0,1}$ correspond to D-branes and anti-D-branes in $\mathbb{C}[x_1 \ldots, x_m]$ and the maps J and E correspond to tachyon profiles that trigger the condensing of $P_{0,1}$ into P [12,25]. Thus we construct a particular D-brane, P, by setting up an action that will generate it via an infra-red boundary flow (*i.e.*, tachyon condensation). In this formulation, J has the interpretation as a boundary superpotential while E appears in a modified chirality condition of fermionic boundary superfields [37,38].

Anti-D-branes are associated with objects commonly denoted by $P[1]$, and look like P in (1.6) except that $J_P \to -E_P$ and $E_P \to -J_P$; we will often use the notation $\overline{P}$ for them.

The trivial object in the category, denoted by V, is described by the simplest factorization, $P_{1\times 1}$, for which $J = \mathbb{1}_{1\times 1}$ and $E = W\mathbb{1}_{1\times 1}$, or vice-versa. It corresponds to the situation where P_0 and P_1 have completely annihilated to leave no net branes at all. Factorizations of different dimension are thus considered to be equivalent if they differ by appending or removing such trivial matrix blocks.[3]

On the cubic curve, Σ, the simplest non-trivial factorizations are two- and three-dimensional [21,16]. The first ones are given by:

[3] In the derived category of matrix factorizations these objects are called perfect complexes and are divided out [8], because they do not have any non-trivial morphisms with any other object in the category. This means that there are no open-string states associated to these perfect complexes, and hence such configurations are isomorphic to the open-string vacuum [7].

$$P_{2\times 2}: \begin{cases} J_{2\times 2} &= \begin{pmatrix} Q_1 & -Q_2 \\ L_2 & L_1 \end{pmatrix} \\ E_{2\times 2} &= \begin{pmatrix} L_1 & Q_2 \\ -L_2 & Q_1 \end{pmatrix} , \end{cases} \tag{1.7}$$

where the linear entries read

$$\begin{aligned} L_1 &= \alpha_3 x_1 - \alpha_2 x_3 \ , \\ L_2 &= -\alpha_3 x_2 + \alpha_1 x_3 \ , \end{aligned} \tag{1.8}$$

and the quadratic entries are[4]

$$\begin{aligned} Q_1 &= \frac{1}{\alpha_1\alpha_2\alpha_3}\left(\alpha_1\alpha_2 x_1^2 + \alpha_2^2 x_1 x_2 - \alpha_1^2 x_2^2 - \alpha_1\alpha_3 x_3^2\right) \ , \\ Q_2 &= \frac{1}{\alpha_1\alpha_2\alpha_3}\left(\alpha_2^2 x_1^2 - \alpha_1^2 x_1 x_2 - \alpha_1\alpha_2 x_2^2 + \alpha_3^2 x_1 x_3\right) \ . \end{aligned} \tag{1.9}$$

There is also the three-dimensional factorization given by:

$$P_{3\times 3}: \begin{cases} J_{3\times 3} = \begin{pmatrix} \alpha_1\, x_1 & \alpha_2\, x_3 & \alpha_3\, x_2 \\ \alpha_3\, x_3 & \alpha_1\, x_2 & \alpha_2\, x_1 \\ \alpha_2\, x_2 & \alpha_3\, x_1 & \alpha_1\, x_3 \end{pmatrix} \\ E_{3\times 3} = \begin{pmatrix} \frac{1}{\alpha_1}x_1^2 - \frac{\alpha_1}{\alpha_2\alpha_3}x_2\, x_3 & \frac{1}{\alpha_3}x_3^2 - \frac{\alpha_3}{\alpha_1\alpha_2}x_1\, x_2 & \frac{1}{\alpha_2}x_2^2 - \frac{\alpha_2}{\alpha_1\alpha_3}x_1\, x_3 \\ \frac{1}{\alpha_2}x_3^2 - \frac{\alpha_2}{\alpha_1\alpha_3}x_1\, x_2 & \frac{1}{\alpha_1}x_2^2 - \frac{\alpha_1}{\alpha_2\alpha_3}x_1\, x_3 & \frac{1}{\alpha_3}x_1^2 - \frac{\alpha_3}{\alpha_1\alpha_2}x_2\, x_3 \\ \frac{1}{\alpha_3}x_2^2 - \frac{\alpha_3}{\alpha_1\alpha_2}x_1\, x_3 & \frac{1}{\alpha_2}x_1^2 - \frac{\alpha_2}{\alpha_1\alpha_3}x_2\, x_3 & \frac{1}{\alpha_1}x_3^2 - \frac{\alpha_1}{\alpha_2\alpha_3}x_1\, x_2 \end{pmatrix} . \end{cases} \tag{1.10}$$

Both (1.7) and (1.10) represent valid matrix factorizations satisfying (1.4) precisely when the parameters, just like the coordinates x_ℓ, satisfy the cubic equation:

$$\left({\alpha_1}^3 + {\alpha_2}^3 + {\alpha_3}^3\right) \ - \ 3\, a\, \alpha_1\alpha_2\alpha_3 \ = \ 0 \ . \tag{1.11}$$

One may therefore uniformize these parameters by taking, once again:

$$\alpha_\ell \ = \ \mu_\ell(\zeta)\,, \tag{1.12}$$

[4] This particular matrix factorization is valid for $\alpha_3 \neq 0$ because then L_1 and L_2 are linear independent. For the singular limit $\alpha_3 = 0$ one needs to go to a different coordinate patch [16]. For $\alpha_1 = 0$ the matrix E also becomes singular. However, this can be fixed by apply a gauge transformation of the form $Q_1 \to Q_1 + \frac{\alpha_2^2}{\alpha_1\alpha_2\alpha_3^2}x_1 L_2$, $Q_2 \to Q_2 - \frac{\alpha_2^2}{\alpha_1\alpha_2\alpha_3^2}x_1 L_1$. Similarly one proceeds for $\alpha_2 = 0$.

for some parameter, ζ. The physical interpretation is that the factorization parameters, α_ℓ, are moduli of the D-branes, and ζ is the associated flat coordinate that labels a point on the curve corresponding to the location of the $D0$-brane component of the $(D2, D0)$ brane configuration.

As mentioned above, (indecomposable) B-type D-branes are labeled by their RR charges and location: $(r(\mathcal{E}), c_1(\mathcal{E}), \zeta) \equiv (N_2, N_0, \zeta)$. In discussing the corresponding holomorphic vector bundles we will adopt a common notation, $\mathcal{E}(r, c_1)$, that suppresses the parameter, ζ. We will also use the notation, $\mathcal{L}^n$, to denote the n^{th} power of the degree-one line bundle, $\mathcal{L}$.

The question naturally arises as to the precise map between these bundle data and the structure of the matrices J and E that define a given factorization, P. This will be discussed in detail in Sections 2 and 3 below. We recall here that the two factorizations under discussion have been shown to correspond each to a triplet of branes with the following charges:

$$
\begin{aligned}
S \equiv P_{2\times 2}: &\qquad (\,r,\, c_1\,)^{LG}(S_a) \;=\; \Big\{(1,0),(0,1),(-1,-1)\Big\} \\
L \equiv P_{3\times 3}: &\qquad (\,r,\, c_1\,)^{LG}(L_a) \;=\; \Big\{(2,1),(-1,1),(-1,-2)\Big\}.
\end{aligned}
\tag{1.13}
$$

As indicated above, we will denote the two factorizations by S and L, each comprising three branes denoted by S_a and L_a.[5] These charge assignments, at least for the 2×2 factorization, were originally computed rather indirectly [26,16]: The intersection matrices were computed using the matrix formulation and then the results were compared with the intersection matrices computed from the conformal field theory and from the geometry. Part of our purpose here is to give a far more direct algorithm for computing these geometric data from the matrices.

Under mirror symmetry, Kähler and complex structure moduli exchange and the B-type branes map into A-type $D1$-branes, which are labeled by the winding numbers

[5] In BCFT language, the L_a are the "Recknagel-Schomerus" branes [39,40] with smallest charges, which correspond to holomorphic vector bundles inherited from the ambient $\mathbb{P}^2$ and as such do not generate the full RR charge lattice. The S_a correspond to the recently-discovered "permutation branes" [26,33,35] that form an integral basis of the charge lattice on the curve.

p and q: $(r, c_1)_B = (p, q)_A$. Moreover, the brane modulus, ζ, maps into a complex modulus comprising both position and Wilson line moduli of the $D1$-brane. Thus, the lattice of brane charges can be drawn on the covering space of Σ, as shown in Fig. 1.

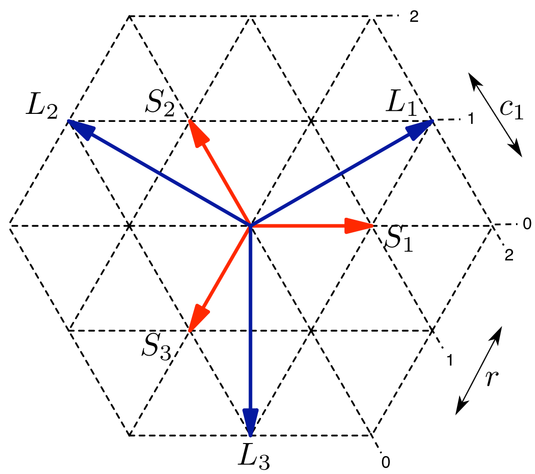

Fig. 1: *The 2×2 and 3×3 factorizations correspond, via mirror symmetry, to $D1$-branes that stretch along the "short" (S) and "long" (L) diagonals on the elliptic curve, respectively. Here we show these $D1$ branes (suppressing the anti-branes with opposite charges, which correspond to factorizations where J and E are exchanged). This figure also provides a useful and simple graphical representation of the RR charge lattice for the B-branes.*

We have listed in (1.13) the charges corresponding to the "Gepner-point" in the Kähler moduli space, which is natural from the Landau-Ginzburg perspective. However, recall that RR charges are ambiguous due to monodromy, or flow of gradings [1,41], in the Kähler moduli. Matrix factorizations, which pertain to the topological B-model and thus depend only on the complex structure moduli, are insensitive to variations of the Kähler moduli and therefore cannot distinguish bundles differing by such monodromies. For example, looping around the large radius limit induces a monodromy that amounts to tensoring with powers of the line bundle $\mathcal{L}^3$, which means that the first Chern number of a rank-r bundle will jump by $\pm 3r$.

On the other hand, performing a "partial monodromy" by moving from the Gepner point to the large radius limit, the charges (1.13) flow according to tensoring with the line bundle[6] $\mathcal{L}^{-2}$, *i.e.*, $(r,\, c_1) \to (r,\, c_1 - 2r)$, and this results in the following list of charges [41,43]:

$$\begin{aligned} 2\times 2: \qquad & (\,r,\, c_1\,)^{LR}(S_a) \;=\; \Big\{(1,-2),(0,1),(-1,1)\Big\} \\ 3\times 3: \qquad & (\,r,\, c_1\,)^{LR}(L_a) \;=\; \Big\{(2,-3),(-1,3),(-1,0)\Big\}. \end{aligned} \tag{1.14}$$

In the following, we will adopt this labeling convention because it refers to the large radius limit, which is semi-classical from the point of view of the sigma-model and coincides with the labeling in the mathematics literature.

In order to discuss topological tachyon condensation, we need to determine the relevant part of the open-string spectrum. Since, in the twisted theory, the tachyons become fermionic operators [44] (coupling to bosonic deformation parameters), we will consider only fermionic operators here. There are two classes of such operators. First, there are boundary preserving operators, represented by $2n_P \times 2n_P$ dimensional, block off-diagonal matrices of the form: $\Omega_P \equiv \Psi_{(P,P)} = \left(\begin{smallmatrix} 0 & \delta J_P \\ \delta E_P & 0 \end{smallmatrix}\right)$, which are tied to a single brane and describe moduli corresponding to infinitesimal deformations of the brane (such as position shifts). Most of the physics literature on open-string TFT deals with this class only. Mathematically these operators correspond to endomorphisms of the object P.

The other class consists of boundary changing operators, $\Psi_{(P,Q)}$, which correspond to open strings stretching between pairs of branes P and Q and thus are localized at their intersection.[7] See Fig. 2. These are the topological version of tachyons, and indeed they typically have R-charges $q < 1$, which means that they are relevant operators inducing a non-trivial boundary RG flow. They can be written in terms of $2n_P \times 2n_Q$ dimensional, block off-diagonal matrices of the form: $\Psi_{(P,Q)} = \left(\begin{smallmatrix} 0 & \psi_0 \\ \psi_1 & 0 \end{smallmatrix}\right)$. The rôle of ψ_0 and ψ_1 as maps between the composite objects P and Q can be visualized by the following diagram:

---

[6] This can be understood from a linear sigma model point of view [9]. Note that this is also closely related to Seiberg dualities [42].

[7] Non-intersecting branes have a trivial topological open-string spectrum between them because the open strings are massive and so are not part of the cohomology.

$$
\begin{tikzcd}
P_1 \arrow[d, bend left, "E_P"] \arrow[drr, "\psi_1" near start] & & Q_1 \arrow[d, bend left, "E_Q"] \\
P_0 \arrow[u, bend left, "J_P"] \arrow[urr, "\psi_0"' near start] & & Q_0 \arrow[u, bend left, "J_Q"]
\end{tikzcd} \quad . \tag{1.15}
$$

There is a similar structure for the bosonic operators, Φ.

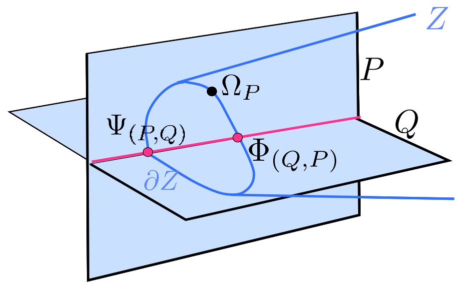

Fig. 2: *This shows schematically where boundary preserving (Ω) and boundary changing (Ψ, Φ) operators are located on intersecting D-branes and on the boundary of the world-sheet, Z.*

The physical operators then correspond to the non-trivial cohomology elements of the BRST operator, which can be written in the form: $\mathcal{Q} = \left(\begin{smallmatrix} 0 & J \\ E & 0 \end{smallmatrix}\right)$. That is, we require them to be closed:

$$
\begin{aligned}
0 &= E_Q\,\psi_0 + \psi_1\,J_P \ , \\
0 &= J_Q\,\psi_1 + \psi_0\,E_P \ ,
\end{aligned} \tag{1.16}
$$

modulo exactness

$$
\begin{aligned}
\psi_0^{\rm ex} &= J_Q\,\varphi_0 - \varphi_1\,J_P \ , \\
\psi_1^{\rm ex} &= E_Q\,\varphi_1 - \varphi_0\,E_P \ ,
\end{aligned} \tag{1.17}
$$

for any choice of matrices φ_0 and φ_1.

The open-string spectrum pertaining to the 2×2 and 3×3 matrix factorizations can be represented by the quiver diagram shown in Fig. 3. The number of arrows indicates the number of inequivalent fermionic cohomology elements, and coincides with the number of intersection points of the mirror A $D1$-branes on the curve. For each arrow there is implicitly another one running in the opposite direction, which corresponds to the Serre dual, bosonic operator and which we do not show.[8] Moreover, closed loops denote the boundary preserving deformations, Ω_P.

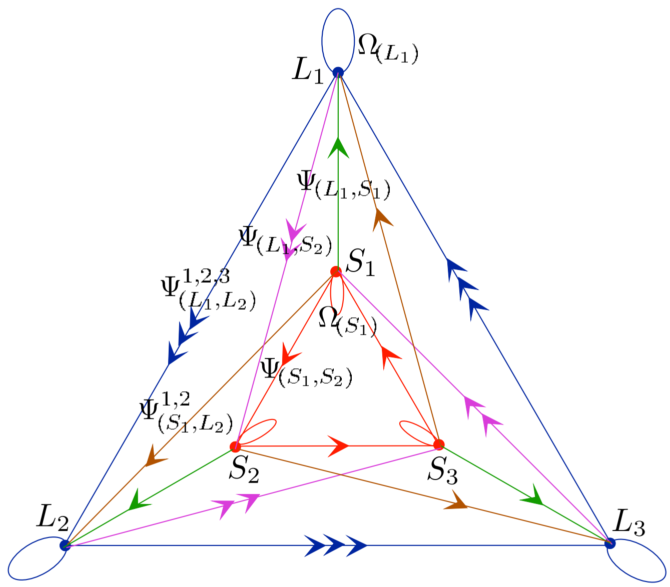

Fig. 3: *The quiver diagram displaying the fermionic open string states related to the short- and long-diagonal branes, S_a and L_a. The arrows depict the multiplicities. For simplicity, we do not show the anti-branes $\overline{S}_a$, $\overline{L}_a$.*

1.3. Examples of tachyon condensation

To illustrate one of the basic techniques we use in this paper, we sketch the two simplest possible examples of tachyon condensation; a more extensive analysis will be presented in the subsequent sections. To recapitulate the basic point: One wants to find, and turn on, a suitable boundary changing operator $\Psi_{(P,Q)}$, whose effect is to condense two sets of branes, P and Q, to form some composite, R. This process

[8] Fermionic operators $\Psi_{(P,Q)}$ correspond to $\mathrm{Ext}(P,Q)$ while bosonic operators $\Phi_{(Q,P)}$ correspond to $\mathrm{Hom}(Q,P)$. Serre duality implies $\mathrm{Hom}(Q,P) \sim \mathrm{Ext}(P,Q)$.

can be visualized by collapsing the two-brane system shown in eq. (1.15) into a single object as follows:

$$\begin{tikzcd} P_1 \arrow[d, bend left, "E_P"] \arrow[rrd, "\psi_1" near start] & & Q_1 \arrow[d, bend left, "E_Q"] \\ P_0 \arrow[u, bend left, "J_P"] \arrow[rru, "\psi_0"' near start] & & Q_0 \arrow[u, bend left, "J_Q"] \end{tikzcd} \quad \Longrightarrow \quad \left[\begin{tikzcd} R_1 \arrow[d, bend left, "E_R"] \\ R_0 \arrow[u, bend left, "J_R"] \end{tikzcd} \right] \equiv R \,, \tag{1.18}$$

where the composite maps, J_R and E_R, can be thought of as $(n_P + n_Q) \times (n_P + n_Q)$ block matrices of the form:

$$J_R = \begin{pmatrix} J_Q & \psi_0 \\ 0 & J_P \end{pmatrix} \,, \qquad E_R = \begin{pmatrix} E_Q & \psi_1 \\ 0 & E_P \end{pmatrix} \,. \tag{1.19}$$

If Ψ is a non-trivial cohomology element, these matrices satisfy the factorization condition (1.4) and represent a new B-type of D-brane.[9] In the following, we will often use the following shorthand notation to denote the process of tachyon condensation:

$$P \succ_{\Psi} Q \Longrightarrow R \,. \tag{1.20}$$

Note that this construction is well-known in the mathematical literature and goes by the name of the "cone construction". That is, one writes a sequence of maps in the form of a "distinguished triangle":

$$P[1] \xrightarrow{\Psi[1]} Q \longrightarrow R \longrightarrow P \;\;, \tag{1.21}$$

where the composite R coincides with what is called the "mapping cone" (for details see, for example, ref. [7], and for subtleties concerning off-shell versus on-shell physics, see refs. [2,6]).

In practice, one would like to find a simple way to determine exactly what this new D-brane is. From the point of view of matrix factorization, the obvious, but rather impractical method is to try to reduce the matrices to some standard set of canonical forms. The most efficient way of achieving this end is, in fact, to determine the

[9] Note that in principle the maps need not be upper triangular. Factorization then becomes a highly non-trivial condition, which in general is satisfied only on a sub-locus of the combined open/closed string moduli space; for examples, see ref. [30].

underlying bundle data for the new brane, R. Of course, as far as the K-theoretic data are concerned, ranks and first Chern classes of bundles are additive under condensation and thus can be trivially determined. It is, however, not so obvious how to determine data beyond RR charges, that is, the extent to which R is decomposable, and how the parameters of P and Q combine into the parameter(s) of R. Here we illustrate this issue with two examples.

The simplest possible example is combining a pair of 2×2-matrix factorizations. There are two ways to achieve this, namely either combining a pair of branes, or a brane with an anti-brane. We start with the second possibility and take the first D-brane to be given by the 2×2-matrix factorization $S(\alpha) = (J(\alpha), E(\alpha))$ in eq. (1.7), while the anti-D-brane is represented by the 2×2-matrix factorization $\overline{S}(\beta) = (-E(\beta), -J(\beta))$. The outcome of the condensation depends on which precise members, S_a and $\overline{S}_b$, of the two factorizations S and $\overline{S}$ we choose to condense, and this is tied to which specific tachyonic operator in the cohomology between the factorizations we choose to switch on.

From the vector addition of RR charges shown in Fig. 4 we expect that there should be two types of tachyonic perturbations, which either lead to complete annihilation, or to a composite corresponding to a 3×3 factorization, L or $\overline{L}$. The simplest possibility is of course the complete annihilation of the brane/anti-brane pair, which we will discuss momentarily; the other, more involved situation will be analyzed in Section 5.

