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Sub-leading contributions to the black hole entropy in the brick wall approach

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The brick wall model is a semi-classical approach to understand the microscopic origin of black hole entropy. In this approach, the black hole geometry is assumed to be a fixed classical background on which matter fields propagate, and the entropy of black holes supposedly arises due to the canonical entropy of matter fields outside the black hole event horizon, evaluated at the Hawking temperature. Apart from certain lower dimensional cases, the density of states of the matter fields around black holes cannot be evaluated exactly. As a result, often, in the brick wall model, the density of states and the resulting canonical entropy of the matter fields are evaluated at the leading order (in terms of $\hbar$) in the WKB approximation. The success of the approach is reflected by the fact that the Bekenstein-Hawking area law—viz. that the entropy of black holes is equal to one-quarter the area of their event horizon, say, $\mathcal{A}_{\rm H}$—has been recovered using this model in a variety of black hole spacetimes. In this work, we compute the canonical entropy of a quantum scalar field around static and spherically symmetric black holes through the brick wall approach at the higher orders (in fact, up to the sixth order in $\hbar$) in the WKB approximation. We explicitly show that the brick wall model generally predicts corrections to the Bekenstein-Hawking entropy in all spacetime dimensions. In four dimensions, we find that the corrections to the Bekenstein-Hawking entropy are of the form $\left[\mathcal{A}_{\rm H}^n \log \mathcal{A}_{\rm H}\right]$, while, in six dimensions, the corrections behave as $\left[\mathcal{A}_{\rm H}^m + \mathcal{A}_{\rm H}^n \log \mathcal{A}_{\rm H}\right]$, where $(m, n) < 1$. We compare our results with the corrections to the Bekenstein-Hawking entropy that have been obtained through the other approaches in the literature, and discuss the implications.

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I. INTRODUCTION AND MOTIVATION

The concept of black hole entropy was originally introduced by Bekenstein [1–4] to resolve certain thermodynamical paradoxes that arise in the presence of the black holes and, in particular, to preserve the universal applicability of the second law of thermodynamics. Soon after Bekenstein's proposal—based on their classical, macroscopic behavior—the thermodynamic properties of black holes were formalized as the four laws of black hole mechanics [5]. Specifically, it was argued that, as the area theorem of classical general relativity closely resembles the statement of the second law of thermodynamics, the area of the black hole event horizon ($\mathcal{A}_{\rm H}$) can be interpreted as the physical entropy associated with the black hole. This association, in turn, led to the identification of the surface gravity (κ) of the black hole (which, for a stationary black hole, is a constant all over the horizon) as the temperature of the hole.

The laws of black hole mechanics were placed on a firm footing when, a year or two later, Hawking [6, 7] showed that, in the presence of quantum matter fields, a body that collapses into a black hole emits thermal radiation at the temperature

$$T_{\rm H} = \left(\frac{\hbar\, c}{k_{\rm B}}\right)\left(\frac{\kappa}{2\pi}\right), \tag{1}$$

where $\hbar$, c and $k_{\rm B}$ denote the Planck constant, the speed of light and the Boltzmann constant, respectively. The above Hawking temperature fixes the constant of proportionality between the temperature of the black hole and its surface gravity and, therefore, between the entropy and the area of the hole. One finds that the entropy of black holes are given by the following Bekenstein-Hawking area law:

$$S_{\rm BH} = \left(\frac{k_{\rm B}}{4}\right)\left(\frac{\mathcal{A}_{\rm H}}{\ell_{\rm Pl}^2}\right), \tag{2}$$

where $\ell_{\rm Pl} = \left(G\,\hbar/c^3\right)^{1/2}$ denotes the Planck length with G being the Newton's constant.

Black hole entropy assumes considerable importance due to the fact that it may provide us with an insight to the microscopic structure of the gravitational theory through the microcanonical, Boltzmann relation $S = (k_{\rm B} \ln \Omega)$, where Ω is the total number of quantum states that are accessible to a black hole that is described by a small set of classical parameters. The different

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approaches that have been adopted in the literature to understand the microscopic origin of black hole entropy can be broadly classified into two categories. (i) Count the "microstates" by assuming a fundamental structure like D-branes, spin networks or conformal symmetry [8–11]. (ii) Associate the black hole entropy to the quantum fields propagating in the fixed black hole spacetime. Count the microstates of these quantum fields [12–17].

Although none of the above approaches can be considered to be complete; all of them — within their domains of applicability — by counting certain microscopic states yield the semiclassical result (2) in all spacetime dimensions $d \geq 3$. However, all these approaches seem to lead to different sub-leading corrections to the Bekenstein-Hawking entropy. For instance, (i) the prefactor to the logarithmic corrections obtained using the spin-networks and conformal symmetry [18–22] are different from the one obtained using the statistical fluctuations around thermal equilibrium [23]. (ii) the power-law corrections obtained using the Noether charge approach [15] are different from those via entanglement of the modes between inside and outside the horizon [24]. In other words, even though different degrees of freedom lead to the universal Bekenstein-Hawking entropy — *quite naturally* — they lead to different sub-leading terms. This indicates that the key to the understanding of the statistical mechanical interpretation of Bekenstein-Hawking entropy may lie in the origin of the sub-leading contributions.

Physically, it is natural to expect corrections to (2): Bekenstein-Hawking entropy is a semi-classical result and there are strong indications that this is valid for large black holes [i.e. when horizon radius is much larger than Planck length]. However, it is not clear, whether this relation will continue to hold for the Planck size black holes. Besides, there is no reason to expect that the Bekenstein-Hawking entropy to be the whole answer for a correct theory of quantum gravity.

In this work, we calculate the higher order WKB contributions to the Bekenstein-Hawking entropy from the brick-wall model [12, 25, 26]. We extend the zeroth-order ($\hbar^0$) WKB analysis to higher order and show that (i) The contribution to the entropy from the higher-order WKB modes is of the same order as the leading order WKB modes. In other words, our analysis shows that it may be incomplete to calculate the contribution only from the leading order WKB modes. (ii) The brick-wall entropy ($S_{\rm BW}$) leads to generic corrections to area of the form:

$$S_{\rm BW} = S_{\rm BH} + \mathcal{G}(\mathcal{A}_{\rm H}) + \mathcal{F}(\mathcal{A}_{\rm H}) \log\left(\frac{\mathcal{A}_{\rm H}}{\ell^2_{\rm Pl}}\right) , \qquad (3)$$

where $\mathcal{G}(\mathcal{A}_{\rm H})$, and $\mathcal{F}(\mathcal{A}_{\rm H})$ are polynomial functions of $\mathcal{A}_{\rm H}$. In the case of four-dimensions, we show explicitly that the brick-wall entropy (upto sixth-order) has the form given above with $\mathcal{G}(\mathcal{A}_{\rm H}) = 0$. In the case of six-dimensions, $\mathcal{G}(\mathcal{A}_{\rm H}) \neq 0$. (iii) We show that, only in the case of Schwarzschild, $\mathcal{F}(\mathcal{A}_{\rm H})$ is a constant.

The brick wall approach is a semi-classical approach, wherein the background geometry is assumed to be a fixed classical background in which quantum fields propagate. The entropy of the black hole is identified with the statistical mechanical entropy arising from a thermal bath of quantum fields propagating outside the horizon. The entropy computed in this way turns out to be proportional to the area of the horizon. This approach has been very popular in obtaining the leading order to the black hole entropy in different dimensions (for an incomplete list of references, see Refs. [27–42]).

The problem with the brick-wall model [as is the case of any semiclassical approach] is that due to the infinite growth of density of states close to the horizon, one has to impose ultra-violet cutoff near the horizon and hence, the brick-wall entropy depends on the cut-off scale. [See section (III), for discussion on various aspects of brick-wall model.] Clearly, this is an undesirable feature. However, there are several advantages of the brick-wall model over other approaches: (i) Unlike the Noether charge approach [15], the brick-wall entropy depends only on the kinematical properties of the metric close to the horizon and does not depend on the dynamics. Hence, the brick-wall entropy (and the corrections computed in this work) can directly be mapped to the horizon properties. In the case of Noether charge approach, since, such a mapping is not possible, the power-law corrections does not provide any new information about the statistical mechanical properties of black hole entropy. (ii) Unlike entanglement entropy [24], the brick-wall entropy can be computed analytically for any spherically symmetric spacetimes to all orders. Also, it is not possible to compute the entanglement entropy for spacetime dimensions $d > 4$ — the entropy is divergent. (iii) In the conformal field theory approach [10, 19], the black hole horizon is treated as boundary. However, the vector fields (which generate the symmetries) do not have a well-defined limit at the horizon [44]. If one requires that vector fields generating symmetries be smooth at the horizon, then the central charge should be zero. In other words, the conformal field theory analysis can only be performed close to the horizon like a brick-wall.

The remainder of this paper is organized as follows. In the following section, we shall sketch some essential properties of static, spherically symmetric black holes in arbitrary spacetime dimensions. In Section III, we shall discuss the assumptions and approximations involved in evaluating the brick wall entropy, and outline the procedure for extending the calculation to the higher orders (in terms of $\hbar$) in the WKB approximation. In Section IV, in addition to the zeroth order, we shall evaluate the contributions to the brick wall entropy of four dimensional black holes at the second, the fourth and the sixth orders (in terms of $\hbar$) in the WKB approximation. In Section V, we extend the analysis to case of black holes in six dimensions. In Section VI, we explicitly write down the results for a few specific black hole solutions in four and six dimensions. Finally, in Section VII, after a rapid summary of the results we have obtained, we shall discuss as to how the sub-leading contributions we have evaluated compare

with the results obtained from the other approaches.

Before we proceed further, let us briefly outline the conventions and notations we shall adopt. We shall, in general, consider a $(D+2)$-dimensional[1], spherically symmetric, black hole spacetime. We shall work with the metric signature $(-,+,+,\cdots)$, and use the geometric units wherein $k_{\rm B} = c = G = 1$. We shall denote the derivative of any function with respect to the radial coordinate r of the black hole by an overprime. The quantum field Φ we shall consider will be a minimally coupled scalar field.

II. KEY PROPERTIES OF STATIC, SPHERICALLY SYMMETRIC BLACK HOLES

Consider the following $(D+2)$-dimensional static and spherically symmetric line element

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{g(r)} + r^2\,d\Omega^2_{_D}\,, \tag{4}$$

$$= f(r)\,\left[-dt^2 + dx^2\right] + r^2\,d\Omega^2_{_D}\,, \tag{5}$$

where $f(r)$ and $g(r)$ are arbitrary (but, continuous and differentiable) functions of the radial coordinate r, $d\Omega^2_{_D}$ is the metric on a D-dimensional unit sphere, and

$$x = \int \frac{dr}{\sqrt{f(r)\,g(r)}} \tag{6}$$

denotes the tortoise coordinate. Throughout this work, we shall assume that the line-element (4) contains a singularity (say, at $r = 0$) and *one*, non-degenerate, event horizon (located at, say, $r = r_{\rm H}$)[2]. But, we shall not assume any specific form of $f(r)$ or $g(r)$. In the rest of this section, we shall discuss some generic properties of the spacetime (4) near the horizon at $r = r_{\rm H}$.

In almost all approaches that evaluate the entropy of spherically symmetric black holes, their line-element close to the event horizon is approximated to be that of a Rindler spacetime (see, for instance, Ref. [39]). For the line-element (4), the Rindler behavior near the horizon can be arrived at by first carrying out the following transformation of the radial coordinate:

$$\gamma = \left(\frac{1}{\kappa}\right)\sqrt{f}, \tag{7}$$

where κ is a constant that denotes the surface gravity of the black hole and is defined as (see, for example, Ref. [46])

$$\kappa = \left[\sqrt{\frac{g(r)}{f(r)}}\,\left(\frac{f'(r)}{2}\right)\right]_{r=r_{\rm H}}. \tag{8}$$

In terms of the coordinate γ, the line-element (4) can be expressed as

$$ds^2 = -\kappa^2\,\gamma^2\,dt^2 + 4\,\left(\frac{f}{g}\right)\left(\frac{\kappa}{f'}\right)^2\,d\gamma^2 + r^2\,d\Omega^2_{D}\,. \tag{9}$$

Close to the horizon (i.e. near $r = r_{\rm H}$), this line-element reduces to

$$ds^2 \to -\kappa^2\,\gamma^2\,dt^2 + d\gamma^2 + r^2_{\rm H}\,d\Omega^2_{D} \tag{10}$$

which describes the Rindler spacetime with a horizon that is located at $\gamma = 0$. It should be stressed here that such a behavior is exhibited by all non-degenerate black hole horizons in all dimensions.

The above derivation of the Rindler line-element near the horizon is essentially equivalent to expanding the metric components $f(r)$ and $g(r)$ in (4) about $r_{\rm H}$ up to the linear order in the Taylor series. However, we find that, when evaluating the contributions to the brick wall entropy at the higher orders in the WKB approximation, we need to expand the quantities $f(r)$ and $g(r)$ to higher orders as follows:

$$f(r) = f'(r_{\rm H})\,(r-r_{\rm H}) + \left(\frac{f''(r_{\rm H})}{2}\right)(r-r_{\rm H})^2 + \left(\frac{f'''(r_{\rm H})}{6}\right)(r-r_{\rm H})^3 + \ldots\,, \tag{11a}$$

$$g(r) = g'(r_{\rm H})\,(r-r_{\rm H}) + \left(\frac{g''(r_{\rm H})}{2}\right)(r-r_{\rm H})^2 + \left(\frac{g'''(r_{\rm H})}{6}\right)(r-r_{\rm H})^3 + \ldots\,. \tag{11b}$$

As we shall see, in four dimensions, in addition to the surface gravity of the black hole, the corrections to the Bekenstein-Hawking entropy $S_{\rm BH}$ also depend on the second derivative of the metric evaluated at the horizon. And, in six dimensions, we find that the sub-leading contributions to $S_{\rm BH}$ involve the third derivative of the metric as well.

Another quantity which we shall require in our calculations is the proper or the coordinate invariant distance of the brick wall from the horizon. The proper radial distance to the brick wall, say, h_c, that is located at $r = h$ is given by

$$h_c = \int_{r_{\rm H}}^{r_{\rm H}+h} \frac{dr}{\sqrt{g(r)}}\,. \tag{12}$$

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[1] A comment on this uncommon notation may be in order. We do *not* work with two time coordinates! We find it convenient to assume the spherically symmetric event horizon to be a D-dimensional sphere. The radial and the time coordinates make the dimension of spacetime to be $(D+2)$.

[2] Since the event horizon is a null hypersurface, its location can be determined by the condition $(g^{\mu\nu}\,\partial_\mu N\,\partial_\nu N) = 0$, where N is a scalar quantity that characterizes the hypersurface. For the line-element (4), N proves to be a function of the radial coordinate r. The above equation then leads to $g(r_{\rm H}) = 0$, and the roots of this *algebraic* equation in turn determine the location of the horizon.

On using the expansion (11) for $g(r)$ up to the second order in this integral, we obtain the following relation between h and h_c:

$$h^{1/2} = \sqrt{\frac{2\,g'(r_{\rm H})}{g''(r_{\rm H})}}\,\sinh\left[\sqrt{\frac{g''(r_{\rm H})}{2}}\left(\frac{h_c}{2}\right)\right]. \qquad (13)$$

For small h_c, this relation simplifies to

$$h_c = \sqrt{\frac{4h}{g'(r_{\rm H})}}\,, \qquad (14)$$

and, for convenience, we shall use this expression for the proper distance to the brick wall.

III. EXTENSION OF THE BRICK WALL MODEL TO HIGHER ORDERS IN THE WKB APPROXIMATION

In this section, after a rapid sketch of the assumptions and approximations that are involved in evaluating the black hole entropy using the brick wall model, we go on to outline the procedure for computing the brick wall entropy at the higher orders in the WKB approximation.

A. Basic assumptions

There are two crucial assumptions in the brick wall approach to black hole entropy. The first assumption concerns the modeling of the microscopic origin of the black hole entropy, and the second is regarding the handling of the divergences that arise close to the event horizon.

As we have mentioned before, the brick wall model is a semi-classical approach wherein the black hole is assumed to be described by a fixed classical geometry. It is further assumed that the black hole is in equilibrium with a thermal bath of quantum matter fields at the Hawking temperature. Moreover, it is the canonical entropy (actually, a specific component) of the quantum matter fields that are propagating outside the black hole horizon that is identified to be the entropy of the black hole.

In the process of calculating the canonical entropy of a matter field outside the black hole horizon, we need to evaluate the density of states of the field. However, one finds that, due to the infinite blue shifting of the modes in the vicinity of the event horizon, the density of states actually diverges. This divergence is regulated in the model by introducing a cut-off by hand above the horizon. The cut-off—popularly referred to as the brick-wall—is basically a static, spherical mirror at which the matter fields are assumed to satisfy, say, the Dirichlet boundary conditions. One finds that the leading component of the brick wall entropy diverges as h_c^{-2}, where h_c is the proper distance to the brick wall defined in Eq. (12). (The other component is essentially a volume dependent term that arises even in flat space.) It is this contribution that is identified to be the entropy of the black hole. Moreover, a specific choice for the cut-off h_c has to be made (this depends on the number of fields, the dimension of the spacetime, etc., but is generally of the order of the Planck length $\ell_{\rm Pl}$), in order to reproduce the Bekenstein-Hawking area law (2). As we mentioned, the area law (2) has been recovered in this approach for a variety of black hole spacetimes and matter fields [27–43].