For generic values of the moduli, α_i and β_i, of the branes $S_1(\alpha)$ and $\overline{S}_1(\beta)$, the relevant cohomology of fermionic open-string operators (determined by the physical state condition (1.16) modulo (1.17)) turns out to be empty. This reflects the fact that, in the A-model mirror picture, the anti-parallel $D1$-brane/anti-brane pair does not intersect, so that there is no operator that can be used to form a tachyon condensate. However, upon tuning $\alpha_i = \beta_i$ it is easy to check that the cohomology jumps and now contains the fermionic operator $\Psi_{(S_1,\overline{S}_1)} \sim (\psi_0, \psi_1)$ given by

$$\psi_0 \;=\; \mathbb{1}_{2\times 2} \;, \qquad \psi_1 \;=\; \mathbb{1}_{2\times 2} \;. \tag{1.22}$$

This tuning of the open-string moduli corresponds to the situation where the anti-parallel brane and anti-brane move on top of each other and where the Wilson line

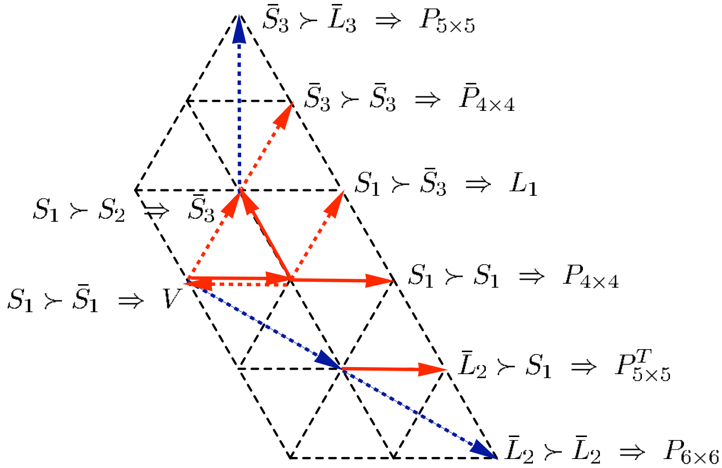

Fig. 4: *In addition to the two simple condensation processes discussed in this section, we show here how the rank two composites can be generated by condensing S and L branes or anti-branes, $\overline{S}$ and $\overline{L}$, respectively, by switching on suitable tachyons. The anti-branes are denoted by dashed arrows. Details will be discussed in Section 7.*

moduli of the branes are tuned to match. In fact the cohomology then also contains an extra bosonic operator, so that the intersection index $\chi_{S_1,\overline{S}_1} = \mathrm{Tr}_{S_1,\overline{S}_1}(-1)^F = \dim(\mathrm{Hom}(S_1,\overline{S}_1)) - \dim(\mathrm{Ext}(S_1,\overline{S}_1)) = 0$ does not change.[10]

To see that the result of the condensation induced by (1.22) is indeed the expected trivial ground state, one can make use of the gauge transformations (1.5), and most particularly, of the constant entries of the $\psi_{0,1}$, to make elementary row and column reductions of J_R and E_R to show that they are gauge equivalent to the trivial factorization (given by matrices with either 1 or W on the diagonal). See Appendix B for more discussion of gauge transformations and row and column reduction.

The next-to-simplest possibility is to condense two of the same type of matrix factorization, but with different values of the moduli: $S(\alpha)$ condensed with $S(\beta)$. We find that there are two distinct choices for the tachyon, and the choice of the tachyon again relates to which particular branes, $S_a(\alpha)$ and $S_b(\beta)$, in the families described

[10] The appearance of such pairs is not a special feature of the 2×2 factorizations, but instead this operator, as well as its bosonic partner, appear for any anti-parallel brane/anti-brane pair as long as the open-string moduli of both branes coincide.

by the factorizations, $S(\alpha)$ and $S(\beta)$, participate in the condensation. In section 4 we will analyze the interrelation between the tachyonic spectrum and the choice of brane pairs in complete detail, using equivariant R-symmetry. For the present we will choose the tachyon that leads to the condensation of S_1 and S_2 to form a composite anti-brane, $\overline{S}_3$ (*c.f.* Fig. 4).

This tachyon, represented by the boundary changing fermionic operator $\Psi_{(S_1,S_2)}(\alpha,\beta)$, is found to be [11]

$$\psi_0(\alpha,\beta) \;=\; \begin{pmatrix} H_1(\alpha,\beta) & H_2(\alpha,\beta) \\ G_1(\alpha,\beta) & G_2(\alpha,\beta) \end{pmatrix}, \quad \psi_1(\alpha,\beta) \;=\; \begin{pmatrix} -G_2(\beta,\alpha) & H_2(\beta,\alpha) \\ G_1(\beta,\alpha) & -H_1(\beta,\alpha) \end{pmatrix}. \tag{1.23}$$

The constant entries are given by[12]

$$\begin{aligned} G_1(\alpha,\beta) \;&=\; \alpha_1\beta_3^2 - \alpha_3\beta_1\beta_3 \ , \\ G_2(\alpha,\beta) \;&=\; -\alpha_2\beta_3^2 + \alpha_3\beta_2\beta_3 \ , \end{aligned} \tag{1.24}$$

while the linear entries are:

$$\begin{aligned} H_1(\alpha,\beta) \;&=\; \left(\alpha_2 - \frac{\alpha_1\beta_2}{\beta_1}\right)x_1 - \alpha_1\left(\frac{\alpha_1}{\alpha_2} - \frac{\beta_1}{\beta_2}\right)x_2 - \alpha_3\left(\frac{\alpha_3}{\alpha_2} - \frac{\beta_3}{\beta_2}\right)x_3 \ , \\ H_2(\alpha,\beta) \;&=\; \left(\frac{\alpha_1^2}{\alpha_2} + \frac{\alpha_2\beta_2}{\beta_1} - \frac{\alpha_3\beta_1^2}{\beta_2\beta_3} - \frac{\alpha_3\beta_2^2}{\beta_1\beta_3}\right)x_1 + \left(\alpha_1 - \frac{\alpha_2\beta_1}{\beta_2}\right)x_2 \ . \end{aligned} \tag{1.25}$$

The fermionic operator (1.23) can be used to construct a condensate R via the cone construction mentioned above, and one obtains the following 4×4-matrix factorization:

$$J_R(\alpha,\beta) \;=\; \begin{pmatrix} J(\beta) & \psi_0(\alpha,\beta) \\ 0 & J(\alpha) \end{pmatrix} \ , \qquad E_R(\alpha,\beta) \;=\; \begin{pmatrix} E(\beta) & \psi_1(\alpha,\beta) \\ 0 & E(\alpha) \end{pmatrix} \ . \tag{1.26}$$

Since the tachyon operator, $\Psi_{(S_1,S_2)}$, contains the constant entries (1.24), the matrix-factorization can again be simplified by the process of row and column elimination.

[11] These expressions have also been obtained by J. Walcher who participated in early stages of this project.

[12] These vanish for $\alpha_i = \beta_i$. However, at $\alpha_i = \beta_i$ there exists a cohomology element of different form, for which the following arguments hold analogously.

After a few straightforward steps of algebra, the matrix factorization (1.26) can be cast into its gauge-equivalent form:

$$J_R(\gamma) = \begin{pmatrix} 0 & -L_1(\gamma) & 0 & -Q_2(\gamma) \\ 0 & 0 & 1 & 0 \\ W & 0 & 0 & 0 \\ 0 & L_2(\gamma) & 0 & -Q_1(\gamma) \end{pmatrix} , \quad E_R(\gamma) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -Q_1(\gamma) & 0 & 0 & Q_2(\gamma) \\ 0 & W & 0 & 0 \\ -L_2(\gamma) & 0 & 0 & -L_1(\gamma) \end{pmatrix} , \tag{1.27}$$

where

$$\begin{aligned} \gamma_1 &= \alpha_2^2\beta_1\beta_2 - \alpha_1\alpha_3\beta_3^2 \ , \\ \gamma_2 &= \alpha_3^2\beta_2\beta_3 - \alpha_1\alpha_2\beta_1^2 \ , \\ \gamma_3 &= \alpha_1^2\beta_1\beta_3 - \alpha_2\alpha_3\beta_2^2 \ . \end{aligned} \tag{1.28}$$

The parameters γ_i also satisfy the cubic torus equation, just like the parameters α_i in (1.11). Indeed, the γ_i become much simpler when written in terms of the uniformizing variables and, as we will discuss in later sections, (1.28) merely represents the addition formula for theta functions.

If we now drop the two trivial D-branes in the matrix factorization (1.27) (each corresponding to $J = \mathbb{1}_{1\times 1}$ and $E = W\mathbb{1}_{1\times 1}$, or vice versa), we finally obtain for the condensate R the following 2×2 factorization:

$$J_R(\gamma) \ = \ \begin{pmatrix} -L_1(\gamma) & -Q_2(\gamma) \\ L_2(\gamma) & -Q_1(\gamma) \end{pmatrix} \ , \qquad E_R(\gamma) \ = \ \begin{pmatrix} -Q_1(\gamma) & Q_2(\gamma) \\ -L_2(\gamma) & -L_1(\gamma) \end{pmatrix} \ . \tag{1.29}$$

We can readily identify it with $\overline{S}(\gamma) = (-E(\gamma), -J(\gamma))$, which corresponds to an anti-brane (*c.f.* (1.7)) depending on the open-string modulus, γ.

2. Holomorphic vector bundles and matrix factorizations

As we discussed earlier, the classification of B-type branes is given by that of holomorphic vector bundles[13] and this is equivalent to finding matrix factorizations.

[13] More generally, B-type branes are described in terms of coherent sheaves, which is a notion more general than vector bundles. As we will explain later and as pointed out in [26], in order to describe sheaves that are not vector bundles, the construction detailed below needs to be refined by using equivariant R-symmetry.

Since the primary focus of this paper is upon matrix factorization, and how to generate new matrix factorizations from old ones via tachyon condensation, we will discuss in some detail how one can extract the bundle data from the matrices. The converse construction is also possible [19], but we will not address it directly here.

More concretely, we will see in the following how one can associate a bundle with each of the matrix factors, J and E. We will denote these bundles by $\mathcal{E}_J$ and $\mathcal{E}_E$, respectively. The charges of the branes will then be determined by $\mathcal{E}_J$ and the charges of the anti-branes by $\mathcal{E}_E$. The impatient reader, who would like to skip the technical details, is invited to move on to Section 3.4, where we summarize our findings with regard to the brane charges.

Before starting with the discussion, one should also note that the bundle data that we seek has a natural ambiguity coming from the fact that the coordinates, x_j, are themselves sections of the line bundle, $\mathcal{L}^3$, and since we allow ourselves to multiply and divide vectors by the x_j, any bundle data that we get will be ambiguous up to tensoring by powers of this line bundle. Therefore, $\mathcal{E}(r,d)$, will be indistinguishable from $\mathcal{E}(r,d\pm 3r)$; note that this is compatible with the ambiguity induced by the large radius monodromy in the Kähler moduli space.

There is a closely related issue with the definitions of $J(x)$ and $E(x)$: The fact that the x_j are sections of a line bundle means that the matrices will generically have non-trivial transformations between patches on Σ. That is, suppose that between two patches one has $x_j \to g\, x_j$ for some transition function, g. Then one has $W(x) \to g^3 W(x)$ and

$$J(gx) \;=\; g^k\, G_1(g)\, J(x)\, G_2(g)^{-1}\,, \qquad E(gx) \;=\; g^{3-k}\, G_2(g)\, E(x)\, G_1(g)^{-1}\,, \tag{2.1}$$

where $k \in \mathbb{Z}$ is arbitrary and G_1 and G_2 are, in fact, matrices that describe the action of the R-symmetry. We will discuss R-symmetry in more detail in Section 5, but here we note that if the matrix elements of J and E each have a well-defined (but possibly different) degree, then G_1 and G_2 will be diagonal with integer powers of g.

2.1. The direct approach

We start by taking the most naive approach to the problem and seeing how far we can get. Indeed, the most elementary way to exhibit the holomorphic vector bundles associated with a matrix factorization is to look at the kernels of the matrix factors. Given an $n \times n$-matrix factorization of the form (1.4), it follow that

$$\det(J) \;=\; W^{n-r}\,, \qquad \det(E) \;=\; W^{r}\,, \tag{2.2}$$

for some $r \in \mathbb{Z}_{+}$. On the surface, Σ, defined by $W = 0$ in a complex projective space, the kernel of E is thus a rank-r vector bundle, $\mathcal{E}_J$, on Σ and it is spanned by the columns of J. Conversely, the columns of E span the rank-$(n-r)$ vector bundle, $\mathcal{E}_E$, that is the kernel of J. In terms of the objects, P, in (1.6), the idea is that we are extracting the data about the condensation that leads to P by passing to the kernels of the maps. The two bundles extracted in this way correspond to the “constituent” D-brane and its anti-D-brane, P_0 and P_1 in (1.6). In more physical terms, the same, naive argument [45] that leads from the bulk Landau-Ginzburg model to the surface $W = 0$ in projective space, when extended to the boundary suggests that one should look at zeroes of J and E, for branes and anti-branes respectively, on the boundary. A proper justification of this argument may, however, require the boundary linear sigma-model [9].

There is a technical problem with the naive construction above: The matrices may have non-trivial transition properties (2.1) and so taking the linear span of columns may not be well-defined upon Σ. The simplest remedy is to consider the kernels of:

$$\hat{J}(x) \;\equiv\; G_1(x_p^{-1})\, J(x)\, G_2(x_p^{-1})^{-1}\,, \qquad \hat{E}(x) \;\equiv\; G_2(x_p^{-1})\, E(x)\, G_1(x_p^{-1})^{-1}\,, \tag{2.3}$$

for some $p = 1, 2, 3$ and where G_1 and G_2 are defined in (2.1). This is similar to the gauge transformation in the sense of (1.5) but it is not invertible in the polynomial ring. I does, however, make the entries of $\hat{J}$ and $\hat{E}$ into rational functions of the x_j. However, $\hat{J}$ and $\hat{E}$ have well-defined kernels and images on Σ (because $\hat{G}_1 = \hat{G}_2 = \mathbb{1}$) and thus $\mathcal{E}_{\hat{J}}$ and $\mathcal{E}_{\hat{E}}$ are well defined in Σ. This is what we really mean by “multiplying and dividing by powers of x_j,” and this is the reason for the ambiguity, $\mathcal{E}(r, d + 3jr)$ for $j \in \mathbb{Z}$, in the bundle data. We will proceed with the mild abuse in terminology by thinking of $\mathcal{E}_J$ and $\mathcal{E}_E$ as vector bundles, with the understanding that they can be turned into vector bundles using the foregoing construction.

Since the matrices are polynomials in the x_j, the spanning sets (the matrix columns) of these vector bundles are holomorphic. These vector bundles will clearly have transition functions induced from those of the x_j and it is tempting to assume that these will be sufficient to define the vector bundles $\mathcal{E}_J$ and $\mathcal{E}_E$. However, it is not that simple: The columns of J (or E) are not linearly independent and therefore do not constitute a basis. The obvious remedy is to choose a linearly independent subset of columns and use this as a basis, but the problem is that such a choice of basis cannot be done globally on Σ: One needs to introduce further patches on Σ and use different sets of columns in each of these new patches. There will also be non-trivial transition functions between such patches. Mathematically, this amounts to constructing an explicit local trivialization of the vector bundle. These transition functions can be written as rational functions of the matrix elements and the patches can be arranged so that these transition functions are holomorphic on the intersections. Thus one can easily see that $\mathcal{E}_J$ and $\mathcal{E}_E$ are holomorphic vector bundles, and one can, in principle, compute their properties in this manner.

For example, consider the 3×3 factorization given by (1.10). Every matrix element of J has degree one and every matrix element of E has degree two and so $G_1 = G_2 = \mathbb{1}$. Since $\det(J) = W$, $\mathcal{E}_E$ has rank 1, and so the columns of E must all be multiples of one another, which can easily be verified (mod $W = 0$). Thus $\mathcal{E}_E$ must be a line bundle on Σ, and the matrices G_1 and G_2 are trivial. Now observe that the j^{th} column of E vanishes identically if $x_j = \alpha_1$, $x_{j+1} = \alpha_3$ and $x_{j+2} = \alpha_2$, where the subscripts are taken mod 3. One therefore needs to use at least two of the columns in two different patches, U_1 and U_2, if one is to define the line bundle globally. The transition functions between these two patches will be a ratio of quadratic functions in the x_j, and thus meromorphic on Σ. Indeed, the patches can be chosen so that the transition function on $U_1 \cap U_2$ is biholomorphic.

From this simple example, we see that to characterize the line bundle we not only need G_1, G_2 and the properties of the x_j, but also the non-trivial transition functions between patches in which different sets of columns are linearly independent. Thus the naive construction of the holomorphic vector bundles $\mathcal{E}_J$ and $\mathcal{E}_E$ works nicely, but it may not result in the simplest, canonical description of the bundle.

To simplify and generalize this discussion, we start with a brief review of holomorphic vector bundles on a torus.

2.2. Holomorphic vector bundles on an elliptic curve

To define the elliptic curve, Σ_q, corresponding to the surface Σ with complex structure modulus, q, it is most convenient to think of it as an annulus in the complex plane with the interior and exterior edges identified. That is, one considers $\mathbb{C}^* \equiv \mathbb{C}\backslash\{0\}$ with the identification $z \sim qz$ and where $q \in \mathbb{C}^*$ is a parameter. It is also useful to use the "additive parametrization" where $q = e^{2\pi i\tau}$, $z = e^{2\pi i\xi}$ and $\xi \sim \xi + 1 \sim \xi + \tau$. There is a natural projection map, $\pi : \mathbb{C}^* \to \Sigma_q$, and given a holomorphic vector bundle on Σ_q, one can pull it back to $\mathbb{C}^*$. One can then show (see, for example, [18,46]) that the pull-back must be a trivial bundle on $\mathbb{C}^*$, and so the bundle on Σ_q is determined entirely by the "gluing matrix" under $z \sim qz$. The trivial, rank-r bundle on $\mathbb{C}^*$ is simply $\mathbb{C}^* \times \mathbb{C}^r$, and to obtain a rank r bundle on Σ_q one must specify an invertible $r \times r$ matrix, $A(z)$, of holomorphic functions on $\mathbb{C}^*$ and then one has:

$$\mathcal{V}_r(A) \equiv \{(z,v) \in \mathbb{C}^* \times \mathbb{C}^r : \quad (z,v) \ \sim \ (q\,z\,, A(z)\,v)\}\,. \tag{2.4}$$

Conversely, the set of non-trivial bundles is determined by the choices of $A(z)$ up to gauge equivalence: $A(z) \to B(qz)A(z)B^{-1}(z)$ for some invertible $r \times r$ holomorphic matrix, $B(z)$. Thus we have a complete characterization of the vector bundles on Σ_q.

There is also a well-known result of Atiyah [17] that states that every indecomposable holomorphic vector bundle on Σ_q is characterized by three parameters: (i) the rank, r, (ii) the first Chern number, c_1, and (iii) a point, y on Σ_q. Indeed, one can always write $\mathcal{V}_r(A)$ as $\tilde{\mathcal{L}} \otimes \mathcal{V}_r(e^N)$ where $\tilde{\mathcal{L}}$ is an appropriately chosen line bundle and N is a *constant*, indecomposable, nilpotent $r \times r$ matrix [18]. We will discuss and illustrate this result extensively in subsequent sections, but here we will merely note that the classical theta function:

$$\hat{\theta}(z,y) \ \equiv \ \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n(n-1)}\,(y\,z)^n\,, \tag{2.5}$$

is a global holomorphic section of the line bundle $\mathcal{V}_1(y^{-1}z^{-1})$ on Σ_q, where $A_y(z) = y^{-1}z^{-1}$. Note that this transition function involves a factor of z^{-1}, which has winding number -1, and thus $c_1 = 1$. Also observe that the transition matrix for $\hat{\theta}(z,q\,y)$ becomes $A_{qy}(z) = q^{-1}A_y(z)$, which is gauge equivalent to $A_y(z)$ with $B(z) = z^{-1}$.

Hence both $\hat{\theta}(z, y)$ and $\hat{\theta}(z, qy)$ are global sections of the same line bundle $\mathcal{V}_1(y^{-1}z^{-1})$ and therefore the parameter y may be thought of as living in Σ_q.

Returning to the line bundle spanned by the columns of the 3×3 matrix, E, of (1.10), an elementary theta-function identity reveals that the matrix elements of E may be rewritten in terms of theta functions:

$$E_{ij} \;=\; \frac{\eta^2(\tau)}{\alpha_1\alpha_2\alpha_3}\,\mu_i(\xi - \zeta)\,\mu_j(\xi + \zeta)\,, \tag{2.6}$$

where $x_\ell = \mu_\ell(\xi)$, $\alpha_\ell = \mu_\ell(\zeta)$ and $\eta(\tau)$ is the Dedekind η-function. Thus the columns of E are all *holomorphic* multiples of the single (nowhere-vanishing) basis vector:

$$v_1 \;\equiv\; e^{3\pi i(\xi-\zeta)} \begin{pmatrix} \mu_1(\xi - \zeta) \\ \mu_2(\xi - \zeta) \\ \mu_3(\xi - \zeta) \end{pmatrix} . \tag{2.7}$$

The phase pre-factor has been included so as to render v_1 periodic under $\xi \to \xi + 1$. Under $\xi \to \xi + \tau$ one has $v_1 \to -y^3 z^{-3} v_1$, where $z = e^{2\pi i\xi}$ and $y = e^{2\pi i\zeta}$, and therefore the holomorphic transition function $A(z)$ reads

$$A(z) \;=\; -y^3\, z^{-3} \;. \tag{2.8}$$

Thus the kernel of J leads to the line bundle, $\mathcal{V}_1(y^{-3}z^{-3}) = (\mathcal{V}_1(y^{-1}z^{-1}))^3$, or $\mathcal{E}(1,3)$, with $c_1 = 3$ and a parameter, y. This bundle is, modulo the ambiguity outlined at the beginning of the section, equivalent to $\mathcal{E}(1,3k), k \in \mathbb{Z}$ and both $\mathcal{E}(1,0)$ and $\mathcal{E}(1,-3)$ are the "anti-bundles" of two of the bundles whose charges were listed in (1.14). We will discuss the relation with all the charges listed in (1.14) in more detail after we have identified all the matrix bundles $\mathcal{E}_J$, $\mathcal{E}_E$, associated with the 2×2 and 3×3 factorizations.