B. Essential approximations

Two approximations turn out to be essential to make the computation of the brick wall entropy tractable. The first approximation is required in evaluating the density of states of matter fields around black holes, and the second involves expanding the metric near the event horizon.

As we pointed out above, in order to evaluate the brick wall entropy, one needs to evaluate the density of states of matter fields around black holes. However, apart from some lower dimensional cases, the density of states cannot be evaluated exactly. As a result, in the brick wall model, the density of states is usually evaluated at the leading order in $\hbar$ in the WKB approximation.

Moreover, barring a few special cases, one finds that, even after the WKB approximation, the brick wall entropy cannot be evaluated exactly. Recall that the dominant contribution to the entropy arises due to the modes close to the horizon. Motivated by this feature, one Taylor expands the metric functions $f(r)$ and $g(r)$ near the horizon in order to obtain a closed form expression for the brick wall entropy.

C. The methodology

Having discussed the assumptions and approximations involved in the brick wall approach, in the remainder of this sub-section, we shall outline the procedure for evaluating the brick wall entropy at the higher orders in the WKB approximation.

The key assumption of the brick wall model, as we have pointed out above, is that the black hole is in equilibrium with a bath of thermal radiation at the Hawking temperature of the hole. The free energy F of a scalar field at the inverse temperature β is given by (see, for example, Ref. [12])

$$\begin{aligned} F &= \left(\frac{1}{\beta}\right)\int_0^\infty dE\,\left(\frac{d\Gamma(E)}{dE}\right)\,\ln\left[1-\exp-(\beta E)\right]\,, \\ &= -\int_0^\infty dE\,\left(\frac{\Gamma(E)}{\exp(\beta E)-1}\right)\,, \end{aligned} \qquad (15)$$

where $\Gamma(E)$ denotes the total number of modes of the field with energy less than E. We have integrated the first of the above equation by parts to arrive at the second

and have assumed that the boundary term vanishes. The canonical entropy associated with the free energy F is given by

$$S_{\rm C}(\beta) = \beta^2 \left(\frac{\partial F}{\partial \beta}\right), \tag{16}$$

and, it is this entropy, evaluated at the Hawking temperature, that will be identified to be the entropy of the black hole.

Consider a massive and minimally coupled scalar field Φ that is propagating in the the line-element (4). Such a field satisfies the differential equation

$$\left(\Box - m^2\right)\, \Phi = 0, \tag{17}$$

where m denotes the mass of the field. The rotational symmetry of the line-element (4) allows us to decompose the normal modes $u_{_{E\ell m_i}}$ of the field Φ as follows (see, for instance, Ref. [47]):

$$u_{_{E\ell m_i}}(x^\mu) = \left(\frac{R(r)}{r^{D/2}\, G^{1/2}(r)}\right) Y_{\ell m_i}(\theta, \phi_i)\; e^{-(iEt/\hbar)}\,, \tag{18}$$

where E, ℓ and m_i (with $i \in [1, (D-1)]$) are the energy, angular momentum and the azimuthal angular momenta associated with the modes, respectively, the quantity $G(r)$ is given by

$$G(r) = \sqrt{f(r)\; g(r)}\,, \tag{19}$$

and $Y_{lm_i}(\theta, \phi_i)$ denote the hyper-spherical harmonics. On substituting the mode (18) in the equation of motion (17) and using the properties of the hyper-spherical harmonics, we find that the function $R(r)$ satisfies the differential equation

$$R''(r) + \left[\frac{V^2(r)}{\hbar^2} - \Delta(r)\right]\, R(r) = 0\,, \tag{20}$$

where the quantities $V^2(r)$ and $\Delta(r)$ are given by

$$V^2(r) \;=\; \left(\frac{1}{G^2(r)}\right)\, \left(E^2 - f(r)\, \left[m^2 + \left(\frac{\ell\,(\ell + D - 1)\,\hbar^2}{r^2}\right)\right]\right), \tag{21a}$$

$$\Delta(r) \;=\; \left(\frac{G''(r)}{2\, G(r)}\right) - \left(\frac{G'^2(r)}{4\, G^2(r)}\right) + \left(\frac{D}{2r}\right)\, \left(\frac{G'(r)}{G(r)}\right) + \left(\frac{D\,(D-2)}{4\, r^2}\right). \tag{21b}$$

The total number of modes $\Gamma(E)$ of the field Φ with energy less than E can be evaluated exactly if the solution to the differential equation (20) can be written down explicitly. However, apart from some simple $(1+1)$-dimensional example[35], it proves to be difficult to obtain an exact analytical solution for the function $R(r)$. As a result, the WKB approximation is almost always resorted to in the literature [27–42], and it is the leading order WKB solution for $R(r)$ that is utilized to evaluate the number of states $\Gamma(E)$, and the resulting free energy F and the entropy of $S_{\rm C}$ of the quantum field. Our goal here is to extend the analysis to the higher orders in the WKB approximation.

Let us begin by expressing the function $R(r)$ in the following WKB form:

$$R(r) = \left(\frac{c_0}{\sqrt{P(r)}}\right)\, \exp\left[\frac{i}{\hbar}\int^r d\tilde{r}\, P(\tilde{r})\right], \tag{22}$$

where c_0 is a constant. On substituting this expression in Eq. (20), we find that the function $P(r)$ satisfies the differential equation

$$\left(\frac{1}{\hbar^2}\right)\, \left[P^2(r) - V^2(r)\right] = \left(\frac{3}{4}\right)\, \left(\frac{P'(r)}{P(r)}\right)^2 - \left(\frac{1}{2}\right)\, \left(\frac{P''(r)}{P(r)}\right) - \Delta(r)\,. \tag{23}$$

Let us now expand the function $P(r)$ in a power series in $\hbar^2$ as follows (see, for instance, Ref. [48]):

$$P(r) = \sum_{n=0}^{\infty} \hbar^{2n}\; P_{2n}(r)\,. \tag{24}$$

On substituting this series in the differential equation (22) and collecting the terms of a given order in $\hbar^2$, we obtain following expressions for $P_{2n}(r)$ upto $n = 3$:

$$P_0(r) = \pm V(r) = \pm \left(\frac{1}{G(r)}\right) \left[E^2 - f(r) \left(m^2 + \left[\frac{\ell\,(\ell + D - 1)\,\hbar^2}{r^2}\right]\right)\right]^{1/2}, \tag{25a}$$

$$P_2(r) = \left(\frac{3}{8P_0(r)}\right) \left(\frac{P_0'(r)}{P_0(r)}\right)^2 - \left(\frac{4\,P_0''(r)}{{P_0}^2(r)}\right) - \left(\frac{\Delta(r)}{2\,P_0(r)}\right), \tag{25b}$$

$$P_4(r) = -\left(\frac{5\,{P_2}^2(r)}{2\,V(r)}\right) - \left(\frac{4\,P_2(r)\,\Delta(r) + P_2''(r)}{4\,V^2(r)}\right) + \left(\frac{3\,{P_2}'(r)\,V'(r) - P_2(r)\,V''(r)}{4\,V^3(r)}\right), \tag{25c}$$

$$P_6(r) = -\left(\frac{5\,P_2(r)\,P_4(r)}{V(r)}\right) - \left(\frac{8\,{P_2}^3(r) + 4\,P_4(r)\,\Delta(r) + P_4''(r)}{4\,V^2(r)}\right) - \left(\frac{\Delta(r){P_2}^2(r)}{2\,V^3(r)}\right) - \left(\frac{2\,P_2''(r)\,P_2(r) + 2\,P_4(r)\,V''(r) - 3\,{P_2'}^2(r) - 6\,P_4'(r)\,V'(r)}{8\,V^3(r)}\right). \tag{25d}$$

Note that the function $P_0(r)$ is related *algebraically* to the quantities $V(r)$ and $\Delta(r)$. It is evident that the higher order functions $P_{2n}(r)$ (with $n > 0$) can be expressed in terms of the functions at the lower orders and their derivatives and, eventually, in terms of the function $P_0(r)$.