2.3. MCM modules

We started this section by specializing to the surface, $W = 0$, and then identifying the holomorphic vector bundles associated with the matrix factorization. There is another, somewhat more abstract, approach that realizes these vector bundles in terms of Maximal Cohen-Macaulay (MCM) modules. Since this description is the standard approach to matrix factorizations in the mathematics literature, it is useful to relate it to our discussion here. Moreover, the formulation in terms of MCM modules can be used to show that holomorphic vector bundles are sufficient to reconstruct the matrix factorization (see, for example, ref. [19]).

The first step is to introduce a local ring, $\mathcal{R}$, defined by the superpotential, W. That is, let $\mathcal{P} = \mathbb{C}[x_1, \ldots, x_m]$ be the ring of complex polynomials in the x_j, and let $[W]$ be the ideal of $\mathcal{P}$ generated by the superpotential, W. One then defines a new ring, $\mathcal{R}$, by:

$$\mathcal{R} \;\equiv\; \mathbb{C}[x_1, \ldots, x_m]/[W] \,, \tag{2.9}$$

which is simply the ring of polynomials taken modulo W. Once again one considers the kernels and images of the matrices E and J, but this time one thinks of these kernels and images as *modules* over the ring $\mathcal{R}$. Specifically, J and E are now thought of as maps from $\mathcal{R}^n$ to $\mathcal{R}^n$, and the columns of the matrices, of course, generate the images of these maps. For instance, the columns of J generate a module, $\mathcal{M}$, however, this module is not necessarily freely generated. That is, elements of $\mathcal{M}$ can generically be written only as a non-trivial linear combination (over the ring $\mathcal{R}$) of *all* the columns, and yet the columns are not linearly independent. There are thus relations between the columns of J, and the set of such relations are called the first syzygy, $\Omega_1(\mathcal{M})$, of $\mathcal{M}$, which is also an $\mathcal{R}$-module. Indeed, the columns of the matrix E exactly generates this set of relations. Again, the module, $\Omega_1(\mathcal{M})$, is generically not freely generated and so there are relations between the relations and this defines the second syzygy, $\Omega_2(\mathcal{M})$, of $\mathcal{M}$. In a matrix factorization the set of such relations between the columns of E is of course generated by the columns of J again. In other words, we have $\Omega_2(\mathcal{M}) = \mathcal{M}$. This identity, in fact, defines an MCM module.

One can summarize the foregoing by stating that the following is an exact sequence:

$$\cdots \xrightarrow{E} \mathcal{R}^n \xrightarrow{J} \mathcal{R}^n \xrightarrow{E} \mathcal{R}^n \xrightarrow{J} \mathcal{R}^n \xrightarrow{E} \mathcal{R}^n \longrightarrow \mathcal{M} \longrightarrow 0 \,, \tag{2.10}$$

with a similar sequence with E and J interchanged.

The ideas and techniques of MCM modules are very useful if one is going to generate matrix factorizations from bundle data [19], but since this is not our focus here, we will not need to discuss these ideas any further.

3. Some examples of matrix factorizations and their vector bundles

Here we will describe, in detail, the vector bundles associated with the 2×2 and 3×3 factorizations. Before we proceed to the examples, we need to summarize some of the properties of the classical theta and Appell functions that arise in the description of holomorphic vector bundles on a torus. More details may be found in Appendix A.

3.1. Classical elliptic functions

In the present paper we mainly deal with line bundles and rank two vector bundles. We therefore briefly discuss their canonical sections, which are given by Riemann theta functions and Appell functions, respectively. The standard definitions are:

$$\vartheta(\xi) \;=\; \vartheta(\xi|\tau) \;\equiv\; \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\,n^2}\, z^n\,, \tag{3.1}$$

$$\kappa(\rho,\xi) \;=\; \kappa(\rho,\xi|\tau) \;\equiv\; \sum_{n\in\mathbb{Z}} \frac{q^{\frac{1}{2}\,n^2}\, z^n}{q^n - y}\,, \tag{3.2}$$

where $y \neq q^m, m \in \mathbb{Z}$ and

$$q \;\equiv\; e^{2\pi i\,\tau}\,, \quad z \;\equiv\; e^{2\pi i\,\xi}\,, \quad y \;\equiv\; e^{2\pi i\,\rho}\,. \tag{3.3}$$

These functions satisfy the following periodicity relations:

$$\vartheta(\xi+1) \;=\; \vartheta(\xi)\,, \qquad \vartheta(\xi+\tau) \;=\; q^{-\frac{1}{2}}\, z^{-1}\, \vartheta(\xi)\,, \tag{3.4}$$

$$\begin{aligned} \kappa(\rho,\xi+1) \;&= \kappa(\rho+1,\xi) \;=\; \kappa(\rho,\xi)\,, \qquad \kappa(\rho,\xi+\tau) \;=\; y\,\kappa(\rho,\xi) \;+\; \vartheta(\xi)\,, \\ \kappa(\rho+\tau,\xi) \;&= q^{-\frac{1}{2}}\, z\,(y\,\kappa(\rho,\xi) \;+\; \vartheta(\xi))\,. \end{aligned} \tag{3.5}$$

Consider these functions on the torus whose fundamental cell in $\mathbb{C}$ is defined by $\xi \sim \xi+1 \sim \xi+\tau$. It is a fairly familiar fact that the theta functions provide global holomorphic sections of line bundles on this torus: They are periodic under $\xi \to \xi+1$ and have a transition function of $q^{-\frac{1}{2}} e^{-2\pi i\,\xi}$ under $\xi \to \xi+\tau$. The fact that this function has winding number -1 on the circle defined by $\xi \in [0,1]$ means $\vartheta(\xi)$ is a holomorphic section of the line bundle, $\mathcal{V}_1(q^{-\frac{1}{2}} z^{-1})$, with $c_1 = +1$ on the torus.

The Appell function provides a global holomorphic section of a non-trivial rank two vector bundle, $\mathcal{F}$. In particular, the vector:

$$\begin{pmatrix} \kappa(\rho,\xi) \\ \vartheta(\xi) \end{pmatrix} \tag{3.6}$$

is globally holomorphic, is periodic under $\xi \to \xi+1$, but has the non-trivial transition matrix under $\xi \to \xi + \tau$:

$$A(z) \;=\; \begin{pmatrix} y & 1 \\ 0 & q^{-\frac{1}{2}}\, z^{-1} \end{pmatrix} . \tag{3.7}$$

This vector bundle, $\mathcal{F}$, is a non-trivial extension of $\mathcal{V}_1(q^{-\frac{1}{2}} z^{-1})$ by $\mathcal{V}_1(y)$:

$$0 \longrightarrow \mathcal{V}_1(y) \longrightarrow \mathcal{F} \longrightarrow \mathcal{V}_1(q^{-\frac{1}{2}} z^{-1}) \longrightarrow 0\,. \tag{3.8}$$

On the cubic curve, Σ, we need the theta functions appearing in (1.3) (see Appendix A for more details). The functions, μ_ℓ, have the following periodicity properties:

$$\mu_\ell(\xi+1) \;=\; -\mu_\ell(\xi)\,, \qquad \mu_\ell(\xi+\tau) \;=\; -q^{-\frac{3}{2}}\, z^{-3}\, \mu_\ell(\xi)\,, \tag{3.9}$$

but it is also useful to note that:

$$\mu_\ell(\xi+\tfrac{1}{3}) \;=\; -\omega^{-(\ell-1)}\, \mu_\ell(\xi)\,, \qquad \mu_\ell(\xi-\tfrac{1}{3}\,\tau) \;=\; -q^{-\frac{1}{6}}\, z\, \mu_{\ell+1}(\xi)\,, \tag{3.10}$$

where the subscript on $\mu_{\ell+1}$ is taken mod 3. Holomorphy and the periodicity properties on the torus defined by $\xi \sim \xi + \frac{1}{3} \sim \xi + \tau$ uniquely define the functions $\mu_\ell(\xi)$. Note that (3.9) implies that $\mu_\ell(\xi)$ is not a function on Σ, but is a section of the line bundle, $\mathcal{L}^3$. To be more precise, $e^{3\pi i \xi}\mu_\ell(\xi)$ is periodic under $\xi \to \xi+1$, and is a section of $\mathcal{L}^3 = \mathcal{V}_1(z^{-3})$, where $\mathcal{L} \equiv \mathcal{V}_1(z^{-1})$.

Since the functions $\mu_\ell(\xi)$ are global holomorphic sections of $\mathcal{L}^3$, each of the $\mu_\ell(\xi)$ has three zeroes. In particular, one has:

$$\mu_1(0) \;=\; \mu_1(\tfrac{1}{3}) \;=\; \mu_1(\tfrac{2}{3}) \;=\; 0\,, \qquad \mu_2(0) \;=\; -\mu_3(0) \;=\; i\,\eta(\tau)\,. \tag{3.11}$$

The values of the μ_ℓ at other third-periods can then be deduced from this using (3.10).

One can define Appell functions, Λ_ℓ, associated with the μ_ℓ (see Appendix A for details). These Appell functions are defined by their periods:

$$\Lambda_\ell(\rho,\xi+\tfrac{1}{3}) \;=\; -\omega^{-(\ell-1)}\, \Lambda_\ell(\rho,\xi)\,, \qquad \Lambda_\ell(\rho,\xi+\tau) \;=\; y^3\, \Lambda_\ell(\rho,\xi) \;+\; \mu_\ell(\xi)\,. \tag{3.12}$$

3.2. The 3×3 factorization revisited

Consider, once again, the matrix factorization defined by (1.10). We have seen that $\mathcal{E}_E$ is a line bundle and that a global, non-vanishing holomorphic section can be taken to be[14]:

$$v_1 \;\equiv\; \begin{pmatrix} \mu_1(\xi-\zeta) \\ \mu_2(\xi-\zeta) \\ \mu_3(\xi-\zeta) \end{pmatrix} . \tag{3.13}$$

The bundle $\mathcal{E}_J$ has rank two, and may be defined, via the kernel of E, as the set of vectors orthogonal to:

$$\left(\mu_1(\xi+\zeta)\,,\mu_2(\xi+\zeta)\,,\mu_3(\xi+\zeta)\right). \tag{3.14}$$

Given that the columns of J are expressed as sections of a line bundle, one might, at first, expect that the rank two, holomorphic vector bundle, $\mathcal{E}_J$, is itself a trivial sum of line bundles. For example, one might try taking the last two columns of J as a basis. However, for $\xi = \zeta + \frac{n}{3}$ one has:

$$x_\ell \;=\; -\omega^{-n\,(\ell-1)}\,\alpha_\ell\,, \tag{3.15}$$

and thus the last two columns of J are multiples of one another, and so they do not represent a good global basis for the vector bundle. In the neighborhood of such points one must use a different pair of columns as a basis, and the transition function is:

$$\begin{pmatrix} \alpha_3\,x_2 \\ \alpha_2\,x_1 \\ \alpha_1\,x_3 \end{pmatrix} \;=\; -\frac{\mu_1(\xi+\zeta)}{\mu_3(\xi+\zeta)} \begin{pmatrix} \alpha_1\,x_1 \\ \alpha_3\,x_3 \\ \alpha_2\,x_2 \end{pmatrix} \;-\; \frac{\mu_2(\xi+\zeta)}{\mu_3(\xi+\zeta)} \begin{pmatrix} \alpha_2\,x_3 \\ \alpha_1\,x_2 \\ \alpha_3\,x_1 \end{pmatrix} . \tag{3.16}$$

This change of basis is singular at the three zeroes of $\mu_3(\xi+\zeta)$ (*i.e.* at $\xi = \zeta + \frac{2\tau}{3} + \frac{n}{3}$). One therefore has to break the torus into patches if one is to use the columns of J as a basis.

On the other hand, one can use Appell functions to obtain a basis of holomorphic sections for the two-dimensional vector bundle, $\mathcal{E}_J$. Consider the vectors:

$$v_1 \;\equiv\; \begin{pmatrix} \alpha_1\,x_1 \\ \alpha_3\,x_3 \\ \alpha_2\,x_2 \end{pmatrix} , \qquad v_2 \;\equiv\; \begin{pmatrix} \alpha_1\,\Lambda_1(\rho,\xi) \\ \alpha_3\,\Lambda_3(\rho,\xi) \\ \alpha_2\,\Lambda_2(\rho,\xi) \end{pmatrix} . \tag{3.17}$$

[14] For simplicity, we have dropped the pre-factor in (2.7).

One can show that

$$\alpha_1\,\mu_1(\xi+\zeta)\,\Lambda_1(\rho,\xi) \;+\; \alpha_3\,\mu_2(\xi+\zeta)\,\Lambda_3(\rho,\xi) \;+\; \alpha_2\,\mu_3(\xi+\zeta)\,\Lambda_2(\rho,\xi) \;=\; 0\,, \tag{3.18}$$

provided that:

$$\rho \;=\; \tfrac{1}{2}\tau \;-\; \zeta\,. \tag{3.19}$$

This means that v_2 is in the kernel of E. One can prove this identity by considering the periodicity properties of the function, $F(\xi,\zeta)$, defined to be the left-hand side of (3.18). One first notes that the "anomalous" shift term in the Appell functions amounts to shifting v_2 by v_1 and since v_1 is in the kernel of E, the shift term does not contribute to $F(\xi,\zeta)$ under $\xi\to\xi+\tau$. This means that, considered either as a function of ξ or as a function of ζ, $F(\xi,\zeta)$ represents a global section of a line bundle. One then uses holomorphy and standard theta-function methods to write it in terms of theta functions or, in this instance, conclude that it is zero provided that one chooses the Appell function parameter according to (3.19).

Using the same kind of argument one can establish the following identities:

$$\mu_2(\xi-\zeta)\,v_1 \;-\; e^{6\pi i\zeta}\,\Lambda_2(\tfrac{\tau}{2}\,,\xi-\zeta)\,v_2 \;=\; -iq^{-\frac{9}{8}}\,\frac{\eta^3(3\tau)}{\eta(\tau)}\,e^{3\pi i(\xi+\zeta)}\begin{pmatrix}\alpha_3\,x_2\\ \alpha_2\,x_1\\ \alpha_1\,x_3\end{pmatrix}, \tag{3.20}$$

$$\mu_3(\xi-\zeta)\,v_1 \;-\; e^{6\pi i\zeta}\,\Lambda_3(\tfrac{\tau}{2}\,,\xi-\zeta)\,v_2 \;=\; iq^{-\frac{9}{8}}\,\frac{\eta^3(3\tau)}{\eta(\tau)}\,e^{3\pi i(\xi+\zeta)}\begin{pmatrix}\alpha_2\,x_3\\ \alpha_1\,x_2\\ \alpha_3\,x_1\end{pmatrix}. \tag{3.21}$$

In other words, the set of columns of J are given by holomorphic linear combinations of the holomorphic vectors, v_1 and v_2.

Finally, observe that the vectors $u_j\equiv e^{3\pi i\xi}v_j$ are periodic under $\xi\to\xi+1$ and under $\xi\to\xi+\tau$ they have the transition matrix:

$$A(z) \;=\; \begin{pmatrix}q^3\,e^{-6\pi i\zeta} & 1\\ 0 & z^{-3}\end{pmatrix}. \tag{3.22}$$

Summarizing, the bundle, $\mathcal{E}_J$, is the non-trivial, non-split bundle with $(r(\mathcal{E}_J),c_1(\mathcal{E}_J))=(2,3)$. Up to the ambiguity of tensoring with $\mathcal{L}^{3n}$, this is equivalent to $\mathcal{E}(2,-3)$, which indeed belongs to the charges listed in (1.14).

3.3. Vector bundles of the 2×2 factorization

Consider the 2×2 factorization given by (1.7). The determinants of E and J are both W and so the kernels of both are one-dimensional. Because the matrix elements have different degrees, the matrices in (2.1) are now non-trivial, that is, taking $g = -q^{-3/2} e^{-6\pi i \xi}$ one has:

$$\begin{aligned} J(x(\xi+\tau)) &= g^k \, G_1(g) \, J(x(\xi)) \, G_2(g)^{-1} \,, \\ E(x(\xi+\tau)) &= g^{3-k} \, G_2(g) \, E(x(\xi)) \, G_1(g)^{-1} \,, \end{aligned} \tag{3.23}$$

where

$$G_1 \ = \ \begin{pmatrix} g^2 & 0 \\ 0 & g \end{pmatrix} , \qquad G_2 \ = \ \mathbb{1}_{2\times 2} \,. \tag{3.24}$$

Since $G_2 = \mathbb{1}$ there are no technical issues in defining $\mathcal{E}_E$. The columns of E must be proportional to each other, and by looking at the common zeroes of the L_j and Q_j one can argue that the kernel should be spanned by theta functions of characteristic 2. However there is a simpler way to obtain the kernels of the matrices using the functions μ_ℓ.

By taking constant (*i.e.* independent of ξ) linear combinations of the rows of J in (1.10), one can show that

$$\widetilde{L}_1 \, \mu_1(\xi - \zeta) \ + \ \widetilde{L}_2 \, (\mu_2(\xi - \zeta) + \mu_3(\xi - \zeta)) \ = \ 0 \,, \tag{3.25}$$

where

$$\widetilde{L}_1 \ \equiv \ (\alpha_1^2 \, x_1 + \alpha_2^2 \, x_2 + \alpha_3^2 \, x_3) \,, \qquad \widetilde{L}_2 \ \equiv \ (\alpha_2 \, \alpha_3 \, x_1 + \alpha_1 \, \alpha_3 \, x_2 + \alpha_1 \, \alpha_2 \, x_3) \,. \tag{3.26}$$

One can similarly derive relations between μ_1 and $\mu_2 + \omega^k \mu_3$. Taking the latter relationships for $k = 1, 2$ one can multiply by each by factors that are linear in the x_j so as to obtain a relationship of the form:

$$\widetilde{Q}_1 \, (\mu_2(\xi - \zeta) + \mu_3(\xi - \zeta)) \ - \ \widetilde{Q}_2 \, \mu_1(\xi - \zeta) \ = \ 0 \,, \tag{3.27}$$

where the $\widetilde{Q}_j$ are quadratic in the x_ℓ. Thus we get a 2×2 matrix with a null vector of the form:

$$\begin{pmatrix} \tilde{s}_1(\xi) \\ \tilde{s}_2(\xi) \end{pmatrix} \ \equiv \ \begin{pmatrix} \mu_2(\xi - \zeta) + \mu_3(\xi - \zeta) \\ \mu_1(\xi - \zeta) \end{pmatrix} , \tag{3.28}$$

This matrix is gauge equivalent to $J_{2\times 2}$ in (1.7), except one must replace α_j in (1.7) according to:

$$\alpha_1 \to \alpha_2\,(\alpha_3^3 - \alpha_1^3)\,, \quad \alpha_2 \to \alpha_1\,(\alpha_2^3 - \alpha_3^3)\,, \quad \alpha_3 \to \alpha_3\,(\alpha_1^3 - \alpha_2^3)\,. \tag{3.29}$$

Using (2.6) at $\xi = \zeta$, along with (3.11) and (3.10), one can see that the replacement (3.29) is equivalent to

$$\zeta \;\to\; 2\,\zeta + \tfrac{1}{3}\,\tau\,. \tag{3.30}$$

To understand the underlying vector bundle, one should note that this derivation of the matrix factorization implicitly relies on the fact that $\tilde{s}_1(\xi)$ and $\tilde{s}_2(\xi)$ have a common zero at $\xi = \zeta$ and so (3.28) vanishes identically at one point. To cover this vanishing point we could find another section and the transition function, but it is simpler to note that there is a global, *nowhere-vanishing* section of this bundle given by:

$$\begin{pmatrix} s_1(\xi) \\ s_2(\xi) \end{pmatrix} \;\equiv\; \frac{e^{3\pi i\xi}}{\vartheta((\xi-\zeta)-\frac{1}{2}(1+\tau))\,)}\,\begin{pmatrix} \mu_2(\xi-\zeta)+\mu_3(\xi-\zeta) \\ \mu_1(\xi-\zeta) \end{pmatrix}\,. \tag{3.31}$$

The s_j are thus theta functions of characteristic 2 and satisfy:

$$s_j(\xi+1) \;=\; s_j(\xi)\,, \qquad s_j(\xi+\tau) \;=\; z^{-2}\,s_j(\xi)\,, \tag{3.32}$$

and so the bundle $\mathcal{E}_E$ is simply $\mathcal{E}(1,2)$.

Conversely, given the s_j, the ratio s_2/s_1 is an elliptic function with two poles and two zeroes on the torus, and so there is a unique way to write it in the form:

$$\frac{s_2}{s_1} \;=\; \frac{a_1x_1+a_2x_2+a_3x_3}{b_1x_1+b_2x_2+b_3x_3}\,, \tag{3.33}$$

and this defines $\widetilde{L}_1$ and $\widetilde{L}_2$. Similarly, one can write s_2/s_1 as a ratio of quadratics in the x_j. Each quadratic, $\widetilde{Q}_j$, has six parameters, one of which is a scale and the other five can be used to adjust the locations of the six zeroes on the torus.[15] The locations of two zeroes in each $\widetilde{Q}_j$ are fixed by the s_j, while the other zeroes of

[15] The sum of the zeroes of Q_j must be zero modulo the lattice $\xi \sim \xi+1$, $\xi \sim \xi+\tau$.