On using the series expansion (24) in the standard semiclassical quantization procedure [12] , we can express the total number of states $\Gamma(E)$ of the field with energy less than E as follows:

$$\Gamma(E) = \sum_{n=0}^{\infty} \Gamma_{2n}(E), \tag{26}$$

where we have defined $\Gamma_{2n}(E)$ as

$$\Gamma_{2n}(E) = \left(\frac{\hbar^{2n-1}}{\pi}\right) \int_{r_{\rm H}+h_c}^{L} dr \int_0^{\ell_{max}} d\ell\ (2\ell + D - 1) \times \mathcal{W}(\ell)\, P_{2n}(r)\,, \tag{27}$$

with the quantity $\mathcal{W}(\ell)$ being given by

$$\mathcal{W}(\ell) = \left(\frac{(\ell + D - 2)!}{(D-1)!\ \ell!}\right). \tag{28}$$

It should be mentioned that, in the above expression for $\Gamma_{2n}(E)$, we have approximated the sum over the angular quantum numbers ℓ as an integral with a degeneracy factor $\mathcal{W}(\ell)$. Such an approximation is often made in the literature, and the approximation is considered to be valid since the separation between the states are expected to be small [33]. Moreover, the upper limit $\ell_{_{max}}$ on the integral over ℓ is a function of energy E of the mode and the radial coordinate r, and it has to be chosen such that $P_0(r)$ is real[3]. Furthermore, the lower limit on the integral over radial coordinate, viz. h_c, is the invariant thickness of the 'brick-wall' defined in (12), and the upper limit L is the infra-red cutoff which we shall assume to be much larger than the horizon radius.

A few clarifying remarks are in order at this stage of our discussion. In the semi-classical quantization of, say, a one-dimensional non-relativistic quantum particle, the integral over the coordinate will be carried out over the range wherein P_0 is real [48]). In the case of bounded systems, these limits will prove to be the turning points of the potential, whereas in the case of potential barriers the limits will be between one of the turning points and infinity. In the context of black holes, the effective potential turns out to be a barrier and the integral over the radial coordinate is to carried out between the event horizon of the black hole (which is an infinity in terms of the tortoise coordinates) and the first turning point that is located on the barrier. But, one finds that, most of the contribution to the density of states of the quantum field arises due to the modes close to the event horizon of the black hole, while the upper limit located on the barrier leads to a volume dependent contribution to the entropy. As a result, the contribution to the number of states and the free energy and the entropy of the quantum field due to the upper (infra-red) limit is usually ignored in the literature.

We should emphasize the point that, apart from replacing the sum over ℓ by an integral, we have not made any approximations until now. Hereafter, we shall make two approximations that we had discussed is some detail in the last subsection. Firstly, we shall approximate the line-element (4) near the event horizon of a spherically symmetric black hole to be that of Rindler spacetime,

[3] Actually, the limits have to be chosen such that $P_{2n}(r)$ are real for all n. However, since, for $n > 0$, the functions $P_{2n}(r)$ can be expressed in terms of $P_0(r)$ and the real functions $V(r)$ and $\Delta(r)$, when $P_0(r)$ is real, $P_{2n}(r)$ are real as well. Therefore, the limits on ℓ proves to be the same for all n.

viz. Eq. (10). It should be pointed out that such an approximation is always made in the literature to arrive at closed form expressions for the free energy and the entropy of the quantum field. Secondly, we shall truncate the series (24) at a particular order (we shall work until the sixth order in $\hbar$), and evaluate the density of states and the associated free energy and the entropy of the quantum field around the black hole. It is important to note that, in the literature, it is only the leading term in the series (26) that has *always* been taken into account ignoring the higher orders when evaluating the brick-wall entropy[4].

In the following two sections, we shall evaluate the contributions to the brick wall entropy at the higher order for four and six dimensional black holes, respectively. As we shall see, the contributions to the entropy from the higher orders turn out to be of the same order as the leading order in the WKB approximation. In other words, it may be incomplete to calculate the contribution only from the leading term in the WKB expansion. Moreover, we show that the brick wall entropy leads to generic corrections to Bekenstein-Hawking entropy (2). For instance, in the case of four-dimensions, we find that the brick-wall entropy has the form:

$$S_{\rm BW}^{\rm (4D)} = S_{\rm BH} + \mathcal{F}^{\rm (4D)}(\mathcal{A}_{\rm H})\, \log\left(\frac{\mathcal{A}_H}{\ell_{\rm Pl}^2}\right), \qquad (29)$$

where $\mathcal{F}^{\rm (4D)}(\mathcal{A}_{\rm H})$ depends on the surface gravity and the second derivative of the metric at the horizon.

Before we proceed with the calculations, there is yet another point concerning the WKB approximation at the sub-leading orders that we need to discuss. As we mentioned above, the limits on the integral over ℓ has been chosen such that $P_0(r)$ is real. This condition essentially identifies the turning points of the potential. Notice that, in Eq. (25), all the higher order WKB terms—i.e. $P_{2n}(r)$ for $n > 0$—contain $P_0(r)$ in the denominator. Obviously, these functions will diverge at the turning points, or equivalently, at the upper limits ℓ. Such a divergence is a well-known feature of the WKB approximation at the higher orders[48], and we shall devise a systematic procedure to isolate these divergences. We shall outline this procedure in the next section and relegate some of the details to Appendix (B).

IV. HIGHER ORDER CONTRIBUTIONS IN FOUR DIMENSIONS

In this section, we shall evaluate the brick wall entropy for spherically symmetric, four dimensional black holes by considering the contributions up to the $n = 3$ term in the series expansion (26) for the number of states of the quantum field. For simplicity, we shall consider here the case of $f(r) = g(r)$ in the line-element (4) and restrict ourselves to a massless scalar field (i.e. $m = 0$). In Appendix (C), we shall extend the second order results we obtain in this section for the general case wherein $f(r) \neq g(r)$ and, in Appendix (D), we extend the analysis to a massive field, but restrict ourselves to the case $f(r) = g(r)$.

A. Second order

Let us now evaluate the contribution due to the $n = 1$ term in the series (26). For $f(r) = g(r)$, we find that the expression (25b) for second order 'momentum' $P_2(r)$ can be written as

$$P_2(r) = \left(\frac{P_2^{(0)}(r)}{\mathcal{G}(\mathcal{E}, r)}\right) + \lambda(r)\left(\frac{P_2^{(1)}(r)}{\mathcal{G}^3(\mathcal{E}, r)}\right) + \lambda^2(r)\left(\frac{P_2^{(2)}(r)}{\mathcal{G}^5(\mathcal{E}, r)}\right), \qquad (30)$$

where the functions $P_2^{(0)}(r)$, $P_2^{(1)}(r)$ and $P_2^{(2)}(r)$ are given by

$$\begin{aligned} P_2^{(0)}(r) &= -\left(\frac{g'}{2\,r}\right), \\ P_2^{(1)}(r) &= \left(\frac{g'^2(r)}{8\,g(r)^2}\right) - \left(\frac{3\,g'(r)}{4\,r\,g(r)}\right) + \left(\frac{g''(r)}{8\,g(r)}\right) + \left(\frac{3}{4\,r^2}\right), \\ P_2^{(2)}(r) &= \left(\frac{5}{32}\right)\left(\frac{g'(r)}{g^2(r)}\right)^2 - \left(\frac{5\,g'(r)}{8\,r\,g(r)}\right) + \left(\frac{5}{8\,r^2}\right), \end{aligned} \qquad (31)$$

and, for convenience, we have defined

$$\mathcal{G}(\mathcal{E}, r) = [\mathcal{E} - \lambda(r)]^{1/2} \qquad (32)$$

with $\mathcal{E} = E^2$ and $\lambda(r)$ being given by

$$\lambda(r) = \left[\ell\,(\ell+1)\,\hbar^2\right]\left(\frac{g(r)}{r^2}\right). \qquad (33)$$

We now need to substitute the above expression for $P_2(r)$ in Eq. (27) and evaluate the number of modes Γ_2 with the upper limit $\ell_{\rm max}$ on the integral over ℓ being determined by the condition that the term $\mathcal{G}(\mathcal{E}, r)$ vanishes. Clearly, the integral over ℓ will diverge in such a case. In order to isolate the finite contribution due to these higher order WKB modes, it is necessary that we follow a systematic procedure. The procedure we shall adopt is as follows. We shall first rewrite all the terms containing inverse powers of $\mathcal{G}(\mathcal{E}, r)$ in terms of derivatives of $\mathcal{E}$ as

[4] The higher order WKB procedure we use is different compared to the approach used in the quasi-normal modes [49–51]. In Ref. [49], the Regge-Wheeler potential is expanded around the maxima, and the modes close to the maxima are matched to the one close to the horizon.

follows:

$$\left(\frac{1}{\mathcal{G}(\mathcal{E},r)}\right) = 2\left(\frac{\partial\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}}\right), \tag{34a}$$

$$\left(\frac{1}{\mathcal{G}^3(\mathcal{E},r)}\right) = -4\left(\frac{\partial^2\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^2}\right), \tag{34b}$$

$$\left(\frac{1}{\mathcal{G}^5(\mathcal{E},r)}\right) = \left(\frac{8}{3}\right)\left(\frac{\partial^3\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^3}\right). \tag{34c}$$

Then, before evaluating the ℓ integral, we shall make use of the Leibnitz's rule, viz.

$$\begin{aligned}\frac{\partial}{\partial x}\int\limits_{a(x)}^{b(x)} dt\, f[x,t] \\ = f[x,a(x)]\left(\frac{da(x)}{dx}\right) - f[x,b(x)]\left(\frac{db(x)}{dx}\right) \\ + \int\limits_{a(x)}^{b(x)} dt\left[\frac{\partial f(x,t)}{\partial x}\right].\end{aligned} \tag{35}$$

and interchange the order of differentiation and integration over the energy E and ℓ. When we do so, we find that the divergences occur at the turning point. We have checked the procedure up to the 6^{th}-order WKB modes and, indeed, systematically separates the non-divergent part from the divergent. For completeness, in Appendix (B), we give the details of the above procedure. (The procedure involves calculating the contour integral around the branch cut that joins the turning points. For details, see Sec. (10.7) in Ref. [48].)

Having obtained the non-divergent part of the mode-functions as a function of E, our next step is to evaluate the contribution of these modes to the density of states $\Gamma_2(E)$. Using the general expression (27), we have

$$\Gamma_2(E) = \frac{\hbar}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr \int\limits_{0}^{\ell_{max}} d\ell\,(2\ell+1)\,P_2(r)\,. \tag{36}$$

Substituting for $P_2(r)$ from Eq. (30) and using the relations (34), we get

$$\begin{aligned}\hbar\Gamma_2(E) &= \frac{1}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr\frac{r^2P_0^{(2)}(r)}{2}\int\limits_0^{\mathcal{E}} d\lambda\,\frac{\partial\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}} \\ &- \frac{1}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr\,r^2P_1^{(2)}(r)\int\limits_0^{\mathcal{E}} d\lambda\,\lambda\,\frac{\partial^2\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^2} \\ &+ \frac{1}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr\,\frac{3r^2P_2^{(2)}(r)}{2}\int\limits_0^{\mathcal{E}} d\lambda\,\lambda^2\,\frac{\partial^3\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^3}\,.\end{aligned} \tag{37}$$

Using the Leibniz rule (35) and following the steps discussed in Appendix (B), we get

$$\Gamma_2(E) = \frac{E}{\hbar\pi}\int\limits_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left\{\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right\}\right]. \tag{38}$$

Following points are worth noting regarding the above expression: (i) In the case of leading order WKB modes, the density of states goes as E^3 [see Eq. (A2)]. However, for the second-order WKB modes the density of states scales as E. (ii) As in the leading-order, most of the contributions to the entropy come close to the horizon. (iii) The expression for the density of state (C3) for the general spherically symmetric spacetime is same as for the special case discussed in this section. Hence, the dependence on the entropy with area is identical to the special case discussed in this section.

Substituting the above expression in Eq. (15), and integrating over E, the free-energy is

$$F_2 = -\frac{\pi}{6\hbar\beta^2}\int\limits_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]. \tag{39}$$

Using the relation (16), the entropy is given by

$$S_2 = \frac{\pi}{3\hbar\beta}\int\limits_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]. \tag{40}$$

As mentioned above, maximum contribution to the entropy is from the modes close to the horizon. Hence, using the expansion (11) close to the horizon and the definition of surface gravity (8), we get,

$$S_2 = \frac{1}{9}\frac{r_{\rm H}^2}{h_c^2} - \left[\frac{g''(r_{\rm H})r_{\rm H}^2}{72} + \frac{\kappa}{9}r_{\rm H}\right]\log\left(\frac{r_{\rm H}^2}{h_c^2}\right) \tag{41}$$

where h_c is given by Eq. (14). This is the first result of this paper, regarding which we would like to stress the following points:

1. The dependence of the entropy on area (from the second-order WKB modes) is similar to that from the zeroth order WKB modes (A6). Also the contribution to the entropy from the second order WKB modes contribute more as compared to the leading order WKB modes. This result has two immediate consequences:

   (a) To associate the brick-wall entropy to $S_{\rm BH}$ it is *necessary* to calculate all the higher order WKB mode contribution to the brick-wall entropy.

   (b) The sub-leading corrections (at the zeroth and second order WKB) depend only on the surface gravity and second derivative of the metric functions. They are of the form $\mathcal{F}(\mathcal{A}_{\rm H})\log(\mathcal{A}_{\rm H}/h_c^2)$. To confirm the generic structure for higher-order,

in the next two subsections we evaluate fourth and sixth order contributions to the brick-wall entropy[5].

2. If the surface gravity is inversely proportional to horizon radius and $g''(r_{\rm H})$ is inversely proportional to the square of the horizon radius, then second term in the RHS of (41) is a constant. In this case, the corrections to $S_{\rm BH}$ are purely logarithmic and does not contain any power-law dependence. This uniquely corresponds to Schwarzschild spacetime.

   In the case of Schwarzschild, we have

$$f(r) = g(r) = 1 - \frac{2M}{r} \tag{42}$$

   where M is the mass of the black hole. The horizon is at $r_{\rm H} = 2M$, $\kappa = 1/(4M)$ and $g''(r_{\rm H}) = -1/(2M^2)$. Substituting the above expressions in Eq. (41), we get

$$S_2 = \frac{4}{9}\frac{M^2}{h_c^2} - \frac{1}{36}\log\left(\frac{r_{\rm H}^2}{h_c^2}\right). \tag{43}$$

   This result shows that, at least, in the zeroth and second order, there are no power-law corrections to $S_{\rm BH}$ for the four-dimensional Schwarzschild black hole, while, for all other black holes — since κ and $g''(r)$ has a more non-trivial structure – there are power-law corrections to the Bekenstein-Hawking entropy. This leads to the following conclusion: *The power-law corrections to the entropy occur for any non-vacuum solutions.* In Sec. (VI) we obtain the entropy for some known black hole solutions.

B. Fourth order

Using the expression (25c), we get,

$$\begin{aligned} P_4(r) &= \frac{P_4^{(0)}(r)}{\mathcal{G}^3(\mathcal{E},r)} + \frac{\lambda(r)P_4^{(1)}(r)}{\mathcal{G}^5(\mathcal{E},r)} + \frac{\lambda^2(r)P_4^{(2)}(r)}{\mathcal{G}^7(\mathcal{E},r)} \\ &+ \frac{\lambda^3(r)P_4^{(3)}(r)}{\mathcal{G}^9(\mathcal{E},r)} + \frac{\lambda^4(r)P_4^{(4)}(r)}{\mathcal{G}^{11}(\mathcal{E},r)}, \end{aligned} \tag{44}$$

where, the complete form of $P_4^{(i)}(r)$ [where $i = 0\cdots4$] are given in Appendix (E).

Rewriting the above expressions in terms of the derivatives of energy and following the procedure discussed in Appendix (B), the contribution to the density of states by the fourth-order WKB modes is given by:

$$\begin{aligned} \Gamma_4(E) &= \frac{\hbar}{\pi}\int_{r_{\rm H}+h}^{L} dr \left[-4P_4^{(0)}(r)\frac{\partial^2}{\partial\mathcal{E}^2}\int_0^{\mathcal{E}} d\lambda\, \mathcal{G}(\mathcal{E},r) \right. \\ &+ \frac{8}{3}P_4^{(1)}(r)\frac{\partial^3}{\partial\mathcal{E}^3}\int_0^{\mathcal{E}} d\lambda\,\lambda\,\mathcal{G}(\mathcal{E},r) \\ &- \frac{16}{15}P_4^{(2)}(r)\frac{\partial^4}{\partial\mathcal{E}^4}\int_0^{\mathcal{E}} d\lambda\,\lambda^2\,\mathcal{G}(\mathcal{E},r) \\ &+ \frac{32}{105}P_4^{(3)}(r)\frac{\partial^5}{\partial\mathcal{E}^5}\int_0^{\mathcal{E}} d\lambda\,\lambda^3\,\mathcal{G}(\mathcal{E},r) \\ &\left. - \frac{64}{945}P_4^{(4)}(r)\frac{\partial^6}{\partial\mathcal{E}^6}\int_0^{\mathcal{E}} d\lambda\,\lambda^4\,\mathcal{G}(\mathcal{E},r) \right]. \end{aligned} \tag{45}$$