$\widetilde{Q}_1$ must coincide with those of $\widetilde{Q}_2$, but are otherwise free. Thus there is a three-parameter family of ways of writing s_2/s_1 as ratio of quadratics. On the other hand, one can trivially generate such a quadratic ratio by multiplying the numerator and denominator of (3.33) by the same arbitrary linear function of the x_j. This choice of linear function has three parameters, and so the quadratic ratio is unique up to gauge transformations. Thus one can reconstruct the 2×2-matrix factorization uniquely from (3.31).

By keeping track of the locations of the zeroes in the foregoing argument, it is easy to check that one can write the columns of E as:

$$\begin{pmatrix} \widetilde{L}_1 \\ -\widetilde{L}_2 \end{pmatrix} = \chi_1(\xi) \begin{pmatrix} s_1(\xi) \\ s_2(\xi) \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{Q}_2 \\ \widetilde{Q}_1 \end{pmatrix} = \chi_2(\xi) \begin{pmatrix} s_1(\xi) \\ s_2(\xi) \end{pmatrix}, \tag{3.34}$$

for some global holomorphic sections, $\chi_1(\xi)$ and $\chi_2(\xi)$, of $\mathcal{E}(1,1)$ and $\mathcal{E}(1,4)$ respectively. From this one can write the columns of J as:

$$\begin{pmatrix} \widetilde{Q}_1 \\ \widetilde{L}_2 \end{pmatrix} = s_2(\xi) \begin{pmatrix} \chi_2(\xi) \\ -\chi_1(\xi) \end{pmatrix}, \qquad \begin{pmatrix} -\widetilde{Q}_2 \\ \widetilde{L}_1 \end{pmatrix} = -s_1(\xi) \begin{pmatrix} \chi_2(\xi) \\ -\chi_1(\xi) \end{pmatrix}, \tag{3.35}$$

and hence the columns of $\mathcal{E}_J$ correspond to the bundle $\mathcal{E}(1,1)$ (or $\mathcal{E}(1,4)$).

3.4. Summary

In this section, we considered both the 2×2 and 3×3 factorizations and determined the data of the bundles $\mathcal{E}_J$ and $\mathcal{E}_E$, associated with the matrix factors, J and E. We now like to match these data to the RR charges of the B-branes. Specifically, as mentioned before, the charges of the branes will be associated with $\mathcal{E}_J$, while the anti-branes will be associated with $\mathcal{E}_E$.

To proceed, recall that for extracting the bundle data from the matrix factorization, we had to impose $W \equiv 0$, and therefore we implicitly characterized the branes in the large radius limit. Thus we should refer to the RR charges listed in (1.14), which apply to the large radius limit. Indeed we find a perfect match between the data of $\mathcal{E}_J$, and the charges of *one* of the members of $\{S_a\}$ or $\{L_a\}$, respectively, provided we tensor uniformly with the line bundle $\mathcal{L}^{-3}$ (recall that brane charges are defined only up to such shifts, reflecting the monodromy around the large radius limit). There is also an analogous match between the data of $\mathcal{E}_E$, and the charge of one of the anti-branes in $\{\overline{S}_a\}$ or $\{\overline{L}_a\}$, respectively. We have summarized these findings in Table 1.

	2×2 factorization		3×3 factorization	
Bundle	$\mathcal{E}_J$	$\mathcal{E}_E$	$\mathcal{E}_J$	$\mathcal{E}_E$
Matrix bundle $\mathcal{E}$	$\mathcal{E}(1,1)$	$\mathcal{E}(1,2)$	$\mathcal{E}(2,3)$	$\mathcal{E}(1,3)$
$\mathcal{E}\otimes\mathcal{L}^{-3}\simeq(r,c_1)^{LR}$	$(1,-2)^{LR}$	$(1,-1)^{LR}$	$(2,-3)^{LR}$	$(1,0)^{LR}$
Brane & anti-brane in the positive cone	S_1	$\overline{S}_3$	L_1	$\overline{L}_3$

Table 1. *Here we summarize some properties of the short- and long-diagonal branes, associated to the 2×2 and the 3×3-matrix factorizations. In particular the relationship between the bundle data of the matrix factors $\mathcal{E}_J$ and $\mathcal{E}_E$, and the large radius RR charges is shown.*

Note that if we view the charge lattice in Fig. 1 as the $SU(3)$ root lattice, the specific charges of the vector bundles at large radius lie within what one may call the positive cone, or fundamental region, spanned by the "simple roots" S_1 and S_2; this is indeed natural from the point of view of quiver representations (see the appendix of ref. [41]).

What then about the other charges in the lists (1.14), some of which are related to more general objects than vector bundles (like $D0$-branes described by point-like sheaves)? The point is, of course, that in order to obtain the TFT on the elliptic curve from the Landau-Ginzburg model, one must implement a $\mathbb{Z}_3$ orbifold projection, and it is only then that the full (quantum) $\mathbb{Z}_3$ orbits of the branes will emerge. This is what we will discuss in Section 5, however before doing so, we will first complete the discussion of $P_{3\times 3}$ by analyzing its open-string moduli space.

4. Compactifying the moduli space of the 3×3 factorization.

We now return to the 3×3-matrix factorization, $P_{3\times3}$, in order to examine in greater detail its open-string moduli space. This will eventually lead us to an exceptional 4×4 matrix factorization, which appears only at a certain point in that moduli space, and which compactifies the moduli space of the 3×3 factorization $P_{3\times3}$. It is this 4×4 matrix factorization which naturally appears in the boundary fermion construction.

First of all one observes that the 3×3 matrices, $J = J_{3\times3}$ and $E = E_{3\times3}$, given in (1.10) are quasi-(double-)periodic for $\zeta \to \zeta + 1$ and $\zeta \to \zeta + \tau$ due to the quasi-periodicity (3.9) of the parameters α_ℓ in terms of the uniformizing open-string modulus ζ of eq. (1.12). In other words, such a shift in ζ results in a multiplication of both J and E by an overall factor. However, the factors of J and E are inverse to each other, and hence they are easily compensated by a gauge transformation (1.5). Therefore, modulo gauge transformations the 3×3-matrix factorization is indeed periodic for $\zeta \to \zeta + 1$ and $\zeta \to \zeta + \tau$.

However, it is conceivable that there are further identifications in the open-string moduli space of ζ due to additional gauge equivalences: According to eq. (3.10) the uniformizing functions $\mu_\ell(\zeta)$ of α_ℓ have also nice periodicity properties for one-third of a period. In particular, for the 3×3 matrices J and E, we readily find:

$$J(\zeta + \tfrac{1}{3}) \,=\, C_1 \begin{pmatrix} \omega^2\,\alpha_1\,x_1 & \omega\;\;\alpha_2\,x_3 & \alpha_3\,x_2 \\ \alpha_3\,x_3 & \omega^2\,\alpha_1\,x_2 & \omega\;\;\alpha_2\,x_1 \\ \omega\;\;\alpha_2\,x_2 & \alpha_3\,x_1 & \omega^2\,\alpha_1\,x_3 \end{pmatrix} , \tag{4.1}$$

and

$$J(\zeta + \tfrac{\tau}{3}) \,=\, C_2 \begin{pmatrix} \alpha_3\,x_1 & \alpha_1\,x_3 & \alpha_2\,x_2 \\ \alpha_2\,x_3 & \alpha_3\,x_2 & \alpha_1\,x_1 \\ \alpha_1\,x_2 & \alpha_2\,x_1 & \alpha_3\,x_3 \end{pmatrix} , \tag{4.2}$$

with overall factors C_1 and C_2. Hence, apart from these factors, for $\zeta \to \zeta + \frac{1}{3}$ the entries of J are multiplied by cube roots of unity, whereas for $\zeta \to \zeta + \frac{\tau}{3}$ the coefficients of x_ℓ in J are permuted. However, both transformations can be compensated by suitable gauge transformations, *i.e.* $J(\zeta) = U_{L,1}\, J(\zeta + \frac{1}{3})\, U_{R,1}$ and $J(\zeta) = U_{L,2}\, J(\zeta + \frac{\tau}{3})\, U_{R,2}$ with[16]

$$U_{L,1} \;=\; \mathrm{Diag}\left(1,\, \omega^2,\, \omega\right) \;, \qquad U_{R,1} \;=\; \tfrac{1}{C_1}\, \mathrm{Diag}\left(\omega,\, \omega^2,\, 1\right) \;, \tag{4.3}$$

and

$$U_{L,2} \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \;, \qquad U_{R,2} \;=\; \tfrac{1}{C_2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \;. \tag{4.4}$$

We conclude that, modulo gauge transformations, the 3×3 matrix factorization is double-periodic in the open-string modulus ζ with $\zeta \sim \zeta + \frac{1}{3}$ and $\zeta \sim \zeta + \frac{\tau}{3}$, and thus obtain for the 3×3-matrix factorization the following toroidal open-string moduli space:

$$\mathfrak{M}_\zeta^{3\times 3} = \{\zeta \in \mathbb{C} \,|\, \zeta \sim \zeta + \tfrac{1}{3}\,,\, \zeta \sim \zeta + \tfrac{\tau}{3}\ \} \;. \tag{4.5}$$

One might suspect that there are other identifications in the open-string moduli space (4.5) of the 3×3-matrix factorization and that the space $\mathfrak{M}_\zeta^{3\times 3}$ is yet again just the covering space of the true moduli space. However, the construction of the vector bundles encoded in the columns of the matrix E and in the columns of the matrix J shows that the corresponding vector bundle transition matrices (2.8) and (3.22) contain both the factor $e^{2\pi i \cdot 3\zeta}$, which independently confirms the stated periodicity of J and E. Thus $\mathfrak{M}_\zeta^{3\times 3}$ is indeed the open-string moduli space for the 3×3-matrix factorization.

The reduction of the periodicity by one third will be an important ingredient in the construction of the 3×3-matrix factorization via tachyon condensation from 2×2 matrix building blocks as discussed in Section 5.

Our next task is to examine the moduli-space (4.5) of the factorization $P_{3\times 3}$ in greater detail. For $\zeta \to 0$ the uniformizing function, α_1, approaches zero, *c.f.* eq. (3.11), for which the 3×3 matrix factorization becomes singular.[17] Note that, in contrast to the singularity in the matrix factorization $P_{2\times 2}$, this singularity is not a

[16] One also has $E(\zeta) = U_{R,1}^{-1}\, E(\zeta + \frac{1}{3})\, U_{L,1}^{-1}$ and $E(\zeta) = U_{R,2}^{-1}\, E(\zeta + \frac{\tau}{3})\, U_{L,2}^{-1}$.

[17] At first sight there seem to be nine singularities, namely $\alpha_\ell = 0$ for the three choices of ℓ, which, according to (3.10) and (3.11), have each three zeroes. However, eqs. (4.1) and (4.2) show that all nine choices are gauge-equivalent.

mere gauge artifact, because it cannot be removed by a gauge transformation (1.5). This observation, however, is at first puzzling since there is no obvious physical reason for a singularity in the toroidal open-string moduli space.

However, we should keep in mind that the topological B-type D-branes really are objects in a category with certain equivalence relations, and a given matrix-factorization is just a particular realization of a topological B-type D-brane. In other words, singularities of matrix factorizations can also be an artifact of using the wrong representative for a D-brane in a given patch in the open-string moduli space. As has been discussed in ref. [21], the apparent singularity encountered in the 3×3-matrix factorization is indeed of this type, and this is what we want to make manifest in the following.

Following ref. [21], we replace as a first step the 3×3 matrix factorization by a 4×4-matrix factorization that is obtained by adding a trivial brane-anti-brane pair to (1.10), *i.e.*:

$$J_{4\times4} = \begin{pmatrix} \frac{W}{\alpha_1} & \\ & J(\alpha) \end{pmatrix} , \qquad E_{4\times4} = \begin{pmatrix} \alpha_1 & \\ & E(\alpha) \end{pmatrix} . \tag{4.6}$$

The next task is to perform an appropriate gauge transformation, (1.5), so that the singularity disappears for $\zeta \to 0$. This is achieved by first rewriting the open-string modulus, ζ, by $\zeta = \lambda - \rho$.[18] Using eq. (2.6) with x_ℓ, α_ℓ replaced by β_ℓ and γ_ℓ respectively, we may re-express α_ℓ by[19]

$$\begin{aligned} \alpha_1 &= \beta_2^2\gamma_1\gamma_3 - \beta_1\beta_3\gamma_2^2 , \\ \alpha_2 &= \beta_1^2\gamma_1\gamma_2 - \beta_2\beta_3\gamma_3^2 , \\ \alpha_3 &= \beta_3^2\gamma_2\gamma_3 - \beta_1\beta_2\gamma_1^2 , \end{aligned} \tag{4.7}$$

[18] This step is a little *ad hoc* but it is motivated by viewing the brane $L(\zeta)$ as a composite of the two branes $\overline{L}(\lambda)$ and $\overline{L}(-\rho)$.

[19] Strictly speaking the parameters, α_ℓ, must be rescaled by a homogeneous factor, *i.e.* $\alpha_\ell \to \eta^2(\tau)\mu_3(\lambda+\rho)\,\alpha_\ell$.

where α_ℓ and β_ℓ are uniformized by $\beta_\ell = \mu_\ell(\lambda)$ and $\gamma_\ell = \mu_\ell(\rho)$. Now the limit $\alpha_1 \to 0$ becomes $\gamma_\ell \to \beta_\ell$, which in terms of the uniformizing parameters translates into $\rho \to \lambda$. In terms of these auxiliary variables we perform the gauge transformation

$$U_L = \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1\gamma_1}x_1 & \frac{\beta_2}{\alpha_1\gamma_2}x_2 & \frac{\beta_3}{\alpha_1\gamma_3}x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad U_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\gamma_1}{\alpha_1\beta_1}x_1 & 1 & 0 & 0 \\ -\frac{\gamma_2}{\alpha_1\beta_2}x_2 & 0 & 1 & 0 \\ -\frac{\gamma_3}{\alpha_1\beta_3}x_3 & 0 & 0 & 1 \end{pmatrix}, \tag{4.8}$$

which generates the following 4×4-matrix factorization:

$$\widetilde{J}_{4\times4} = \begin{pmatrix} 0 & \frac{\beta_1}{\gamma_1}x_1^2 - \frac{\beta_1\gamma_1}{\gamma_2\gamma_3}x_2x_3 & \frac{\beta_2}{\gamma_2}x_2^2 - \frac{\beta_2\gamma_2}{\gamma_1\gamma_3}x_1x_3 & \frac{\beta_3}{\gamma_3}x_3^2 - \frac{\beta_3\gamma_3}{\gamma_1\gamma_2}x_1x_2 \\ \frac{\beta_1\gamma_1}{\beta_2\beta_3}x_2x_3 - \frac{\gamma_1}{\beta_1}x_1^2 & \alpha_1\, x_1 & \alpha_2\, x_3 & \alpha_3\, x_2 \\ \frac{\beta_2\gamma_2}{\beta_1\beta_3}x_1x_3 - \frac{\gamma_2}{\beta_2}x_2^2 & \alpha_3\, x_3 & \alpha_1\, x_2 & \alpha_2\, x_1 \\ \frac{\beta_3\gamma_3}{\beta_1\beta_2}x_1x_2 - \frac{\gamma_3}{\beta_3}x_3^2 & \alpha_2\, x_2 & \alpha_3\, x_1 & \alpha_1\, x_3 \end{pmatrix},$$

$$\widetilde{E}_{4\times4} = \begin{pmatrix} \alpha_1 & -\frac{\beta_1}{\gamma_1}x_1 & -\frac{\beta_2}{\gamma_2}x_2 & -\frac{\beta_3}{\gamma_3}x_3 \\ \frac{\gamma_1}{\beta_1}x_1 & -\frac{\alpha_1}{\alpha_2\alpha_3}x_2x_3 & \frac{1}{\alpha_3}x_3^2 + \frac{\beta_3\gamma_3}{\alpha_2\beta_1\gamma_2}x_1x_2 & \frac{1}{\alpha_2}x_2^2 + \frac{\beta_2\gamma_2}{\alpha_3\beta_1\gamma_3}x_1x_3 \\ \frac{\gamma_2}{\beta_2}x_2 & \frac{1}{\alpha_2}x_3^2 + \frac{\beta_3\gamma_3}{\alpha_3\beta_2\gamma_1}x_1x_2 & -\frac{\alpha_1}{\alpha_2\alpha_3}x_1x_3 & \frac{1}{\alpha_3}x_1^2 + \frac{\beta_1\gamma_1}{\alpha_2\beta_2\gamma_3}x_2x_3 \\ \frac{\gamma_3}{\beta_3}x_3 & \frac{1}{\alpha_3}x_2^2 + \frac{\beta_2\gamma_2}{\alpha_2\beta_3\gamma_1}x_1x_3 & \frac{1}{\alpha_2}x_1^2 + \frac{\beta_1\gamma_1}{\alpha_3\beta_3\gamma_2}x_2x_3 & -\frac{\alpha_1}{\alpha_2\alpha_3}x_1x_2 \end{pmatrix}, \tag{4.9}$$

which we will denote by $\widetilde{P}_{4\times4}$. We want to emphasize again that $\widetilde{P}_{4\times4}$ describes the same physical D-brane as the original factorization $P_{3\times3}$. As we will see in a moment, this factorization is indeed most canonical from the Landau-Ginzburg point of view: while $P_{3\times3}$ is odd-dimensional so that it cannot be given a standard representation in terms of boundary fermions, it is the equivalent factorization $\widetilde{P}_{4\times4}$ that can be given a concise expression in terms of Landau-Ginzburg fields.

In addition, $\widetilde{P}_{4\times4}$ has, by construction, the virtue that it is also well-defined for $\zeta = 0$, and thus we may proceed and take the limit $\alpha_1 \to 0, \gamma_\ell \to \beta_\ell$. Upon doing so, we prefer to perform yet another gauge transformation:

$$\begin{aligned} U_L &= \begin{pmatrix} \beta_2\left(\beta_1^3 - \beta_3^3\right) & -\left(\frac{\beta_3^2}{\beta_1\beta_2} + a\right)x_1 & -2a\,x_2 & \left(\frac{\beta_1^2}{\beta_2\beta_3} - 3a\right)x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ U_R &= \frac{1}{\beta_2(\beta_1^3 - \beta_3^3)}\, U_L^{\mathrm{T}}, \end{aligned} \tag{4.10}$$

in order to obtain the following form of $\widetilde{P}_{4\times4}$ at $\zeta=0$:

$$
\begin{aligned}
\widetilde{J}^0_{4\times4} &= \begin{pmatrix} 0 & x_1^2 - a\,x_2x_3 & x_2^2 - a\,x_1x_3 & x_3^2 - a\,x_1x_2 \\ -x_1^2 + a\,x_2x_3 & 0 & x_3 & -x_2 \\ -x_2^2 + a\,x_1x_3 & -x_3 & 0 & x_1 \\ -x_3^2 + a\,x_1x_2 & x_2 & -x_1 & 0 \end{pmatrix} , \\
\widetilde{E}^0_{4\times4} &= \begin{pmatrix} 0 & -x_1 & -x_2 & -x_3 \\ x_1 & 0 & -x_3^2 + a\,x_1x_2 & x_2^2 - a\,x_1x_3 \\ x_2 & x_3^2 - a\,x_1x_2 & 0 & -x_1^2 + a\,x_2x_3 \\ x_3 & -x_2^2 + a\,x_1x_3 & x_1^2 - a\,x_2x_3 & 0 \end{pmatrix} .
\end{aligned}
\tag{4.11}
$$

We will denote this limit by $\widetilde{P}^0_{4\times4}$. In terms of the boundary BRST operator, $\mathcal{Q}$, we can rewrite this factorization in a very simple form

$$
\widetilde{P}^0_{4\times4}: \quad \mathcal{Q} = \sum_{\ell=1}^{3} \left(x_\ell\,\pi_\ell + \tfrac{1}{3}\partial_{x_\ell} W(x,a)\,\overline{\pi}_\ell \right) \ , \tag{4.12}
$$

where $W(x,a)$ is the Landau-Ginzburg potential in (1.1) and $\pi_\ell, \overline{\pi}_\ell$, $\ell=1,2,3$, are boundary fermions obeying $\{\pi_i,\pi_j\} = \{\overline{\pi}_i,\overline{\pi}_j\} = 0$, $\{\pi_i,\overline{\pi}_j\} = \delta_{ij}$.[20] Indeed, for an appropriately chosen basis of the Clifford algebra, for which the chirality gamma matrix, γ_5, takes the form $\gamma_5 = \mathrm{Diag}\,(1,\ldots,1,-1,\ldots,-1)$, the boundary BRST operator (4.12) is precisely given by the matrix factorization (4.11), *i.e.*,

$$
\mathcal{Q} = \begin{pmatrix} 0 & \widetilde{J}^0_{4\times4} \\ \widetilde{E}^0_{4\times4} & 0 \end{pmatrix} \ . \tag{4.13}
$$

For non-vanishing open-string modulus, ζ, (4.12) gets deformed in leading order to $\mathcal{Q} \to \mathcal{Q} + \zeta\delta\mathcal{Q}$, where

$$
\delta\mathcal{Q} \sim \pi_1\pi_2\pi_3 + \left((a^3-1)x_1x_2x_3\right)\overline{\pi}_1\overline{\pi}_2\overline{\pi}_3 + \ldots \ . \tag{4.14}
$$

The ellipsis indicates terms that are linear and quadratic in x and are multiplied by products of boundary fermions of the type $\overline{\pi}_i\pi_j\pi_k$ and $\pi_i\overline{\pi}_j\overline{\pi}_k$. Note that the deformation, $\delta\mathcal{Q}$, contains a constant, x-independent term, which generates the constant entry, α_1, of the matrix $\tilde{E}_{4\times4}$ in eq. (4.9).

[20] That this represents a valid boundary BRST operator that obeys $\mathcal{Q}^2 = W(x,a)\mathbb{1}$, follows from the fact that, due to homogeneity of the Landau-Ginzburg potential, $W(x,a)$, we can write $W(x,a)\mathbb{1} = \frac{1}{3}\sum_\ell x_\ell\,\partial_{x_\ell}W(x,a)\mathbb{1} \equiv \frac{1}{3}\sum_{i,j} x_i\pi_i\,\partial_j W(x,a)\overline{\pi}_j$.

Let us summarize and analyze the foregoing results. Starting from the matrix factorization, $P_{3\times 3}$, we encountered a singularity. This singularity was removed by taking an equivalent description in terms of the 4×4-matrix factorization $\widetilde{P}_{4\times 4}$ in (4.9), which, however, is *reducible* due to an extra constant entry in the matrix, E. This constant entry disappears as $\zeta \to 0$, which implies that the corresponding factorization $\widetilde{P}^0_{4\times 4}$ becomes *irreducible*. This irreducible factorization is rigid in the sense that it exists only at one point in the open-string moduli space, and specifically is not part of a parametric family of *irreducible* matrix factorizations.[21] For a simple depiction of the situation, see Fig. 5.

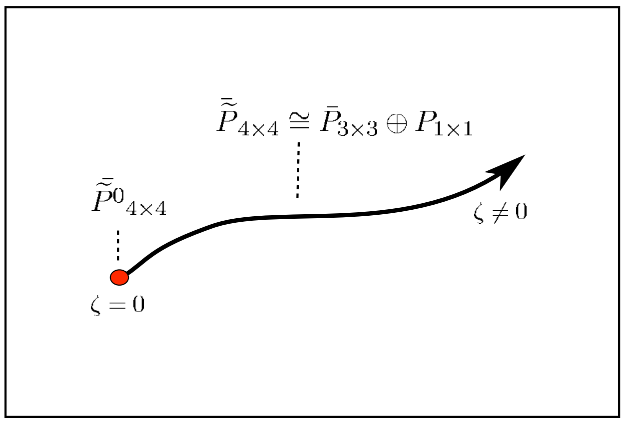

Fig. 5: *This shows the open-string moduli space of the factorization $\widetilde{P}_{4\times 4}$. It is compactified at $\zeta = 0$ by an exceptional indecomposable object, which is rigid in the sense that any deformation of it produces a decomposable object. It is associated with the most canonical factorization, (4.12), of the Landau-Ginzburg superpotential. In physical terms, it can be interpreted as a single rigid anti-$D2$ brane, and the deformation away from it corresponds to adding and pulling apart an extra $D0$-$\overline{D0}$-brane pair.*

In order to give this mathematical description a physical interpretation, we observe that the long-diagonal brane, $\overline{L}_3$, is associated to the line bundle, $\mathcal{E}(1,0)$, and located in the positive cone of the charge lattice in Fig. 1. Furthermore, according to

[21] There does, however, exist another family of irreducible 4×4-matrix factorizations (denoted by $P_{4\times 4}$, without the tilde) which describes D-branes with different charges (*c.f.* Section 7).

(1.14), the brane $\overline{L}_3$ has the (large radius) RR charges of a pure $\overline{D2}$-brane. This, however, does not necessarily imply that the brane $\overline{L}_3$ corresponds to a pure $\overline{D2}$-brane, rather more generally it describes a $\overline{D2}$-brane with a $D0$-$\overline{D0}$-brane pair resolved on its world-volume. We will now argue that the rigid exceptional matrix factorization, $\overline{\widetilde{P}}^0_{4\times 4}$, describes a pure $\overline{D2}$-brane, which also on physical grounds does not depend on an open-string modulus, whereas the matrix factorization, $\overline{P}_{3\times 3}$, captures the situation of a $\overline{D2}$-brane with an additional resolved $D0$-$\overline{D0}$-brane pair.[22] This D-brane configuration is indeed expected to depend on a (relative) open-string modulus.

This line of arguments can be substantiated by using the relationship between line bundles and divisors. Each line bundle in the class, $\mathcal{E}(1,0)$, can be specified by a divisor $\mathcal{O}(\zeta_1 - \zeta_2)$, where ζ_1 and ζ_2 denote points on the torus, Σ. The distance between them corresponds to the open-string modulus ζ that appears in the transition function, $A(z)$, of the line bundle $\mathcal{E}(1,0)$. Physically, the point ζ_1 describes the position of a $D0$-brane, whereas the point ζ_2 refers to the position of a $\overline{D0}$-brane. If, on the other hand, ζ_1 and ζ_2 coincide, or in other words, if the $D0$-brane is on top of the $\overline{D0}$-brane, we arrive at the divisor of a trivial line bundle, which precisely corresponds to the exceptional 4×4-matrix factorization of a pure, rigid $\overline{D2}$-brane.

The appearance of distinguished matrix factorizations at special points in the moduli space has been observed before in ref. [19]. According to the classification of holomorphic vector bundles on the torus Σ [17], each vector bundle of rank r and degree zero is classified by a degree zero line bundle on Σ, namely by the determinant line bundle of the vector bundle in question. Moreover, each line bundle of degree zero has vanishing first Chern class and according to eq. (2.4) is given by the transition function $A(z) = y^{-1}$ on $\mathbb{C}^*$ [46,18]. Note that for $y = 1$, or for $\zeta = 0$, the transition function becomes the identity and hence the determinant bundle becomes the trivial line bundle.[23] In ref. [19] it is shown that such "exceptional" vector bundles in $\mathcal{E}(r,0)$ with trivial determinant bundle, correspond to exceptional MCM modules $\mathcal{M}_r$,[24] and

[22] Note again that such D-brane configurations cannot be distinguished at the level of K-theory.

[23] Those "exceptional" bundles of rank r and degree zero are also distinguished by the fact that they have a global non-trivial section.

[24] The MCM modules $\mathcal{M}_r$ are self-dual $\mathcal{R}$-modules, i.e. $\mathcal{M}_r \simeq \mathrm{Hom}(\mathcal{M}_r, \mathcal{R})$ [19].

furthermore give rise to exceptional matrix-factorizations, which are isolated in the sense that there exist no (open string) deformations that would stay within the class of *irreducible* factorizations.

To conclude this section, we consider how the bundle data encoded in $J_{3\times 3}$ and $E_{3\times 3}$ evolves over the moduli space, $\mathfrak{M}_\zeta^{3\times 3}$. Consider first the anti-D-brane, $\overline{L}$, for which (up to an unimportant sign) the rôles of $J_{3\times 3}$ and $E_{3\times 3}$ are exchanged, *i.e.* the vector bundle data is encoded in the matrix $E_{3\times 3}$ given in (1.10). At a generic point in the open-string moduli space $\mathfrak{M}_\zeta^{3\times 3}$, the vector bundle spanned by the columns of $E_{3\times 3}$ is associated to a line bundle in $\mathcal{E}(1,0)$ (*c.f.* eq. (3.13)). Therefore one might naively expect that the exceptional 4×4-matrix factorization at $\zeta = 0$, corresponds to the exceptional rank one MCM module $\mathcal{M}_1$ (it is actually the anti-version, $\overline{\widetilde{P}}{}^0_{4\times 4}$, that we talk about here, as we started out with the anti-brane, $\overline{L}$). This, however, is not true, because we should keep in mind that this 4×4 matrix factorization has been constructed by adding a trivial brane-anti-brane pair. In particular the new column of $E_{4\times 4}$ in (4.6) has increased the number of linearly independent columns of $E_{3\times 3}$ by one, and thus the 4×4-matrix factorization is correctly identified with the rank two exceptional module, $\mathcal{M}_2$.[25] Comparison with ref. [19] does indeed confirm that the exceptional 4×4-matrix factorization is associated to the exceptional MCM module $\mathcal{M}_2$.

On the other hand, if we now consider the the D-brane, L, the bundle data are encoded in the matrix J of (1.10). As explained in section 3.2, the D-brane L gives rise to the rank two vector bundle $\mathcal{E}(2,3)$. As before, in order to analyze the limit $\zeta \to 0$, we must not neglect the added trivial brane-anti-brane pair. However, the matrix J is only enhanced by the block-diagonal entry, W, (*c.f.* eq. (4.6)), which does not increase the number of linear independent columns of J because $W \equiv 0$ on the torus, Σ. Therefore the exceptional 4×4-matrix factorization assigned to the brane, L, at $\zeta = 0$ must still be associated to a vector bundle in $\mathcal{E}(2,3)$. Once again comparing with ref. [19] reveals that the 4×4-matrix factorization (4.11) is really mapped to the exceptional 4×4-matrix factorization in $\mathcal{E}(2,3)$, which, furthermore, is associated to the first syzygy module, $\Omega_1(\mathcal{M}_2)$, of $\mathcal{M}_2$. Note that this is precisely in accord with the physical picture because passing over to the first syzygy module, $\Omega_1(\mathcal{M}_2)$, of $\mathcal{M}_2$ corresponds to switching from the anti-D-brane, $\overline{L}$, to the D-brane, L.

[25] The self-duality of $\mathcal{M}_2$ is reflected in the fact that the matrices $J_{4\times 4}$ and $E_{4\times 4}$ in (4.11) are antisymmetric.

5. Tachyon condensation

The aim of this section is to generate new matrix factorizations via tachyon condensation. In particular, we will show that any matrix factorization on the elliptic curve may, in principle, be obtained by iteratively condensing building blocks of 2×2-matrix factorizations. So as to generate new matrices in a controlled fashion, we introduce the notion of equivariant matrix factorizations. This enables us to explain how different choices of tachyons for a given pair of matrix factorizations will lead to different condensates. Conversely, if we seek a certain condensate, equivariance provides a very useful set of constraints upon the form of the requisite tachyon.

In the present and the next two sections we will use these ideas to generate explicitly all rank two matrix factorizations.

5.1. Equivariant matrix factorizations

In order to systematically generate, via tachyon condensation, a particular matrix factorization for a given set of RR charges, it is necessary to have control over the RR charges of the constituents of the condensate. For instance, we need to distinguish among the three D-branes comprised in the 2×2-matrix factorization, S, and in the 3×3-matrix factorization, L, in order to realize the different composites illustrated in Fig. 4. This is achieved by refining the description of B-type D-branes in terms of equivariant matrix factorizations along the lines of [26] (see also [32]).

Since our analysis takes place at the Gepner point of the Kähler moduli space, the appropriate Landau-Ginzburg model is specified in terms of the superpotential (1.1) *together* with the natural $\mathbb{Z}_3$ orbifold action $\rho(k)$, $k\in\mathbb{Z}_3$

$$\rho:\rho(k)\,x_\ell\mapsto\omega^k\,x_\ell\quad\text{with}\quad\omega=e^{\frac{2\pi i}{3}}\ . \tag{5.1}$$

Obviously, this orbifold action must also be taken into account in the characterization of B-type D-branes, for which the equivariant formulation of matrix factorizations becomes the appropriate framework. In practice this means that a D-brane, P, represented by a $n\times n$-matrix factorization (1.6) is supplemented by two $\mathbb{Z}_3$ representations R_0 and R_1 of dimension n:

$$P \equiv \left[\begin{array}{c} (P_1, R_1^P) \\ J_P \Big\uparrow \quad \Big\downarrow E_P \\ (P_0, R_0^P) \end{array} \right] \ , \tag{5.2}$$

such that, in addition to the factorization condition (1.4), one also requires the equivariance condition [26]:

$$\begin{aligned} J_P(x) \ &= \ R_1^P(k^{-1})\, J_P(\rho(k)x)\, R_0^P(k) \ , \\ E_P(x) \ &= \ R_0^P(k^{-1})\, E_P(\rho(k)x)\, R_1^P(k) \ . \end{aligned} \tag{5.3}$$

Note that these relations also require an adjustment of the notion of gauge transformations. Namely, it is easy to infer that a gauge transformation (1.5) induces also a conjugation transformation acting on the representations R_0^P and R_1^P by

$$R_0^P(k) \to U_R^{-1}\, R_0^P(k)\, U_R \ , \qquad R_1^P(k) \to U_L\, R_1^P(k)\, U_L^{-1} \ . \tag{5.4}$$

We demonstrate this idea by going through the matrix factorizations that we have encountered so far. The easiest example is the trivial D-brane configuration, V, given by the 1×1-matrix factorization, $J_V = 1$, $E_V = W$, which corresponds to the vacuum. For this configuration the equivariance conditions (5.3) are fulfilled as long as $R_0^V(k) = R_1^V(k) = \omega^{a\,k}$ for any $a = 1,2,3$. The triviality of the D-brane configuration manifests itself in the fact that the two representations $R_0^V(k)$ and $R_1^V(k)$ must be the same.

For the 2×2-matrix factorization, S, the equivariance yields three possible representations R_0^S and R_1^S, which read

$$R_0^S(k) \ = \ \mathrm{Diag}\left(\omega^{(1-a)\,k},\, \omega^{(1-a)\,k}\right) \ , \qquad R_1^S(k) \ = \ \mathrm{Diag}\left(\omega^{-a\,k},\, \omega^{-(a+1)\,k}\right) \ . \tag{5.5}$$

Taking into account the orbifold $\mathbb{Z}_3$ action therefore results in three different 2×2-matrix factorizations, which are distinguished by $a = 1,2,3$. Thus in the orbifolded Landau-Ginzburg model the 2×2-matrix factorization, S_a, gains an additional label, a, which specifies the choice of representations R_0^S and R_1^S.

Analogously, we find for the 3×3-matrix factorization three possible representations

$$R_0^L(k) \,=\, \omega^{(1-a)\,k}\, \mathbb{1}_{3\times 3}\ , \qquad R_1^L(k) \,=\, \omega^{-(a+1)\,k}\, \mathbb{1}_{3\times 3}\ , \tag{5.6}$$

which give rise to the three distinct branes, L_a, labeled by $a = 1, 2, 3$.

Note that the equivariant labels allow us to unambiguously distinguish among all the short- and long-diagonal branes illustrated in Fig. 1. This use of equivariance means that we can go beyond the vector bundles discussed in ref. [19] and enables us to deal with objects that are more general than vector bundles in the large radius limit. In particular, recall that one of the short branes, S_2, is the $D0$-brane and this is not associated to a vector bundle but rather to a point-like sheaf.

Tachyon condensation provides the opportunity to make extensive direct tests of this equivariant formulation of the $\mathbb{Z}_3$ orbits of branes. We will now examine this in detail but first we need the appropriate equivariant modification of the open-string spectrum [26]. Just as in (1.15), a boundary changing operator $\Psi_{(P,Q)}$ of an open string stretching between the brane P and Q can be pictured by the diagram:

$$\begin{tikzcd}
(P_1, R_1^P) \arrow[d, bend left, "E_P"] \arrow[rrd, "\psi_1"] & & (Q_1, R_1^Q) \arrow[d, bend left, "E_Q"] \\
(P_0, R_1^P) \arrow[u, bend left, "J_P"] \arrow[rru, "\psi_0"'] & & (Q_0, R_0^Q) \arrow[u, bend left, "J_Q"]
\end{tikzcd} \ . \tag{5.7}$$

In the equivariant context, the physical state conditions, (1.16), of the maps, ψ_0 and ψ_1, of (the fermionic) operator, $\Psi_{(P,Q)}$, are now supplemented by the equivariance condition[26]

$$\begin{aligned}
\psi_0(x) \,&=\, R_1^Q(k^{-1})\, \psi_0(\rho(k)x)\, R_0^P(k)\ , \\
\psi_1(x) \,&=\, R_0^Q(k^{-1})\, \psi_1(\rho(k)x)\, R_1^P(k)\ .
\end{aligned} \tag{5.8}$$

This condition amounts to a selection rule upon the original (non-equivariant) tachyon spectrum. The selection rule tells us which, if any, tachyon appears between different members of the $\mathbb{Z}_3$ families. It also tells us the $\mathbb{Z}_3$ label of the condensed state. We now illustrate this with some examples.

[26] A similar equivariance condition applies for the bosonic operators.

5.2. A reprise of tachyon condensations of $P_{2\times 2}$

In calculating tachyon condensations, our *modus operandi* consists of two basic steps: First we analyze the boundary changing operators of the constituents in terms of equivariant matrix factorizations. Then we build up the composites via the cone construction, (1.18), and simplify the result (if necessary) using gauge transformations, (1.5) and (5.4). We begin by returning to the examples of tachyon condensation already introduced in section 1.2.

The simplest example is brane/anti-brane annihilation between $S_a(\alpha)$ and $\overline{S}_b(\beta)$. Recall that the fermionic identity operator, (1.22), appears in the open-string spectrum of the boundary changing sector provided that $\alpha = \beta$. Moreover, equivariance imposes an additional constraint on the existence of this operator, *i.e.* due to (5.8) the representation R_0^S and $R_1^{\overline{S}}$ acting on $\psi_0 = \mathbb{1}_{2\times 2}$ and the representations R_1^S and $R_0^{\overline{S}}$ acting on $\psi_1 = \mathbb{1}_{2\times 2}$ must pairwise coincide. In other words the fermionic identity operator, (1.22), can only form a condensate of $S_a(\alpha)$ with $\overline{S}_a(\alpha)$.[27]

Applying the cone construction to the constituents $S_a(\alpha)$ and $\overline{S}_a(\alpha)$ using the boundary changing operator, (1.22), one obtains the following composite:

$$J_V = \begin{pmatrix} J_{2\times 2}(\alpha) & \mathbb{1}_{2\times 2} \\ 0 & -E_{2\times 2}(\alpha) \end{pmatrix} , \qquad E_V = \begin{pmatrix} E_{2\times 2}(\alpha) & \mathbb{1}_{2\times 2} \\ 0 & -J_{2\times 2}(\alpha) \end{pmatrix} . \tag{5.9}$$

Note that the constant entries in the matrices, J_V and E_V, mean that one may make elementary simplifications by "row and column elimination". Here we simply observe that because there are two independent constant entries in J_V and in E_V, the method of row and column elimination shows that the composite, (5.9), is gauge equivalent to four trivial brane-anti-brane pairs. Thus, within the D-brane category of B-type D-branes, the tachyon condensation of S_a and $\overline{S}_a$ describes the annihilation to the vacuum:

$$S_a(\zeta) \succ_\Psi \overline{S}_a(\zeta) \Longrightarrow V , \tag{5.10}$$

where ζ is given by $\alpha_\ell = \mu_\ell(\zeta)$.

---

[27] In equivariant matrix factorizations going from the brane P to the anti-brane, $\overline{P}$, does not only exchange the matrices (J_P, E_P) with $(-E_P, -J_P)$ but is also accompanied by a flip of the representations (R_0^P, R_1^P) to (R_1^P, R_0^P).

The other example considered in Section 1.2 was the condensation of $S_a(\alpha)$ with $S_b(\beta)$. The relevant boundary-changing operator, $\Psi_{(S_a,S_b)}(\alpha,\beta)$, is given in eq. (1.23) and the condensation process has already been described in section 1.3. Here, we want to re-examine the discussion so as to elucidate the selection rules arising from equivariance. First, the form of the tachyon, $\Psi_{(S_a,S_b)}(\alpha,\beta)$, constraints the equivariance labels, a and b. That is to say, that evaluating condition (5.8) together with the representations (5.5) of S_a and S_b yields the relation $b = a + 1$. Hence, the boundary changing operator $\Psi_{(S_a,S_b)}(\alpha,\beta)$ only appears in the open-string cohomology for the "equivariant pair" $S_a(\alpha)$ and $S_{a+1}(\beta)$, and only for such a pair can this fermionic boundary operator be used to construct the composite, $\overline{S}_c(\gamma)$, determined in eq. (1.29).

Furthermore, we can also determine the equivariance label, c, of the resulting composite, $\overline{S}_c(\gamma)$. The representations, (5.5), of the constituents, $S_a(\alpha)$ and $S_{a+1}(\beta)$, yield, for the untransformed composite, (1.26) the representations

$$\begin{aligned} R_0(k) \;&=\; \mathrm{Diag}\left(\omega^{-a\,k},\omega^{-a\,k},\omega^{(1-a)\,k},\omega^{(1-a)\,k}\right) \ , \\ R_1(k) \;&=\; \mathrm{Diag}\left(\omega^{(2-a)\,k},\omega^{(1-a)\,k},\omega^{-a\,k},\omega^{-(a+1)\,k}\right) \ , \end{aligned} \tag{5.11}$$

As discussed in section 1.3, by acting with a gauge transformation, U_L, U_R, the composite, (1.26), can be cast into the 4×4-matrix factorization (1.27). However, this gauge transformation also induces a corresponding conjugation action on the representations, (5.11), according to (5.8), and hence, after dropping all the trivial brane-anti-brane pairs, we readily read off the resulting equivariant 2×2-matrix factorization, which turns out to be $\overline{S}_{a+2}(\gamma)$.