Integrating over λ, we get

$$\Gamma_4(E) = \frac{c_0^{(4)}}{E}\int_{r_{\rm H}+h}^{L} dr\Sigma^{(4)}(r), \tag{46}$$

where $c_0^{(4)}$ is a constant and $\Sigma^{(4)}(r)$ is given in Eq. (E6). Using the expansion (11) close to the horizon, we get

$$\Gamma_4(E) = \frac{c_0^{(4)}\kappa}{E}\left[\frac{323\,r_{\rm H}\kappa}{2520\,(r-r_{\rm H})^2} + \frac{5\,r_{\rm H}^2\,g''(r) - 20\,\kappa r_{\rm H}}{16(r-r_{\rm H})}\right]. \tag{47}$$

This is the second result of the paper, regarding which we would like to stress the following points: (i) The fourth order contributions to the density of states goes as $1/E$. Using the expression (15), it is easy to see that the fourth order contribution to the free energy is independent of β and, hence, the contribution to the entropy vanishes[6]. (ii) The density of states contribution close to the horizon again depends only on the first and second order derivatives of the metric. (ii) Comparing the fourth order contribution to the density of states with the leading and second order, it is clear that the density of states scales as E^{3-2n} where n is the order of the WKB modes.

C. Sixth order

Using the expression (25d), we get,

$$\begin{aligned} P_6(r) &= \frac{P_6^{(0)}(r)}{\mathcal{G}^5(\mathcal{E},r)} + \frac{\lambda(r)P_6^{(1)}(r)}{\mathcal{G}^7(\mathcal{E},r)} + \frac{\lambda^2(r)P_6^{(2)}(r)}{\mathcal{G}^9(\mathcal{E},r)} \\ &+ \frac{\lambda^3(r)P_6^{(3)}(r)}{\mathcal{G}^{11}(\mathcal{E},r)} + \frac{\lambda^4(r)P_6^{(4)}(r)}{\mathcal{G}^{13}(\mathcal{E},r)} + \frac{\lambda^5(r)P_6^{(5)}(r)}{\mathcal{G}^{15}(\mathcal{E},r)} \\ &+ \frac{\lambda^6(r)P_6^{(6)}(r)}{\mathcal{G}^{17}(\mathcal{E},r)} \end{aligned} \tag{48}$$

[5] It should be noted that, in the case of sixth order WKB modes, the integral over E is divergent near $E \to 0$. However, the near-horizon contribution of the entropy in identical to the one obtained in this subsection. The fourth order WKB modes do not contribute to the brick-wall entropy.

[6] Note that, as mentioned earlier, the free-energy integral has an infra-red ($E \to 0$) divergence.

where, the complete form of $P_6^{(i)}(r)$ [where $i = 0 \cdots 6$] are given in Appendix (F).

Rewriting the above expressions in terms of the derivatives of energy and following the procedure discussed in Appendix (B), the contribution to the density of states by the sixth-order WKB modes is given by:

$$\begin{aligned}\Gamma_6(E) &= \frac{\hbar^3}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr\left[\frac{8}{3}P_6^{(0)}(r)\frac{\partial^3}{\partial\mathcal{E}^3}\int\limits_0^{\mathcal{E}} d\lambda\,\mathcal{G}(\mathcal{E},r)\right.\\ &-\frac{16}{15}P_6^{(1)}(r)\frac{\partial^4}{\partial\mathcal{E}^4}\int\limits_0^{\mathcal{E}} d\lambda\,\lambda\,\mathcal{G}(\mathcal{E},r)\\ &+\frac{32}{105}P_6^{(2)}(r)\frac{\partial^5}{\partial\mathcal{E}^5}\int\limits_0^{\mathcal{E}} d\lambda\,\lambda^2\,\mathcal{G}(\mathcal{E},r)\\ &-\frac{64}{945}P_6^{(3)}(r)\frac{\partial^6}{\partial\mathcal{E}^6}\int\limits_0^{\mathcal{E}} d\lambda\,\lambda^3\,\mathcal{G}(\mathcal{E},r)\\ &+\frac{128}{10395}P_6^{(4)}(r)\frac{\partial^7}{\partial\mathcal{E}^7}\int\limits_0^{\mathcal{E}} d\lambda\,\lambda^4\,\mathcal{G}(\mathcal{E},r)\\ &-\frac{256}{135135}P_6^{(5)}(r)\frac{\partial^8}{\partial\mathcal{E}^8}\int\limits_0^{\mathcal{E}} d\lambda\,\lambda^5\,\mathcal{G}(\mathcal{E},r)\\ &\left.+\frac{512}{2027025}P_6^{(6)}(r)\frac{\partial^9}{\partial\mathcal{E}^9}\int\limits_0^{\mathcal{E}} d\lambda\,\lambda^6\,\mathcal{G}(\mathcal{E},r)\right],\end{aligned} \tag{49}$$

Integrating over λ, we get

$$\Gamma_6(E) = \frac{c_0^{(6)}}{E^3}\int\limits_{r_{\rm H}+h}^{L} dr\Sigma^{(6)}(r) \tag{50}$$

where $c_0^{(6)}$ is a constant $\Sigma^{(6)}(r)$ is given by Eq. (F8). Repeating the steps i. e. using the relation (Eq. (15)) obtaining the free-energy, substituting the free-energy in (16) and expanding the metric close to horizon using Eq. (11), we get,

$$\begin{aligned}\frac{S_6}{\epsilon} &= -\frac{13892\pi^2}{45045}\frac{r_{\rm H}^2}{h_c^2}\\ &+\left[\frac{9\pi^2}{77}g''(r_{\rm H})r_{\rm H}^2+\frac{30\pi^2}{77}\kappa\, r_{\rm H}\right]\log\left(\frac{r_{\rm H}^2}{h_c^2}\right).\end{aligned} \tag{51}$$

This is the third result of the paper, regarding which we would like to stress the following points: (i) The sub-leading corrections (like the zeroth and second order WKB) depend only on the surface gravity and second derivative of the metric functions. This indeed implies that brick-wall entropy does indeed provide generic corrections to the Bekenstein-Hawking entropy at all orders. We have shown this to be the case upto sixth order. It is natural to expect this to be valid for all higher orders. (ii) As mentioned above, the density of states in each order is given by E^{3-n}. (iii) ϵ in the above expression is due to the fact that the E divergences as $E \to 0$. Thus, the above expression for the entropy depends on the infra-red cutoff.

V. HIGHER ORDER CONTRIBUTIONS IN SIX DIMENSIONS

In this section, we obtain the zeroth and second-order WKB mode contributions to the brick-wall entropy in six-dimensional black hole spacetime. The analysis can be extended to any even dimensional spacetime, however, the analysis in odd-dimensional spacetime is more involved[7].

We show that the results of the brick-wall entropy in the zeroth and second order WKB modes have the same structure confirming the results of 4-dimensions and has the following generic form:

$$S_{\rm BW}^{(6{\rm D})} = S_{\rm BH} + \mathcal{G}(\mathcal{A}_{\rm H}) + \mathcal{F}^{(6{\rm D})}(\mathcal{A}_{\rm H})\,\log\left(\frac{\mathcal{A}_{\rm H}}{\ell_{\rm Pl}^2}\right). \tag{52}$$

A. Zeroth order

In the case of $D = 4$, the weight function (28) becomes

$$\mathcal{W}(\ell) = \frac{(\ell+1)(\ell+2)}{6}. \tag{53}$$

Substituting the above expression in (27), the density of states for the zeroth-order WKB modes (for $f(r) = g(r)$) is

$$\Gamma_0^{(6{\rm D})} = \frac{1}{\hbar^3\pi}\int\limits_{r_H+h}^{L} dr\frac{r^2}{g(r)^2}\int\limits_0^{\mathcal{E}} d\lambda\left(\frac{\lambda r^2}{\hbar^2 g(r)}+2\right)\mathcal{G}(\mathcal{E},r) \tag{54}$$

where, $\mathcal{G}(\mathcal{E},r)$ is given by (32) and

$$\lambda = l(l+3)\hbar^2\frac{g(r)}{r^2}. \tag{55}$$

Repeating the procedure discussed in the previous section, the zeroth order brick-wall entropy is given by

$$S_0^{(6{\rm D})} = \frac{32\pi^5}{945\beta^5\hbar^5}\int\limits_{r_{\rm H}+h}^{L} dr\frac{r^4}{g(r)^3}+\frac{8\pi^2}{135\beta^3\hbar^3}\int\limits_{r_{\rm H}+h}^{L} dr\frac{r^2}{g(r)^2}. \tag{56}$$