Before we move on to the next example, we can gain some further insight into the open-string parameters, γ_ℓ, given in (1.28), by rewriting them in terms of uniformizing parameter along the lines of eq. (1.12). Namely, with $\alpha_\ell = \mu_\ell(\zeta)$ and $\beta_\ell = \mu_\ell(\lambda)$ the parameter γ_ℓ can be uniformized by $\gamma_\ell = \mu_\ell(\zeta-\lambda)$, where we used eq. (2.6) after substituting x_ℓ by β_ℓ. Thus schematically the tachyon condensation process is summarized by

$$S_a(\zeta) \succ_{\Psi_{(S_a,S_{a+1})}} S_{a+1}(\lambda) \Longrightarrow \overline{S}_{a+2}(\zeta-\lambda) \ . \tag{5.12}$$

The next task is to construct the long-diagonal D-branes, L, by tachyon condensation of two short-diagonal D-branes S. From Fig. 4 we readily infer that one is required to condense the brane $S_a(\alpha)$ with $\overline{S}_{a+2}(\beta)$ in order to reach L as a condensate. Thus we seek a fermionic boundary changing operator, $\Psi_{(S_a,\overline{S}_{a+2})}(\alpha,\beta)$, in the spectrum of open-strings stretching from S_a to $\overline{S}_{a+2}$.[28] By once again taking advantage of the equivariance condition, (5.8), we readily deduce the degrees of the entries of the fermionic boundary changing operators:

$$\psi_0 \;=\; \begin{pmatrix} l_1(x) & l_2(x) \\ l_3(x) & l_4(x) \end{pmatrix} \;, \qquad \psi_1 \;=\; \begin{pmatrix} l_5(x) & q(x) \\ c & l_6(x) \end{pmatrix} \;. \tag{5.13}$$

Here c has degree zero, $l_1, \ldots, l_6$ are linear and q is quadratic in x_ℓ. A detailed analysis of the corresponding fermionic cohomology element along the lines of eq. (1.16) and (1.17) eventually reveals:

$$\psi_0 \;=\; \begin{pmatrix} G_1(\alpha,\beta) & G_2(\alpha,\beta) \\ G_3(\alpha,\beta) & G_1(\beta,\alpha) \end{pmatrix} \;, \qquad \psi_1 \;=\; \begin{pmatrix} G_4(\alpha,\beta) & H(\alpha,\beta) \\ C(\alpha,\beta) & G_4(\beta,\alpha) \end{pmatrix} \;, \tag{5.14}$$

with

$$C(\alpha,\beta) \;=\; -\alpha_3\beta_3\left(\alpha_3\beta_2-\alpha_2\beta_3\right) \;, \tag{5.15}$$

and

$$\begin{aligned}
G_1(\alpha,\beta) \;&=\; -\left(\alpha_3\beta_1+\alpha_1\beta_3\right)x_2+\frac{\alpha_3\left(\beta_2^3+\beta_3^3\right)}{\beta_1\beta_3}x_3 \;, \\
G_2(\alpha,\beta) \;&=\; \left(\alpha_2\beta_3-\alpha_3\beta_2\right)\,x_2 \;, \\
G_3(\alpha,\beta) \;&=\; \left(\alpha_3\beta_2-\alpha_2\beta_3\right)x_1+\left(\frac{\alpha_3\beta_2^2}{\beta_3}-\frac{\alpha_2^2\beta_3}{\alpha_3}\right)x_3 \;, \\
G_4(\alpha,\beta) \;&=\; \frac{\alpha_3\beta_2\left(\alpha_3\beta_2-\alpha_2\beta_3\right)}{\beta_1\beta_3}x_1+\frac{\alpha_3\left(\alpha_2\beta_1+\alpha_1\beta_2\right)}{\beta_2}x_2-\frac{\alpha_2\alpha_3\left(\beta_2^3+\beta_3^3\right)}{\beta_1\beta_2\beta_3}x_3 \;,
\end{aligned} \tag{5.16}$$

[28] The precise argument is actually a bit more involved: From the A-model mirror picture we know that S_a intersects with $\overline{S}_{a+2}$ once, and hence we expect (at least) either one fermionic operator or one bosonic operator in the spectrum. The orientation of the intersection tells us whether the operator is bosonic or fermionic. Since the intersection of S_a and S_{a+1} has the same orientation as S_a with $\overline{S}_{a+2}$ the fermionic operator, $\Psi_{(S_a,S_{a+1})}$, implies the presence of a fermionic operator $\Psi_{(S_a,\overline{S}_{a+2})}$.

and

$$
\begin{aligned}
H(\alpha,\beta) \;=\; & \frac{\alpha_2\beta_2\left(\alpha_3\beta_2-\alpha_2\beta_3\right)}{\alpha_1\alpha_3\beta_1\beta_3}x_1^2+\frac{\left(\alpha_2\alpha_3\beta_1^2-\alpha_1^2\beta_2\beta_3\right)}{\alpha_2\alpha_3\beta_2\beta_3}x_2^2 \\
& +\frac{\left(\alpha_2^3\beta_1^2\beta_3-\alpha_1^2\alpha_3\beta_2^3\right)}{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}x_1x_2+\frac{\left(\alpha_3^3\beta_2^3-\alpha_2^3\beta_3^3\right)}{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}x_1x_3+\frac{\alpha_3\beta_2-\alpha_2\beta_3}{\alpha_2\beta_2}x_2x_3\ .
\end{aligned}
\tag{5.17}
$$

Next we use this fermionic boundary-changing operator, $\Psi_{(S_a,\overline{S}_{a+2})}(\alpha,\beta)$, to form the composite

$$
\hat{J}_L(\alpha,\beta) \;=\; \begin{pmatrix} -E_{2\times 2}(\beta) & \psi_0(\alpha,\beta) \\ 0 & J_{2\times 2}(\beta) \end{pmatrix}\ , \qquad \hat{E}_L(\alpha,\beta) \;=\; \begin{pmatrix} -J_{2\times 2}(\beta) & \psi_1(\alpha,\beta) \\ 0 & E_{2\times 2}(\beta) \end{pmatrix}\ .
\tag{5.18}
$$

Note that the matrix $\hat{E}_L$ contains a constant entry, which allows us to perform row and column eliminations so as to obtain a 3×3-matrix factorization. Moreover, from the degrees of the entries of $\hat{E}_L$, it is apparent that the row and column operations precisely remove the linear entries in $\hat{E}_L$. Therefore the matrix $\hat{E}_L$ becomes, after removing the trivial brane-anti-brane pair, precisely the matrix $E_{3\times 3}$ with only quadratic entries. Conversely, after row and column elimination, $\hat{J}_L$ contains only linear entries and takes the form the matrix $J_{3\times 3}$.

The final task is to relate the result of this row and column elimination to the standard form of the 3×3-matrix factorization stated in (1.10), and thereby determine the precise form of the brane, L. This is achieved by an appropriate gauge transformation that allows us to determine the open-string parameter of the composite in terms of α_ℓ and β_ℓ. The result of this, straightforward but tedious, analysis yields (in terms of the parameters ζ and λ of the uniformized functions $\alpha_\ell=\mu_\ell(\zeta), \beta_\ell=\mu_\ell(\lambda)$):

$$
S_a(\zeta)\succ_{\Psi_{(S_a,\overline{S}_{a+2})}}\overline{S}_{a+2}(\lambda)\Longrightarrow L_a(\tfrac{1}{3}(\zeta-\lambda))\ .
\tag{5.19}
$$

Before we conclude this section, a few comments are in order: First, note that the factor $\frac{1}{3}$ in the uniformizing function μ_ℓ correctly reproduces the periodicity of the 3×3-matrix factorization observed in eq. (4.5). Furthermore, the enhancement of the 3×3-matrix factorization to the 4×4-matrix factorization observed in section 3.4 also becomes manifest. Namely, the indecomposable 4×4-matrix factorization is obtained for $\lambda\to\zeta$, or $\beta_\ell\to\alpha_\ell$. In this limit, the constant entry (5.15) vanishes and the composite, (5.14), becomes gauge equivalent to the exceptional factorization $\widetilde{P}^0_{4\times 4}$ in (4.11).

6. Constructing more general matrix factorizations

So far we have analyzed some examples of tachyon condensation among the known factorizations $P_{2\times 2}$ and $P_{3\times 3}$, and shown how the latter can be obtained as a condensate of the 2×2 factorization. We have also seen how the 4×4 factorization $\widetilde{P}_{4\times 4}$, which is more naturally associated with boundary fermions, can be produced by condensation and how it connects to the moduli space of the 3×3 factorization.

We now wish to go beyond this and obtain new factorizations. There are several techniques that we can use to do this, and in this section we will describe them and determine the tachyons that we will need to create a number of new factorizations. In the subsequent sections, we will present a list of the resulting factorizations and discuss some of their properties.

6.1. New matrix factorizations and tachyons from transpositions

Given a matrix factorization $P=(J,E)$, one can obtain another matrix factorization by transposing the matrices J and E. We denote the resulting factorization by

$$P^{\mathrm{T}} = (J^{\mathrm{T}}, E^{\mathrm{T}}) \ . \tag{6.1}$$

According to the equivariance condition (5.3), transposition also acts upon the equivariant representations, $R^P_{0,1}$, by:

$$R_0^{P^{\mathrm{T}}}(k) \,=\, R_1^{P^{\mathrm{T}}}(k^{-1}) \ , \qquad R_1^{P^{\mathrm{T}}}(k) \,=\, R_0^{P^{\mathrm{T}}}(k^{-1}) \ . \tag{6.2}$$

Applying the transposition operation to the branes S_a and L_a we find (modulo simple gauge transformations) the corresponding transposed matrix factorizations to be:

$$S_a^{\mathrm{T}}(\zeta) \,\sim\, \overline{S}_{-a-1}(\zeta) \ , \qquad L_a^{\mathrm{T}}(\zeta) \,\sim\, L_{-a}(-\zeta) \ . \tag{6.3}$$

In general the transposed matrix factorization can give rise to new matrix factorizations, which are not related to the original factorization by changing the open-string modulus or by passing over to the anti-brane. We will encounter such an example in with the two 5×5 factorizations, $P_{5\times 5}$ and $P^{\mathrm{T}}_{5\times 5}$, that describe two distinct classes of D-branes.

In the language of MCM modules, the transposed matrix factorization is associated to the dual MCM module. In particular, eq. (6.3) illustrates that the brane, L, at $\zeta = 0$ is invariant under transposition, which reflects the self-duality of the corresponding exceptional MCM module at $\zeta = 0$ in the open-string moduli space as discussed in Section 4.

Suppose we have a known tachyon, $\Psi_{(P,Q)}$, satisfying the physical state condition (1.16). Then the transposed physical state condition reads

$$\begin{aligned} 0 &= \psi_0^{\mathrm{T}}\, E_Q^{\mathrm{T}} + J_P^{\mathrm{T}}\, \psi_1^{\mathrm{T}} \ , \\ 0 &= \psi_1^{\mathrm{T}}\, J_Q^{\mathrm{T}} + E_P^{\mathrm{T}}\, \psi_0^{\mathrm{T}} \ , \end{aligned} \tag{6.4}$$

which we readily identify with the physical state conditions for a fermionic boundary changing operator, $\Psi_{(Q^{\mathrm{T}},P^{\mathrm{T}})} \sim (\psi_0^{\mathrm{T}}, \psi_1^{\mathrm{T}})$, of an open string stretching between Q^{T} and P^{T}.

To illustrate this procedure we choose as an example the fermionic boundary changing operator, $\Psi_{(S_a(\lambda),L_{a+1}(\zeta))}$ of an open string stretching between $S_a(\lambda)$ and $L_{a+1}(\zeta)$. This immediately gives rise to the boundary changing operator, $\Psi_{(L_{a+1}^{\mathrm{T}}(\zeta),S_a^{\mathrm{T}}(\lambda))}$, of an open string stretching between $L_{a+1}^{\mathrm{T}}(\zeta)$ and $S_a^{\mathrm{T}}(\lambda)$, which we can finally convert according to (6.3) to the fermionic boundary changing operator

$$\Psi_{(L_{-a-1}(-\zeta),\overline{S}_{-a-1}(\lambda))} \sim \Psi_{(L_{a+1}^{\mathrm{T}}(\zeta),S_a^{\mathrm{T}}(\lambda))} \ , \tag{6.5}$$

where those two boundary changing operators are related by the same gauge transformation that gives rise to the corresponding identification in eq. (6.3).

6.2. Creating bound states at threshold

In obtaining matrix factorizations via condensation, there is a significant difference between the situation where the D-brane charges, (r, c_1), of the end product are co-prime or have common factors. In particular, when they have a common factor, it turns out that there is a simple canonical procedure for obtaining the matrix factorization as a "bound state at threshold".

To illustrate the basic physical idea it is easiest to use the A-brane mirror picture where the charges (r, c_1) correspond to winding numbers (p, q). The tension m of a BPS-saturated $D1$-brane is then simply proportional to its length on the covering space of Σ, as shown in Fig. 1. That is,

$$m \;=\; |q + \rho\, p| \;, \tag{6.6}$$

where ρ is the complex structure parameter of the mirror torus. It coincides with the Kähler parameter of the original B-model, which decouples in the topologically twisted theory so that we have no control over it.[29] However, things are simple on the torus; in particular, there are no lines of marginal stability on the Kähler moduli space, so all what matters is that the tension is linear in the charges. This implies that if we only know the charges, (p,q), we cannot distinguish a single string with winding numbers (np,nq) from n strings, each with winding number (p,q). Such single-string configurations are commonly referred to as bound states at threshold [47].[30] In general, it is often unclear whether a combination of states with equal charges will actually form a true bound state at threshold, or whether the state is simply a multi-particle superposition of the constituents.

In matrix factorizations, where we have additional information in form of the open-string moduli, we can, in fact, easily see the difference between a bound state at threshold and a mere direct sum of its constituents. In the following, we consider only the condensation of a pair of identical branes but more general configurations can be treated in a similar way.

Consider first a pair of identical branes, P, but with distinct position moduli, α and β. It is clear that as long as $\alpha \neq \beta$, the branes do not intersect and thus there isn't a tachyon that could possibly lead to bound state formation. The combined system is characterized by a block-diagonal matrix factor

$$J_{PP}(\alpha,\beta) \;=\; \begin{pmatrix} J_P(\beta) & 0 \\ 0 & J_P(\alpha) \end{pmatrix} \;, \tag{6.7}$$

[29] In the Landau-Ginzburg model, which corresponds a $\mathbb{Z}_3$ orbifold of the torus CFT, ρ is fixed to be a third root of unity, but in the orbifolded Landau-Ginzburg model, the Kähler modulus ρ becomes a free parameter.

[30] Of course, when (p,q) are co-prime, all multiple string configurations with the same net winding number cannot satisfy the BPS bound because the triangle inequality implies that their total length will not be minimal.

and similarly for E_{PP}. The vector bundles corresponding to J_{PP} and E_{PP} are thus trivial direct sums. What is the moduli space of such a configuration?[31] The naive answer is that it is simply given by the direct product of the individual moduli spaces. However, there exists a gauge transformation that switches the rôles of the block matrices. That is, $J_{PP}(\alpha, \beta)$ is gauge equivalent to $J(\beta, \alpha)_{PP}$, and since gauge equivalences are equivalences in the full category, it follows that the open-string moduli space is given by the symmetrized product of the individual moduli spaces, *i.e.*,

$$\mathfrak{M}_{PP} = \mathrm{Sym}^2(\Sigma) \ , \tag{6.8}$$

which is entirely to be expected because the $D0$-branes are indistinguishable.

The identifying gauge transformations have a fixed point when the branes coincide and this leads to an orbifold singularity in the moduli space when $\beta = \alpha$. This is reflected in the physics in that, when the branes move on top of one another, the cohomology jumps and a tachyon appears (in a similar manner to the results discussed in Section 1).[32] Since the physical state condition at $\beta = \alpha$ is identical to the equation that determines the boundary preserving endomorphism, Ω_P, the new operator $\Omega_{(P,P)} \equiv \Psi_{(P,P)}$ has the exactly the same form as Ω_P, however, due to the implicit Chan-Paton labels it is really a boundary changing operator because it acts between different D-branes.

Because the "tachyon" $\Omega_{(P,P)} = (\Omega^{(J)}_{(P,P)}, \Omega^{(E)}_{(P,P)})$ has charge one, it is a marginal operator of the theory that couples to a dimensionless modulus, ζ_Ω:

$$J_{PP}(\alpha, \zeta_\Omega) = \begin{pmatrix} J_P(\alpha) & \zeta_\Omega\, \Omega^{(J)}_{PP}(\alpha) \\ 0 & J_P(\alpha) \end{pmatrix} \ . \tag{6.9}$$

Switching on ζ_Ω condenses the two-brane system into one indecomposable object, which is a genuine bound state at threshold. Indeed, the off-diagonal terms in (6.9) generically transform non-trivially between coordinate patches on the torus and thereby make the vector bundle defined by (6.9) into a non-split extension of the original vector bundles. This clearly shows the usefulness of the extra moduli information

[31] We thank Robert Helling for asking this question, which then lead to the following discussion.

[32] Again, both a bosonic and a fermionic tachyon appear at $\beta = \alpha$, so that there is no net contribution to the intersection index.

carried by the factorization, as the existence of such an object cannot be inferred from the K-theory charges alone. Mathematically, the interpretation of this extra degree of freedom is that it resolves the singularity of $\mathfrak{M}_{PP}$. We have depicted the situation in Fig. 6.

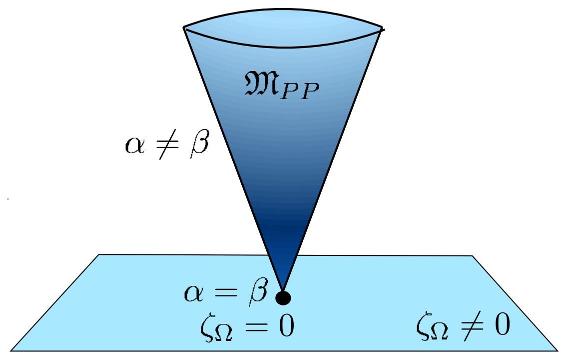

Fig. 6: *The upper part of this diagram shows the moduli space, $\mathfrak{M}_{PP} \simeq \mathrm{Sym}^2(\Sigma)$, of two identical branes but with independent moduli α and β. A new branch emerges at the singularity at $\alpha = \beta$. Switching on ζ_Ω moves the factorization onto this new branch and the reducible two-brane system condenses into an indecomposable bound state at threshold.*

The tachyon, $\Omega_{(P,P)}$, responsible for creating bound states at threshold, can be explicitly obtained by following an observation in ref. [16]. Let $P = (J_P(\alpha), E_P(\alpha))$ be a matrix factorization, where α_ℓ are the usual functions of the brane moduli: $\alpha_\ell = \mu_\ell(\zeta)$. One has

$$E_P(\alpha) \cdot J_P(\alpha) = W \ \mathbb{1} \ . \tag{6.10}$$

Taking the derivative of the above equation with respect to ζ, one obtains

$$E_P(\alpha) \cdot \partial_\zeta J_P(\alpha) + \partial_\zeta E_P(\alpha) \cdot J_P(\alpha) = 0 \ , \tag{6.11}$$

which becomes the physical state condition, (1.16), for the following tachyon between identical branes:

$$\Omega^{(J)}_{(P,P)} \ = \ \partial_\zeta J_P(\alpha), \qquad \Omega^{(E)}_{(P,P)} \ = \ \partial_\zeta E_P(\alpha). \tag{6.12}$$

That this operator is not exact follows from the fact that its boundary preserving version describes the non-trivial marginal operator associated with the open-string modulus, ζ.

For our present purposes, it is simpler to work with derivatives with respect to the α_ℓ, rather than with respect to the flat coordinate, ζ. However, the α_ℓ are not independent since they must lie on the cubic curve. This is easily taken into account by taking a particular linear combinations of derivatives. We find that the

$$\mathcal{D}(\alpha) \equiv \sum_\ell \hat{\alpha}_\ell \frac{\partial}{\partial \alpha_\ell} \ , \tag{6.13}$$

where $\hat{\alpha}_\ell \equiv \mu_\ell(-2\zeta)$, does the job *in lieu* of the ζ-derivative in (6.12).[33] To summarize, we can write the following concise cohomology representative for the "tachyonic modulus":

$$\Omega_{(P,P)}(\alpha) \ = \ \mathcal{D}(\alpha)\mathcal{Q}_P(\alpha) \ . \tag{6.14}$$

6.3. Completing the links in the quiver diagram: Tachyons between S and L

The spectrum of open strings connecting long branes to short branes/anti-branes, falls into four distinct classes (ignoring equivariance labels):

$$\Psi_{(S,L)} \ ; \ \Psi_{(L,S)} \ ; \quad \Psi_{(L,\overline{S})} \text{ and } \Psi_{(\overline{S},L)} \ . \tag{6.15}$$

The tachyons for the last two classes may be obtained from the first two using transposition as outlined in Section 6.1. Thus, we shall focus here on the first two classes alone. The prediction based on the quiver diagram shown in Fig. 3 is that there should be one tachyon of type $\Psi_{(S_a,L_{a+1})}$, one tachyon of type $\Psi_{(S_a,L_a)}$ and two tachyons of type $\Psi_{(L_a,S_{a+1})}$. The degrees of the terms in the relevant tachyon operators are obtained by using the equivariance condition (5.8). We find that the number of solutions to (1.16) is indeed consistent with the quiver diagram.