Expanding the metric near the horizon (11), upto third order, and using the relation (14), the zeroth-order entropy is given by

$$S_0^{(6{\rm D})} = \frac{r_{\rm H}^4}{3780h_c^4}+\mathcal{G}_{(0)}(r_{\rm H})+\mathcal{F}_{(0)}(r_{\rm H})\log\left(\frac{r_H^2}{h_c^2}\right), \tag{57}$$

where

$$\begin{aligned}\mathcal{G}_{(0)}(r_{\rm H}) &= \frac{r_{\rm H}^2}{15120\,h_c^2}\left[-3g''(r_{\rm H})\,r_{\rm H}^2+16\kappa r_{\rm H}+56\right]\\ \mathcal{F}_{(0)}(r_{\rm H}) &= +\frac{r_{\rm H}}{60480}\left[\left(2\kappa g'''(r_{\rm H})-3g''(r_{\rm H})^2\right)r_{\rm H}^3\right.\\ &+\left.24g''(r_{\rm H})\kappa r_{\rm H}^2-224\kappa+\left(56g''(r_{\rm H})-48\kappa^2\right)r_{\rm H}\right].\end{aligned} \tag{58}$$

As in the 4-dimensions, the leading order term in the above expression (57) is proportional to area (Bekenstein-Hawking

[7] This can be traced to the fact that the wave propagation in these spacetimes are non-local. For more discussion, see Refs. [52, 53].

area relation). The sub-leading term has two parts: (a) One that contain a purely power-law corrections $[\mathcal{G}_{_{(0)}}(r_{\rm H})]$ that is absent in the case of 4-dimensions. (b) The logarithmic term contains a prefactor which, in general, is a function of area as like in 4-dimensions. $\mathcal{F}_{_{(0)}}(r_{\rm H})$, as in 4-dimensions, depend upto the second derivative of the metric close to the horizon while $\mathcal{G}_{_{(0)}}(r_{\rm H})$ depend upto the third derivative of the metric close to the horizon.

B. Second order

Substituting Eq. (53) in Eq. (25b), the contribution to the density of states from the second-order WKB modes is given by

$$\begin{aligned}\Gamma_2^{(6D)} &= \frac{1}{\pi\hbar}\int\limits_{r_{\rm H}+h}^{L} dr\left[P_{26}^{(0)}(r)\int\limits_0^{\mathcal{E}} d\lambda\mathcal{W}(\lambda)\frac{\partial\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}}\right.\\ &+ 4P_{26}^{(1)}(r)\int\limits_0^{\mathcal{E}} d\lambda\,\mathcal{W}(\lambda)\,\lambda(r)\,\frac{\partial^2\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^2} \qquad (59)\\ &+ \left.\frac{8}{3}P_{26}^{(2)}(r)\int\limits_0^{\mathcal{E}} d\lambda\,\mathcal{W}(\lambda)\,\lambda(r)^2\,\frac{\partial^3\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^3}\right],\end{aligned}$$

where, the degeneracy factor $\mathcal{W}(\lambda)$ in terms of λ is given by

$$\mathcal{W}(\lambda) = \frac{\lambda}{\hbar^2}\frac{r^2}{g(r)} + 2\,, \qquad (60)$$

and

$$\begin{aligned}P_{26}^{(0)}(r) &= -\frac{g(r)}{r^2} - \frac{g'(r)}{r}\,, \qquad (61)\\ P_{26}^{(1)}(r) &= \frac{3g(r)^2}{4r^4} - \frac{3g(r)g'(r)}{4r^3} + \frac{g'(r)^2}{8r^2} + \frac{g(r)g''(r)}{8r^2}\,,\\ P_{26}^{(2)}(r) &= \frac{5g(r)^3}{8r^6} - \frac{5g(r)^2g'(r)}{8r^5} + \frac{5g(r)g'(r)^2}{32r^4}\,.\end{aligned}$$

Repeating the procedure discussed in Appendix (B), and substituting the resultant in Eq. (15), we get,

$$\begin{aligned}F_2 = \frac{\pi}{\beta^2\hbar}\int\limits_{r_H+h}^{L} dr\left[-\frac{r^2g'(r)^2}{54g(r)^2} + \frac{3g''(r)r^2+14g'(r)r}{108g(r)} + \frac{5}{54}\right] \qquad (62)\\ + \frac{\pi^3}{\beta^4\hbar^3}\int\limits_{r_H+h}^{L} dr\left[\frac{g''(r)r^4+6g'(r)r^3}{135g(r)^2} - \frac{r^4g'(r)^2}{90g(r)^3} - \frac{2r^2}{135g(r)}\right].\end{aligned}$$

Substituting the above expression in Eq. (16) and expanding the metric using (11) the second order WKB mode contribution to the brick-wall entropy is given by

$$S_2^{(6D)} = \frac{r_H^4}{180h_c^4} + \mathcal{G}_{_{(2)}}(r_{\rm H}) + \mathcal{F}_{_{(2)}}(r_{\rm H})\log\left(\frac{r_H^2}{h_c^2}\right), \qquad (63)$$

where

$$\begin{aligned}\mathcal{G}_{_{(2)}}(r_{\rm H}) &= \frac{r_{\rm H}^2}{2160\,h_c^2}\left[80 - r_{\rm H}^2 g''(r_{\rm H})\right]\\ \mathcal{F}_{_{(2)}}(r_{\rm H}) &= -\frac{r_{\rm H}}{8640}\left[\left(g''(r_{\rm H})^2 - 2\kappa g'''(r_{\rm H})\right)r_{\rm H}^3\right. \qquad (64)\\ &\quad \left. -8g''(r_{\rm H})\kappa r_{\rm H}^2 - 800\kappa - (112\kappa^2 + 40g''(r_{\rm H}))r_{\rm H}\right].\end{aligned}$$

This is the fourth result of this paper, regarding which we would like to discuss the following: (i) As in the case of 4-dimensions, the second order WKB modes contribute the same to the brick-wall entropy as the zeroth order modes. This again proves that in-order to associate the brick-wall entropy to the black hole entropy, it is necessary to calculate all the higher order WKB mode contribution. (ii) As in the case of 4-dimensions, the dependence of the entropy on the horizon area is the same in both the orders. (iii) $\mathcal{G}_{_{(2)}}(r_{\rm H})$ (like $\mathcal{G}_{_{(0)}}(r_{\rm H})$) has a generic power-law corrections to $S_{\rm BH}$ and depend only upto the second derivative of the metric near the horizon. $\mathcal{F}_{_{(2)}}(r_{\rm H})$ (like $\mathcal{F}_{_{(0)}}(r_{\rm H})$) has a prefactor which is a function of the area, and — as in 4-dimensions – is a constant only for the Schwarzschild spacetime.

VI. RESULTS FOR SPECIFIC BLACK HOLES

In this section, we shall explicitly write down the brick wall entropy (evaluated upto the second order in the WKB approximation) for a few well-known black hole solutions in four and six spacetime dimensions. We shall restrict ourselves to the cases wherein $f(r) = g(r)$.

A. Four dimensional examples

We find that, in four dimensions, on combining the zeroth order (A6) and the second order (41) terms, the total brick wall entropy can be expressed as

$$S_{\rm BW}^{(4D)} = S_{\rm BH} + \mathcal{F}^{(4D)}(\mathcal{A}_{\rm H})\,\log\left(\frac{\mathcal{A}_{\rm H}}{\ell_{\rm Pl}^2}\right), \qquad (65)$$

where, in order for the leading term to match the Bekenstein-Hawking entropy (2), we have set the brick wall invariant cutoff h_c to be

$$h_c^2 = \left(\frac{11\,\ell_{\rm Pl}^2}{90\pi}\right). \qquad (66)$$

and the quantity $\mathcal{F}^{(4D)}(\mathcal{A}_{\rm H})$ is given by

$$\mathcal{F}^{(4D)}(\mathcal{A}_{\rm H}) = -\left(\frac{1}{60}\right)g''(r_{\rm H})\,r_{\rm H}^2 - \left(\frac{1}{10}\right)\kappa\,r_{\rm H}\,. \qquad (67)$$

1. Schwarzschild black hole

For the Schwarzschild black hole, the metric coefficients are given by Eq. (42) and the event-horizon of the black hole is located at $r_{\rm H} = (2M)$. The surface gravity κ and the second derivative of the metric at the horizon are given by

$$\kappa = \left(\frac{1}{4M}\right), \quad g''(r_{\rm H}) = -\left(\frac{1}{2M^2}\right). \qquad (68)$$

On substituting these expressions in Eq. (65), we obtain that

$$S_{\rm Sch}^{(4D)} = S_{\rm BH} - \left(\frac{1}{60}\right)\log\left(\frac{\mathcal{A}_{\rm H}}{\ell_{\rm Pl}^2}\right). \qquad (69)$$

2. Schwarzschild (anti-)de sitter spacetime

For the Schwarzschild (anti-)de sitter spacetime, the metric function $g(r)$ is given by

$$g(r) = \left(1 - \frac{2M}{\tilde{r}} \pm \frac{\tilde{r}^2}{l^2}\right) = \left(1 - \frac{2}{r} \pm \frac{r^2}{y}\right) \tag{70}$$

where $y = (l/M)^2$, $r = (\tilde{r}/M)$, M is the mass of the black hole, l is related to the positive (negative) cosmological constant and $-(+)$ corresponds to (anti-)de Sitter spacetime. Note that the coordinates y and r are dimensionless. While the Schwarzschild anti-de Sitter spacetime has only one horizon associated with the singularity at the origin, the Schwarzschild de Sitter has two—one event and one cosmological— horizons. Here, we shall focus on the entropy associated with the event-horizon.