We now present the explicit expressions that we find for the tachyons. We denote the moduli associated with the L and S branes by $\alpha_\ell \equiv \mu_\ell(\zeta)$ and $\beta_\ell \equiv \mu_\ell(\lambda)$,

[33] The $\hat{\alpha}_\ell$ satisfy the following two identities, enabling us to show that $\mathcal{D} \propto \partial/\partial\zeta$: $\sum_\ell \hat{\alpha}_\ell \alpha_\ell^2 = 0$ and $\hat{\alpha}_1\alpha_2\alpha_3 + \hat{\alpha}_2\alpha_3\alpha_1 + \hat{\alpha}_3\alpha_1\alpha_2 = 0$.

respectively, and let $\gamma_\ell \equiv \mu_\ell(-\zeta-\lambda)$. We then find the following rectangular matrices for the first kind of tachyons:

$$\Psi_{(S_a,L_{a+1})}:\left\{\begin{array}{l} \psi_0=\begin{pmatrix} \alpha_1\gamma_3 & \alpha_3\gamma_2 \\ \alpha_2\gamma_2 & \alpha_1\gamma_1 \\ \alpha_3\gamma_1 & \alpha_2\gamma_3 \end{pmatrix}\ , \\ \psi_1=\begin{pmatrix} -\beta_3\gamma_3 & \frac{x_1\alpha_2\gamma_1}{\alpha_1\beta_1}+\frac{x_3\alpha_1\gamma_1}{\alpha_3\beta_2}-\frac{x_2\alpha_1\gamma_2}{\alpha_2\beta_2} \\ -\beta_3\gamma_1 & \frac{x_1\alpha_1\gamma_2}{\alpha_3\beta_1}+\frac{x_3\alpha_3\gamma_2}{\alpha_2\beta_2}-\frac{x_2\alpha_3\gamma_3}{\alpha_1\beta_2} \\ -\beta_3\gamma_2 & -\frac{x_2\alpha_2\gamma_1}{\alpha_3\beta_2}+\frac{x_1\alpha_3\gamma_3}{\alpha_2\beta_1}+\frac{x_3\alpha_2\gamma_3}{\alpha_1\beta_2} \end{pmatrix}\ , \end{array}\right. \tag{6.16}$$

where $a=1,2,3$. This tachyon gives rise to the composite as given below:

$$S_a(\lambda)\succ_{\Psi_{(S_a,L_{a+1})}} L_{a+1}(\zeta)\Longrightarrow S_{a+1}(3\zeta+\lambda)\ . \tag{6.17}$$

In arriving at (6.17), we used the constant entries in the tachyon to carry out row and column reductions and discarded three 'trivial' $P_{1\times 1}$ and $\overline{P}_{1\times 1}$ vacuum pieces.

The tachyon $\Psi_{(S_a,L_a)}$ can be shown to be gauge equivalent to x_1 times the tachyon $\Psi_{(S_a,L_{a+1})}$ in (6.16):

$$\Psi_{(S_a,L_a)}\ =\ x_1\ \Psi_{(S_a,L_{a+1})}\ . \tag{6.18}$$

It is easy to see that any function linear in the x_i multiplying $\Psi_{(S_a,L_{a+1})}$ will satisfy the physical state condition. However, it turns out that there is only one non-trivial cohomology element that can be obtained in this way. This is the tachyon that leads to a new 5×5 factorization that will be discussed later.

The last kind of tachyons comes with a multiplicity of two, and we distinguish them by adding a superscript:

$$\Psi^{(1)}_{(L_a,S_{a+1})}:\left\{\begin{array}{l} \psi_0=\begin{pmatrix} \frac{x_2\gamma_3}{\beta_1\beta_3}-\frac{x_1\alpha_3\gamma_1}{\alpha_2\beta_1\beta_3} & \frac{x_2\alpha_2\gamma_1}{\alpha_3\beta_2\beta_3}+\frac{x_3\gamma_2}{\beta_1\beta_2} & -\frac{x_3\gamma_1}{\beta_1\beta_2}+\frac{x_2\alpha_3\gamma_2}{\alpha_2\beta_1\beta_2}+\frac{x_1\alpha_1\gamma_2}{\alpha_3\beta_1\beta_3} \\ 0 & -\gamma_2 & \gamma_1 \end{pmatrix}\ , \\ \psi_1=\begin{pmatrix} \frac{x_1\alpha_3\gamma_1}{\alpha_1\alpha_2\beta_1}+\frac{x_2\gamma_1}{\alpha_3\beta_3}+\frac{x_3\gamma_2}{\alpha_2\beta_1} & \frac{x_1\gamma_3}{\alpha_3\beta_3}-\frac{x_2\gamma_2}{\alpha_1\beta_3} & \frac{x_1\beta_2\gamma_2}{\alpha_3\beta_1\beta_3}-\frac{x_3\gamma_1}{\alpha_1\beta_1} \\ \frac{x_2\left(\alpha_3^2\beta_3\gamma_1-\alpha_2^2\beta_1\gamma_2\right)}{\alpha_1\alpha_2\alpha_3\beta_1\beta_3} & \frac{x_2\gamma_3}{\alpha_3\beta_3}-\frac{x_1\gamma_1}{\alpha_2\beta_3} & \frac{x_1\gamma_2}{\alpha_3\beta_3}-\frac{x_2\gamma_3}{\alpha_2\beta_1} \end{pmatrix}, \end{array}\right. \tag{6.19}$$

and

$$\Psi^{(2)}_{(L_a,S_{a+1})}:\left\{\begin{array}{l} \psi_0=\begin{pmatrix} \frac{x_3\gamma_2}{\beta_1\beta_2}-\frac{x_2\alpha_2\gamma_3}{\alpha_1\beta_1\beta_2}-\frac{x_1\alpha_3\gamma_3}{\alpha_2\beta_1\beta_3} & \frac{x_1\alpha_2\gamma_2}{\alpha_1\beta_1\beta_3}-\frac{x_2\gamma_1}{\beta_1\beta_3} & -\frac{x_2\alpha_1\gamma_2}{\alpha_2\beta_2\beta_3}-\frac{x_3\gamma_3}{\beta_1\beta_2} \\ -\gamma_2 & 0 & \gamma_3 \end{pmatrix}\ , \\ \psi_1=\begin{pmatrix} \frac{x_2\gamma_3}{\alpha_3\beta_3}-\frac{x_1\gamma_1}{\alpha_2\beta_3} & \frac{x_3\gamma_2}{\alpha_3\beta_1}-\frac{x_1\beta_2\gamma_3}{\alpha_2\beta_1\beta_3} & -\frac{x_1\alpha_2\gamma_2}{\alpha_1\alpha_3\beta_1}-\frac{x_2\gamma_2}{\alpha_2\beta_3}-\frac{x_3\gamma_3}{\alpha_1\beta_1} \\ \frac{x_1\gamma_2}{\alpha_1\beta_3}-\frac{x_2\gamma_1}{\alpha_2\beta_3} & \frac{x_2\gamma_2}{\alpha_1\beta_1}-\frac{x_1\gamma_3}{\alpha_2\beta_3} & \frac{x_2\left(\alpha_2\beta_3^2\gamma_3-\alpha_1\beta_1^2\gamma_2\right)}{\alpha_1\alpha_2\beta_1\beta_2\beta_3} \end{pmatrix}. \end{array}\right. \tag{6.20}$$

We can form the same composite with either one of the tachyons, using as before the constant entries in ψ_0 to carry out row and column reductions and thereby discarding one trivial, $P_{1\times 1}$, vacuum piece. The result is the 4×4-matrix factorization, $\overline{P}_{4\times 4}$, as shown in Fig. 4 and discussed in Section 7.

7. All rank two factorizations obtained via tachyon condensation

We will now use the tachyons found in the previous to write down explicitly all rank-two matrix factorizations. Note that these do not describe all rank-two vector bundles, but only those for which the charge vector, (r, c_1), lies in the positive cone. By equivariance, a general rank-two vector bundle for which (r, c_1) is outside of the positive cone, can be mapped to a bundle within the positive cone but with $r > 2$. These correspond to higher rank matrix factorizations.

From Fig. 1 and Fig. 4 one can see that there are six different factorizations to be considered. These are distinguished by their Chern numbers, c_1, and we will take $-2 < c_1 \leq +3$. These factorizations are shown on the right edge of the diagram in Fig. 4.

There is one minor subtlety which arises from exceptional matrix factorizations. Recall from Section 4 that the moduli space of some bundles may have singularities and that their smooth resolution can involve an extension of the matrix factorization and thus, to a corresponding extension of the underlying bundle to one of higher rank. We encountered this with the "exceptional" bundle that appears in the special matrix factorization, $\overline{\widetilde{P}}{}^{0}_{4\times 4}$. Such factorizations are rigid and do not exist at generic moduli. Thus our statement about there being only six different factorizations is meant to be true only for generic moduli.

We have already seen that the vector bundle $\mathcal{E}(2,3)$ is obtained from $P_{3\times 3}$, and thus it remains to construct the factorizations with $c_1 = 0, \pm 1, \pm 2$. Below we will explain how to do this explicitly, and obtain a new 4×4-matrix factorization, denoted by $P_{4\times 4}$ and $\overline{P}_{4\times 4}$, which describes the vector bundles $\mathcal{E}(2,2)$ and $\mathcal{E}(2,-2)$ respectively; a 6×6-matrix factorization, $P_{6\times 6}$, and corresponding to $\mathcal{E}(2,0)$; and a 5×5-matrix factorization, $P_{5\times 5}$, which gives $\mathcal{E}(2,1)$. The bundle $\mathcal{E}(2,-1)$ is associated with the transpose, $P^{\mathrm{T}}_{5\times 5}$.

7.1. *Chern number* ± 2

Since we already know that the 2×2 factorizations correspond to line bundles with $c_1 = \pm 1$, the construction of bound states at threshold will lead us to rank two bundles with $c_1 = \pm 2$ respectively. Using (6.14), we consider the 4×4-matrix factorization given by

$$P_{4\times 4} : \begin{cases} J = \begin{pmatrix} J_{2\times 2}(\alpha) & \psi_0 \\ 0 & J_{2\times 2}(\alpha) \end{pmatrix} , \\ E = \begin{pmatrix} E_{2\times 2}(\alpha) & \psi_1 \\ 0 & E_{2\times 2}(\alpha) \end{pmatrix} . \end{cases} \tag{7.1}$$

The explicit form of the tachyon that we obtain is given below:

$$\begin{aligned} \psi_0 &= \mathcal{D}(\alpha) J_{2\times 2}(\alpha) \\ &= \begin{pmatrix} \left[\frac{2x_1x_2\hat{\alpha}_2}{\alpha_1\alpha_3} - \frac{\hat{\alpha}_3 x_1^2}{\alpha_3^2} - \frac{2x_2^2\hat{\alpha}_1}{\alpha_2\alpha_3} + \frac{x_3^2\hat{\alpha}_2}{\alpha_2^2}\right] & \left[\left(\frac{2x_2\hat{\alpha}_1}{\alpha_2\alpha_3} - \frac{2x_3\hat{\alpha}_3}{\alpha_1\alpha_2}\right)x_1 - \frac{2\hat{\alpha}_2 x_1^2}{\alpha_1\alpha_3} - \frac{x_2^2\hat{\alpha}_3}{\alpha_3^2}\right] \\ x_3\hat{\alpha}_1 - x_2\hat{\alpha}_3 & x_1\hat{\alpha}_3 - x_3\hat{\alpha}_2 \end{pmatrix} , \\ \psi_1 &= \mathcal{D}(\alpha) E_{2\times 2}(\alpha) = \begin{pmatrix} x_1\hat{\alpha}_3 - x_3\hat{\alpha}_2 & \left[\frac{2\hat{\alpha}_2 x_1^2}{\alpha_1\alpha_3} + \left(\frac{2x_3\hat{\alpha}_3}{\alpha_1\alpha_2} - \frac{2x_2\hat{\alpha}_1}{\alpha_2\alpha_3}\right)x_1 + \frac{x_2^2\hat{\alpha}_3}{\alpha_3^2}\right] \\ x_2\alpha\alpha_3 - x_3\hat{\alpha}_1 & \left[-\frac{\hat{\alpha}_3 x_1^2}{\alpha_3^2} + \frac{2x_2\hat{\alpha}_2 x_1}{\alpha_1\alpha_3} - \frac{2x_2^2\hat{\alpha}_1}{\alpha_2\alpha_3} + \frac{x_3^2\hat{\alpha}_2}{\alpha_2^2}\right] \end{pmatrix} , \end{aligned} \tag{7.2}$$

where $\hat{\alpha}_\ell$ denotes $\mu_\ell(-2\zeta)$. The matrix bundles, $\mathcal{E}_J$ and $\mathcal{E}_E$, for the 4×4 factorization are associated with the vector bundles $\mathcal{E}(2,2)$ and $\mathcal{E}(2,-2)$ respectively. Because of the behavior of $P_{2\times 2}$ under transposition, it follows that $P^{\mathrm{T}}_{4\times 4}$ is gauge equivalent to $\overline{P}_{4\times 4}$.

7.2. *Chern number* 0

The 3×3 factorization gives us two matrix bundles, $\mathcal{E}_J$ and $\mathcal{E}_E$, one having rank one and degree zero and the other having rank two and degree 3. Therefore, the composite factorization will consist of one matrix bundle of rank two and degree zero, plus a bundle of rank four and degree 6, of the following form:

$$P_{6\times 6} : \begin{cases} J = \begin{pmatrix} -E_{3\times 3}(\alpha) & \psi_0 \\ 0 & -E_{3\times 3}(\alpha) \end{pmatrix} \\ E = \begin{pmatrix} -J_{3\times 3}(\alpha) & \psi_1 \\ 0 & -J_{3\times 3}(\alpha) \end{pmatrix} . \end{cases} \tag{7.3}$$

The explicit form of the tachyon is then given by (with $\hat{\alpha}_l$ denoting $\mu_l(-2\zeta)$)

$$
\begin{aligned}
\psi_0 = \mathcal{D}(\alpha) E_{3\times 3}(\alpha) &= \begin{pmatrix} -\frac{\hat{\alpha}_1 x_1^2}{\alpha_1^2} - \frac{2x_2x_3\hat{\alpha}_1}{\alpha_2\alpha_3} & -\frac{\hat{\alpha}_3 x_3^2}{\alpha_3^2} - \frac{2x_1x_2\hat{\alpha}_3}{\alpha_1\alpha_2} & -\frac{\hat{\alpha}_2 x_2^2}{\alpha_2^2} - \frac{2x_1x_3\hat{\alpha}_2}{\alpha_1\alpha_3} \\ -\frac{\hat{\alpha}_2 x_3^2}{\alpha_2^2} - \frac{2x_1x_2\hat{\alpha}_2}{\alpha_1\alpha_3} & -\frac{\hat{\alpha}_1 x_2^2}{\alpha_1^2} - \frac{2x_1x_3\hat{\alpha}_1}{\alpha_2\alpha_3} & -\frac{\hat{\alpha}_3 x_1^2}{\alpha_3^2} - \frac{2x_2x_3\hat{\alpha}_3}{\alpha_1\alpha_2} \\ -\frac{\hat{\alpha}_3 x_2^2}{\alpha_3^2} - \frac{2x_1x_3\hat{\alpha}_3}{\alpha_1\alpha_2} & -\frac{\hat{\alpha}_2 x_1^2}{\alpha_2^2} - \frac{2x_2x_3\hat{\alpha}_2}{\alpha_1\alpha_3} & -\frac{\hat{\alpha}_1 x_3^2}{\alpha_1^2} - \frac{2x_1x_2\hat{\alpha}_1}{\alpha_2\alpha_3} \end{pmatrix}, \\
\psi_1 = \mathcal{D}(\alpha) J_{3\times 3}(\alpha) &= \begin{pmatrix} x_1\hat{\alpha}_1 & x_3\hat{\alpha}_2 & x_2\hat{\alpha}_3 \\ x_3\hat{\alpha}_3 & x_2\hat{\alpha}_1 & x_1\hat{\alpha}_2 \\ x_2\hat{\alpha}_2 & x_1\hat{\alpha}_3 & x_3\hat{\alpha}_1 \end{pmatrix}.
\end{aligned}
\tag{7.4}
$$

Recalling the results for $P_{3\times 3}$ from Table 1, we see that the matrix bundles, $\mathcal{E}_J$ and $\mathcal{E}_E$, are associated with the vector bundles $\mathcal{E}(2,0)$ and $\mathcal{E}(4,6)$ respectively. Note that $P^{\mathrm{T}}_{6\times 6}(\zeta)$ is gauge equivalent to $P_{6\times 6}(-\zeta)$, which follows from the known behavior (6.3) of $P_{3\times 3}$ under transposition.

7.3. *Chern number* ± 1

These two composites arise as 5×5-matrix factorizations that are transposes of each other. The composite obtained from the tachyon $\Psi_{(\overline{S}_a, \overline{L}_a)}$ is given by

$$
P_{5\times 5}: \left\{ \begin{aligned} J &= \begin{pmatrix} -E_{3\times 3}(\alpha) & \psi_0(\alpha,\beta) \\ 0 & -E_{2\times 2}(\beta) \end{pmatrix}, \\ E &= \begin{pmatrix} -J_{3\times 3}(\alpha) & \psi_1(\alpha,\beta) \\ 0 & -J_{2\times 2}(\beta) \end{pmatrix}, \end{aligned} \right.
\tag{7.5}
$$

where the tachyon is explicitly given by:

$$
\Psi_{(\overline{S}_a, \overline{L}_a)}: \left\{ \begin{aligned} \psi_0 &= \begin{pmatrix} -\beta_3\gamma_3 x_1 & \frac{\alpha_2\gamma_1 x_1^2}{\alpha_1\beta_1} + \frac{\alpha_1\gamma_1 x_1 x_3}{\alpha_3\beta_2} - \frac{\alpha_1\gamma_2 x_1 x_2}{\alpha_2\beta_2} \\ -\beta_3\gamma_1 x_1 & \frac{\alpha_1\gamma_2 x_1^2}{\alpha_3\beta_1} + \frac{\alpha_3\gamma_2 x_1 x_3}{\alpha_2\beta_2} - \frac{\alpha_3\gamma_3 x_1 x_2}{\alpha_1\beta_2} \\ -\beta_3\gamma_2 x_1 & -\frac{\alpha_2\gamma_1 x_1 x_2}{\alpha_3\beta_2} + \frac{\alpha_3\gamma_3 x_1^2}{\alpha_2\beta_1} + \frac{\alpha_2\gamma_3 x_1 x_3}{\alpha_1\beta_2} \end{pmatrix}, \\ \psi_1 &= \begin{pmatrix} \alpha_1\gamma_3 x_1 & \alpha_3\gamma_2 x_1 \\ \alpha_2\gamma_2 x_1 & \alpha_1\gamma_1 x_1 \\ \alpha_3\gamma_1 x_1 & \alpha_2\gamma_3 x_1 \end{pmatrix}, \end{aligned} \right.
\tag{7.6}
$$

where γ_ℓ denotes $\mu_\ell(-\zeta - \lambda)$. For $P_{5\times 5}$, the matrix bundle $\mathcal{E}_J$ is associated with the vector bundle $\mathcal{E}(2,5) \equiv \mathcal{E}(2,-1)$. The factorization as given in (7.5) seemingly depends on two open-string moduli, ζ and λ, of the 3×3 and 2×2 matrix factorizations that form the composite. However, one can carry out a gauge transformation such that the factorization depends on only the combination $(3\zeta + \lambda)$. This combination is to be identified with the physical open-string modulus of the composite. The form of the factorization after such a gauge transformation has been given in Appendix C.

As we already mentioned in Section 6, the remaining 5×5 matrix factorization, corresponding to $\mathcal{E}(2,1)$, can be obtained by transposition and we will therefore denote it by $P^{\mathrm{T}}_{5\times5}$. This then completes the list of all matrix factorizations related to rank two bundles.

8. Summary and conclusions

Our purpose has been to show how matrix factorizations, physically realized in terms of boundary Landau-Ginzburg models, can be used as an effective computational tool for dealing with topological B-type branes. Once again we stress that to precisely understand these B-branes, one must understand the derived category and not merely the K-theory of the branes. That is, it is not sufficient to classify the branes in terms of their charges, but one must also understand the equivariant maps between the branes (as well as the dependence of all these quantities on the complex structure moduli). In more physical terms, this means classifying the tachyons between branes and understanding the condensates they produce. We have seen that matrix factorizations provide a very explicit way of realizing the branes and of performing such computations. Moreover the description of B-type branes via matrix factorization appears to be complete in that the results of refs. [8,9] show that the derived category of topological B-type D-branes is isomorphic to the category of matrix factorizations [10,11] (at least for a large class of geometries). Thus we have an explicit and relatively simple way of representing the category of topological B-branes, as well as performing computations that capture the dependence on moduli, in a manner that is easily accessible for physicists.

To flesh-out this picture, we have shown how such computations work in practice, and while we have focused on the cubic torus, the ideas involved appear universal. We showed how the bundle data for the branes and anti-branes can be extracted directly from the matrices. These bundle data are ambiguous in a way that is familiar from the bulk linear sigma model: Generically, there is a monodromy around the "large-radius" limit in the Kähler moduli space that makes the bundle data ambiguous up to an overall tensoring with the line bundle $\mathcal{L}^N$. We have seen that this ambiguity is mirrored in the B-model by a corresponding ambiguity in extracting the bundle data from the matrix factors. Moreover, the $\mathbb{Z}_N$ monodromy at the Gepner point in the Kähler moduli space is matched by the $\mathbb{Z}_N$ orbit of brane charges that one can associate with a given matrix factorization.[34] We have seen that these charges can be resolved by refining the matrix factorization in terms of equivariant R-symmetry. In this way, the Landau-Ginzburg model is able to describe general sheaves and not merely vector bundles. A more complete treatment, for instance based on the forthcoming description of the boundary linear sigma model [9], may be needed in order to precisely track states between Landau-Ginzburg and large radius phases, and determine how the bundle data evolve over the Kähler moduli space.

Having explicitly shown how the matrix factorizations encode the details of the underlying vector bundles, we showed, in some detail, how tachyon condensation works within this framework. In particular, we demonstrated how repeated tachyon condensation of the 2×2 factorization can be used to construct, at least in principle, the matrix factorizations corresponding to general vector bundles on the torus. Indeed, we showed how to obtain all previously known factorizations, including their dependence on moduli, from the 2×2 factorization (prior to this paper, the connection with the 3×3 factorization was only known implicitly, based upon the RR charges). We went on to construct all the matrix factorizations corresponding to rank two bundles on the torus.

One of the crucial ingredients in understanding and organizing the process of tachyon condensation was the application of equivariant R-symmetry [26,32]. The resulting orbits of branes are labeled by representations of $\mathbb{Z}_N$, and when translated to the language of matrix factorization, these representations impose selection rules upon tachyons, and so effectively organize the tachyon spectrum between any given pair of branes. This enabled us to determine the correct tachyon to generate each and every possible condensate of the members of different $\mathbb{Z}_N$ multiplets.

[34] Note that these two monodromies are very different: tensoring with a line bundle merely shifts the Chern number c_1, while the $\mathbb{Z}_N$ monodromy at the Gepner point changes all the Chern numbers, including the rank. Indeed, these two monodromies taken together generate a (sub-)group of the full duality group of the theory, and thus a given matrix factorization will describe a whole orbit under this group.

One of the other interesting features we uncovered in this study of tachyon condensation, and the bundles associated with matrix factorization, is how the bundle structure can jump, apparently discontinuously, on the open-string moduli space of the branes. We saw how such behavior is easily, and smoothly, described by matrix factorizations. For example, the 3×3 factorization is apparently singular at one point in its moduli space, however, we showed (following ref. [31]) that this singularity could be resolved smoothly by passing to 4×4 factorization that, at generic points in the moduli space, is equivalent to the 3×3 factorization. We found that the 4×4 factorization corresponds to a pair of rank two vector bundles, and, at generic points in the moduli space, one of these rank two bundles is split, *i.e.*, it is a trivial sum of line bundles, but at the special point, where the 3×3 factorization is singular, both the rank two vector bundles are indecomposable. We interpreted this configuration physically in terms of a pure anti-$D2$-brane, which is rigid; deforming away from the special point corresponds to adding and pulling apart an extra $D0$-$\overline{D0}$-brane pair, which amounts to a reducible bundle configuration depending on an open-string modulus.

A general class of examples that exhibits similar behavior was encountered in studying bound states at threshold. In this instance one combines two parallel branes and, if their moduli do not match, then there is no tachyon and the overall vector bundle is simply the sum of the component bundle for each brane. However when the moduli coincide then there is a new tachyon, formally given by a boundary preserving operator, and this condenses the branes into a true bound state at threshold that corresponds to a non-split extension of the component vector bundles.

Another, related feature that we found interesting is that the gauge symmetry, inherent to any matrix factorization, sometimes leads to identifications in the open-string moduli space; this can impose a non-trivial global structure on it, like an unexpected shortening of periodicities, or orbifold points that connect to new branches.

We believe that many of the ideas and techniques we have used here will be applicable to more general Calabi-Yau manifolds. Once one finds a set of matrix factorizations and suitable tachyonic, boundary changing cohomology elements, one should be able to build more by condensation.[35] However, the obvious approach of trying to find matrix factorizations pertaining to something like the quintic threefold does not seem to be straightforward. One can easily convince oneself by elementary counting arguments that matrix factorization of a generic quintic may be difficult: One simply totals up the degrees of freedom in the matrix elements, taking into account the gauge invariances, and subtracts the constraints imposed by (1.4). The result is generically negative, and thus one should expect interesting new matrix factorizations of the complete quintic only for special branes, and/or for special points in its complex moduli space.

For example, at the Fermat point one has all the matrix factorizations arising from tensoring the matrices corresponding to Recknagel-Schomerus states of the underlying minimal models; the task would be to try to extend these away from the Fermat point. However, while for the torus the sections relevant for the simplest branes are explicitly known to be given by theta and Appell functions, much less is known about sections on threefolds. One the other hand, deformations of branes on threefolds will often be obstructed, so that there are no open string moduli to begin with; this may facilitate the construction of the corresponding matrix factorizations.

Moreover, it might also be rewarding to focus on a kind of "hybrid" description of B-type branes. The same counting arguments that show that matrix factorizations for the complete quintic are rare, also show that if there are only three variables of any degree, then there will generically be matrix factorizations with possibly several moduli. This suggests that it might be productive to look at B-type branes defined via vector bundles on sub-manifolds. For example, by adding two further equations to the function(s) that define the Calabi-Yau manifold, one can recast the Calabi-Yau, at least locally, as a Riemann surface fibered over a base, which may be amenable to a treatment similar to the one described in this paper. Specifically, if one focuses on elliptically fibered Calabi-Yau manifolds, one may make direct use of many of the results presented here; this is presently under investigation.

[35] Of course, a major new ingredient and complication is the problem of stability [1], but since this is tied to the Kähler parameters it is not clear to what extent this can be addressed within the topological B-model; for some ideas in this direction, see ref. [32]. It is also known that some of the permutation branes cross lines of marginal stability, *i.e.*, they can decay, when going to large radius. It is thus natural to carry out Seiberg dualities even in the topological B-model, where they appear as a change of basis.

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Appendix A. Theta functions and Appell functions

The theta functions and Appell functions are defined by:

$$\vartheta(\xi) \ = \ \vartheta(\xi|\tau) \ \equiv \ \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} z^n \,, \tag{A.1}$$

$$\kappa(\rho,\xi) \ = \ \kappa(\rho,\xi|\tau) \ \equiv \ \sum_{n\in\mathbb{Z}} \frac{q^{\frac{1}{2}n^2} z^n}{q^n - y} \,, \tag{A.2}$$

where $y \neq q^m, m \in \mathbb{Z}$ and

$$q \ \equiv \ e^{2\pi i\,\tau} \,, \quad z \ \equiv \ e^{2\pi i\,\xi} \,, \quad y \ \equiv \ e^{2\pi i\,\rho} \,. \tag{A.3}$$

These functions also satisfy the periodicity relations:

$$\vartheta(\xi+1) \ = \ \vartheta(\xi) \,, \qquad \vartheta(\xi+\tau) \ = \ q^{-\frac{1}{2}} z^{-1} \vartheta(\xi) \,, \tag{A.4}$$

$$\begin{aligned} \kappa(\rho,\xi+1) \ &= \kappa(\rho+1,\xi) \ = \ \kappa(\rho,\xi) \,, \qquad \kappa(\rho,\xi+\tau) \ = \ y\,\kappa(\rho,\xi) \ + \ \vartheta(\xi) \,, \\ \kappa(\rho+\tau,\xi) \ &= q^{-\frac{1}{2}} z\,(y\,\kappa(\rho,\xi) \ + \ \vartheta(\xi)) \,. \end{aligned} \tag{A.5}$$

In addition, the Appell functions satisfy an interchange identity:

$$\vartheta\big(\rho - \tfrac{1}{2}(1+\tau)\big)\,\kappa(\rho,\xi) \ = \ -q^{\frac{1}{2}} z\, y^{-1} \vartheta(\xi)\,\kappa\big(\xi + \tfrac{1}{2}(1+\tau), \rho - \tfrac{1}{2}(1+\tau)\big) \,. \tag{A.6}$$

By taking the limit as $\rho \to 0$ on both sides of this identity one can then show that:

$$\kappa(\rho, -\tfrac{1}{2}(1+\tau)) \ = \ \frac{q^{-\frac{1}{8}}\,\eta^3(\tau)}{\vartheta(\rho - \frac{1}{2}(1+\tau))} \,, \tag{A.7}$$

where

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{A.8}$$

is the Dedekind η-function.

To uniformize the curve (1.1), we need the theta functions with characteristics:

$$\Theta\left[\begin{matrix} c_1 \\ c_2 \end{matrix} \middle| \xi\,,\tau\right] = \sum_{m\in\mathbb{Z}} q^{(m+c_1)^2/2} e^{2\pi i(\xi+c_2)(m+c_1)}\,, \tag{A.9}$$

and then define:

$$\begin{aligned} \mu_\ell(\xi) \equiv \mu_\ell(\xi|\tau) &= \omega^{(\ell-1)}\,\Theta\left[\begin{matrix} \frac{1}{3}(1-\ell)-\frac{1}{2} \\ -\frac{1}{2} \end{matrix} \middle| 3\,\xi, 3\,\tau\right], \\ &= i\,(-1)^{(\ell-1)}\, q^{\frac{3}{2}(\frac{1}{2}+\frac{1}{3}\,(\ell-1))^2}\, z^{-(\frac{3}{2}+(\ell-1))}\,\vartheta(3\,\xi - (\tfrac{3}{2}+(\ell-1))\tau - \tfrac{1}{2}|3\,\tau)\,, \end{aligned} \tag{A.10}$$

where $\omega \equiv e^{2\pi i/3}$ and $\ell = 1, 2, 3$. One can then show that (1.1) is uniformized by taking $x_\ell = \mu_\ell(\xi)$ with the modulus, a, related to τ via:

$$\left(\frac{3\,a\,(a^3+8)}{a^3-1}\right)^3 = j(\tau)\,. \tag{A.11}$$

The associated Appell functions are then defined by:

$$\begin{aligned} \Lambda_\ell(\rho,\xi) &\equiv i\,(-1)^{(\ell-1)}\, q^{\frac{3}{2}(\frac{1}{2}+\frac{1}{3}\,(\ell-1))^2}\, z^{-(\frac{3}{2}+(\ell-1))} \sum_{n\in\mathbb{Z}} \frac{\left(q^3\right)^{\frac{1}{2}\,n^2} \left(-\,q^{-(\frac{3}{2}+(\ell-1))}\,z^3\right)^n}{\left(q^3\right)^n - y^3\,q^{(\frac{3}{2}+(\ell-1))}} \\ &= i\,(-1)^{(\ell-1)}\, q^{\frac{3}{2}(\frac{1}{2}+\frac{1}{3}\,(\ell-1))^2}\, z^{-(\frac{3}{2}+(\ell-1))} \\ &\qquad \kappa\big(3\,\rho + (\tfrac{3}{2}+(\ell-1))\tau\,, 3\,\xi - (\tfrac{3}{2}+(\ell-1))\tau - \tfrac{1}{2}\,|\,3\,\tau\big)\,. \end{aligned} \tag{A.12}$$

We now sketch some of the manipulations involving the μ_l that we used in the main text of the paper. A change in the sign of the argument of μ_l is the permutation (up to an overall sign):

$$\mu_1(-\zeta) = -\mu_1(\zeta)\;, \quad \mu_2(-\zeta) = -\mu_3(\zeta)\;, \quad \mu_3(-\zeta) = -\mu_2(\zeta)\;. \tag{A.13}$$

In the following, let us denote $\mu_\ell(\zeta)$, $\mu_\ell(\lambda)$ and $\mu_\ell(-\zeta-\lambda)$ by α_ℓ, β_ℓ and γ_ℓ, respectively. The addition formulae for the $\mu_\ell(\zeta)$ may be deduced from (2.6). For example, by setting $\xi = -\zeta$, one obtains formulae such as the one given below with the argument of the μ_ℓ doubled:

$$\hat{\alpha}_1 \equiv \mu_1(-2\zeta) = \frac{\alpha_1(\alpha_2^3 - \alpha_3^3)}{i\eta(\tau)^3} \ . \tag{A.14}$$

Similar manipulations also lead to (3.30) and (4.7). The following identities also follow from (2.6):

$$\begin{aligned} &\alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_3\beta_3\gamma_3 = 0 \ , \\ &\alpha_1\beta_2\gamma_3 + \alpha_2\beta_3\gamma_1 + \alpha_3\beta_1\gamma_2 = 0 \ , \\ &\alpha_1\beta_3\gamma_2 + \alpha_2\beta_1\gamma_3 + \alpha_3\beta_2\gamma_1 = 0 \ . \end{aligned} \tag{A.15}$$

These identities are useful in solving the physical state condition and in identifying the open-string modulus of condensates.

Appendix B. Row and column elimination of matrix factorizations

Specifying a matrix factorization, given by J and E satisfying (1.4), corresponds to choosing a representation for the $\mathbb{C}[x_1, x_2, x_3]$-module homomorphism J and E. In general, however, there are different representatives for the same module homomorphisms. In physical terms, the choice of representative means that we can act upon a given matrix factorization with a gauge transformation to obtain another matrix factorization that satisfies the defining condition (1.4) and thus describes the same D-brane configuration. From the point of view of the category, such gauge transformations act trivially on the objects and thus lead to an equivalent description.

For a $n \times n$-matrix factorization, the gauge transformations are given by eq. (1.5) with the transformation matrices U_L and U_R as elements of $GL(n, \mathbb{C}[x_1, x_2, x_3])$. That is to say U_L and U_R are invertible matrices in the polynomial ring $\mathbb{C}[x_1, x_2, x_3]$.[36]

In the following we often make use of the gauge freedom (1.5) in order to rewrite a given D-brane configuration in a convenient gauge. There are three basic gauge transformations which allow us to simplify matrix factorizations step by step. The idea is to simplify in each step one of the two matrices of a matrix factorization, *e.g.*,

[36] A matrix is invertible in $\mathbb{C}[x_1, x_2, x_3]$ if and only if its determinant is a non-vanishing constant.

J. A gauge transformation acting on J via U_L and U_R then also defines according to (1.5) the corresponding gauge transformation acting on E.

The first basic gauge transformations are simply given by multiplying a matrix row r and a matrix column t of J by non-zero constants a and b respectively. For a $n \times n$-matrix factorization this gives rise to the diagonal transformations matrices

$$\begin{aligned} U_L^{\times}(a,b) &= \mathrm{Diag}\,(1,\ldots,1,a,1,\ldots,1)\ ,\ U_R^{\times}(a,b) = \mathrm{Diag}\,(1,\ldots,1,b,1,\ldots,1)\ , \\ U_L^{\times}(a,b)^{-1} &= \mathrm{Diag}\left(1,\ldots,1,\frac{1}{a},1,\ldots,1\right)\ ,\ U_R^{\times}(a,b)^{-1} = \mathrm{Diag}\left(1,\ldots,1,\frac{1}{b},1,\ldots,1\right)\ . \end{aligned} \tag{B.1}$$

The second kind of gauge transformations are given by either adding the row r of J, multiplied by a polynomial, $p_r(x)$, to the row s of J, or analogously by adding the column t of J, multiplied by a polynomial, $p_c(x)$, to the column u of J. The appropriate transformation matrices are respectively given by

$$\begin{aligned} U_L^{-}(r,s,p_r(x)) &= \mathbb{1}_{n\times n} + p_r(x)\,\Gamma_{s,r}\ , & U_R^{-}(r,s,p_r(x)) &= \mathbb{1}_{n\times n}\ , \\ U_L^{-}(r,s,p_r(x))^{-1} &= \mathbb{1}_{n\times n} - p_r(x)\,\Gamma_{s,r}\ , & U_L^{-}(r,s,p_r(x))^{-1} &= \mathbb{1}_{n\times n}\ , \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} U_L^{|}(t,u,p_c(x)) &= \mathbb{1}_{n\times n}\ , & U_R^{|}(t,u,p_c(x)) &= \mathbb{1}_{n\times n} + p_c(x)\,\Gamma_{t,u}\ , \\ U_L^{|}(t,u,p_c(x))^{-1} &= \mathbb{1}_{n\times n}\ , & U_L^{|}(t,u,p_c(x))^{-1} &= \mathbb{1}_{n\times n} - p_c(x)\,\Gamma_{t,u}\ , \end{aligned} \tag{B.3}$$

with

$$(\Gamma_{r,s})_{kl} = \begin{cases} 1 & \text{for } r=k, s=l\ , \\ 0 & \text{else}\ . \end{cases} \tag{B.4}$$

With these simple transformation rules we can already deduce an important property of a given matrix factorizations P: If either the matrix J or the matrix E contains a constant, that is to say a non-vanishing entry of degree 0, the matrix factorization P simplifies as follows: For concreteness let us assume that the matrix J of an $n \times n$-matrix factorization has a non-vanishing constant in the top left corner. First, we act on the matrix factorization with a transformation of the type (B.1) in order to normalize this constant to 1. Then we apply $(n-1)$ times the gauge transformations (B.2) with $r=1$, $s=2,\ldots,n$ and appropriate polynomials $p_r(x)$ such that the first row of J_P becomes $(1,0,\ldots,0)$. In a third step we apply the gauge transformation (B.3) $n-1$ times with $t=1$, $u=2,\ldots,n$ and with suitable polynomials $p_c(x)$ so as to also reduce the first column to $(1,0,\ldots,0)$. After this chain of gauge transformations we obtain a gauge equivalent matrix factorization. As a consequence of (1.4) both the first row and the first column of E have automatically been transformed into $(W,0,\ldots,0)$! Hence the original $n\times n$-matrix factorization is really a $(n-1)\times(n-1)$-matrix factorization with a trivial irrelevant summand, $P_{1\times 1}$. This technique, which reduces the dimension of a matrix factorization by applying gauge transformations, is in the main text referred to "row and column elimination".

Appendix C. An explicit form of the 5×5 matrix factorization

We present a form of the 5×5 matrix factorization with the open-string modulus is given by ν_l. This is gauge equivalent to the form given in (7.5) when $\nu_l \sim \mu_l(3\zeta+\lambda)$.

$$J_{5\times 5} = \begin{pmatrix} \frac{x_1^2}{\nu_3} & \frac{x_2^2}{\nu_2} & \frac{x_3^2-3ax_1x_2}{\nu_1} & \frac{x_1x_2}{\nu_3} & -\frac{x_1x_2\nu_2}{\nu_1\nu_3} \\ 0 & x_3\nu_1 & -x_2\nu_2 & x_1\nu_2 & 0 \\ -x_3\nu_1 & 0 & x_1\nu_3 & 0 & x_2\nu_3 \\ x_2\nu_2 & -x_1\nu_3 & 0 & -\frac{x_2\nu_1^2-x_1\nu_2^2+x_3\nu_3^2}{\nu_1} & \frac{x_2\nu_1^2\nu_2-x_1\left(\nu_2^3+\nu_3^3\right)}{\nu_1^2} \\ 0 & 0 & 0 & x_3\nu_1-x_2\nu_3 & x_1\nu_3-x_3\nu_2 \end{pmatrix} \tag{C.1}$$

$$E_{5\times 5} = \begin{pmatrix} x_1\nu_3 & -\frac{x_1x_2}{\nu_2} & \widehat{Q}_2 & \frac{x_2(x_2\nu_3-x_3\nu_1)}{\nu_2\nu_3} & -\frac{x_2\left(x_2\nu_1^2-x_1\nu_2^2+x_3\nu_3^2\right)}{\nu_1\nu_2\nu_3} \\ x_2\nu_2 & \widehat{Q}_1 & \frac{x_1x_2\nu_2^2}{\nu_1\nu_3^2} & \frac{x_1(x_3\nu_2-x_1\nu_3)}{\nu_3^2} & -\frac{x_1\left(x_1\left(\nu_2^3+\nu_3^3\right)-x_2\nu_1^2\nu_2\right)}{\nu_1^2\nu_3^2} \\ x_3\nu_1 & -\frac{x_2^2}{\nu_2} & \frac{x_1^2}{\nu_3} & 0 & -\frac{x_1x_2}{\nu_3} \\ 0 & \left(\frac{x_1^2}{\nu_2}-\frac{x_1x_3}{\nu_3}\right) & \frac{x_2(x_1\nu_3-x_3\nu_2)}{\nu_3^2} & \frac{x_3\nu_1(x_1\nu_3-x_3\nu_2)}{\nu_2\nu_3^2} & \widehat{Q}_3 \\ 0 & \frac{x_1(x_2\nu_3-x_3\nu_1)}{\nu_2\nu_3} & \frac{x_2(x_2\nu_3-x_3\nu_1)}{\nu_3^2} & \frac{x_3\nu_1(x_2\nu_3-x_3\nu_1)}{\nu_2\nu_3^2} & \widehat{Q}_4 \end{pmatrix} \tag{C.2}$$

where

$$\begin{aligned} \widehat{Q}_1 &= \frac{\nu_2x_1^2-3ax_2\nu_3x_1+x_3^2\nu_3}{\nu_1\nu_3} \ , \\ \widehat{Q}_2 &= -\frac{x_2^2}{\nu_3}+\frac{3ax_1x_2}{\nu_1}-\frac{x_3^2}{\nu_1} \ , \\ \widehat{Q}_3 &= \frac{-\nu_1\nu_2\nu_3x_2^2-x_3\nu_1^2\nu_2x_2+x_1x_3\left(\nu_2^3+\nu_3^3\right)}{\nu_1\nu_2\nu_3^2} \ , \\ \widehat{Q}_4 &= \frac{\nu_2\nu_3x_1^2+x_3\nu_2^2x_1-x_2x_3\nu_1^2-x_3^2\nu_3^2}{\nu_2\nu_3^2} \ . \end{aligned}$$

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