Recall that the event horizon is identified by the condition $g(r) = 0$. On substituting the resulting $r_{\rm H}$ corresponding to the above $g(r)$ in Eq. (65), we find that the brick wall entropy upto the second order can be expressed as

$$S^{(4D)}_{\rm Sch-(a)dS} = S_{\rm BH} - \left(\frac{\pi^{1/2}}{15\,\mathcal{A}_{\rm H}^{1/2}} + \frac{\mathcal{A}_{\rm H}}{\pi\, y}\right) \log\left(\frac{M^2\,\mathcal{A}_{\rm H}}{\ell^2_{\rm Pl}}\right), \tag{71}$$

where $\mathcal{A}_{\rm H}$ defined in-terms of the coordinate r is also dimensionless. In contrast to the purely Schwarzschild case wherein the prefactor to the logarithmic correction was a constant, here the factor is a function of the horizon area.

3. Reissner-Nordström black hole

For the Reissner-Nordström black hole, we have

$$g(r) = \left(1 - \frac{2M}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}\right) = \left(\frac{(r - r_-)(r - r_+)}{r^2}\right), \tag{72}$$

where M and Q denotes the mass and the electric charge of the black hole. Also, $r = \tilde{r}/M$ and $r_\pm$ is the outer/inner horizon given by

$$r_\pm = \left(1 \pm \sqrt{1 - \frac{Q^2}{M^2}}\right), \tag{73}$$

where, again, r is a dimensionless variable. It is the outer horizon r_+ that is the event horizon of the black hole.

On substituting the above relations in Eq. (65), we obtain the brick wall entropy upto the second order to be

$$S^{(4D)}_{\rm RN} = S_{\rm BH} - \left(\frac{\pi^{1/2}}{15\,\mathcal{A}_{\rm H}^{1/2}}\right) \log\left(\frac{M^2\,\mathcal{A}_{\rm H}}{\ell^2_{\rm Pl}}\right), \tag{74}$$

where, again, $\mathcal{A}_{\rm H}$ defined in-terms of r is dimensionless. As in the previous example, the prefactor again turns out to be a function of the horizon area $\mathcal{A}_{\rm H}$.

B. Six dimensional examples

On combining the zeroth order (57) and the second order (63) terms, we find that the brick wall entropy for six dimensional black holes can be expressed as

$$S^{(6D)}_{\rm BW} = S_{\rm BH} + \mathcal{G}(\mathcal{A}_{\rm H}) + \mathcal{F}^{(6D)}(\mathcal{A}_{\rm H}) \log\left(\frac{\mathcal{A}_{\rm H}}{\ell^4_{\rm Pl}}\right), \tag{75}$$

where, as in the four dimensional case, we have chosen the invariant cutoff h_c to be such that the leading term matches the Bekenstein-Hawking entropy. The quantities h_c, $\mathcal{G}(\mathcal{A}_{\rm H})$, $\mathcal{F}^{(6D)}(\mathcal{A}_{\rm H})$ are given by

$$h_c^4 = \left(\frac{11\,\ell^4_{\rm Pl}}{1860\pi^2}\right), \tag{76}$$

$$\begin{aligned}\mathcal{F}^{(6D)}(\mathcal{A}_{\rm H}) = \left(\frac{r_{\rm H}}{30240}\right) \Big[&\Big(8\,\kappa\, g'''(r_{\rm H}) - 5\, g''^2(r_{\rm H})\Big)\, r_{\rm H}^3 \\ &+ 40\, g''(r_{\rm H})\,\kappa\, r_{\rm H}^2 + 2688\,\kappa \\ &+ 8\,\left(46\,\kappa^2 + 21\, g''(r_{\rm H})\right)\, r_{\rm H}\Big],\end{aligned} \tag{77}$$

$$\begin{aligned}\mathcal{G}(\mathcal{A}_{\rm H}) = &\left(\frac{31\, r_{\rm H}^4}{165\,\ell^4_{\rm Pl}}\right)^{1/2} \left(\frac{\pi}{252}\right) \\ &\times \left(-5\, g''(r_{\rm H})\, r_{\rm H}^2 + 8\,\kappa\, r_{\rm H} + 308\right).\end{aligned} \tag{78}$$

1. Schwarzschild black hole

In six dimensions, the function $g(r)$ for Schwarzschild black holes is given by

$$g(r) = \left[1 - \left(\frac{r_0}{r}\right)^3\right], \tag{79}$$

where r_0 is related to the black hole mass (M) by the relation

$$M = \left(\frac{2\,\pi\, r_0^3}{3\pi G_6}\right), \tag{80}$$

with G_6 being the six-dimensional Newton's constant (which we shall hereafter set to unity). On using the definition (8) of the surface gravity κ we find that

$$\kappa = \left(\frac{3}{2\, r_{\rm H}}\right), \tag{81}$$

where $r_{\rm H} = r_0$. Substituting the derivatives of the above metric function $g(r)$ in the expression Eq. (75), we obtain the brick wall entropy to be

$$\begin{aligned}S^{(6D)}_{\rm Sch} &= S_{\rm BH} + \frac{19\pi}{63}\sqrt{\frac{155}{33}}\,\frac{r_{\rm H}^2}{\ell^2_{\rm Pl}} - \frac{59}{840}\log\left(\frac{\mathcal{A}_{\rm H}}{\ell^4_{\rm Pl}}\right) \\ &= S_{\rm BH} + \frac{19}{63}\sqrt{\frac{155}{88}}\,\frac{\mathcal{A}_{\rm H}^{1/2}}{\ell^2_{\rm Pl}} - \frac{59}{840}\log\left(\frac{\mathcal{A}_{\rm H}}{\ell^4_{\rm Pl}}\right)\end{aligned} \tag{82}$$

where in deriving the above expression we have used the expression for the area of the 4-sphere i.e. $\mathcal{A}_{\rm H} = (8\pi^2)/3r_{\rm H}^4$.

Unlike four-dimension, there is a pure power-law correction term to the Bekenstein-Hawking entropy.

2. Schwarzschild (Anti-)De sitter black hole

The line-element for Schwarzschild (Anti-)De sitter spacetime is given by (4) with

$$f(r) = g(r) = 1 - \left(\frac{r_0}{\tilde{r}}\right)^3 \pm \frac{\tilde{r}^2}{l^2} = 1 - \frac{1}{r^3} \pm \frac{r^2}{y_6} \tag{83}$$

where $y_6 = (l/r_0)^2$, $r \to (\tilde{r}/r_0)$, l is related to the positive (negative) cosmological constant and $-(+)$ corresponds

to asymptotic (Anti-)de Sitter. Here again, for Schwarzschild-de Sitter, we consider only the event-horizon.

The event horizon is given by the condition $g(r = r_{\rm H}) = 0$. Substituting these in Eq. (75), we get,

$$S^{(6D)}_{(A)dS} = S_{\rm BH} - \frac{\pi}{252}\sqrt{\frac{31}{165}}\frac{r_0^2}{\ell^2_{\rm Pl}}\left[\frac{2r_{\rm H}^4}{y_6} - 308r_{\rm H}^2 - \frac{72}{r_{\rm H}}\right] \quad (84)$$
$$- \left[\frac{19}{1512}\frac{r_{\rm H}^4}{y_6^2} + \frac{r_{\rm H}^2}{10y_6} + \frac{11}{252y_6 r_{\rm H}} + \frac{1}{15r_{\rm H}^3}\right]\log\left[\frac{r_0^4}{\ell^4_{\rm Pl}}\mathcal{A}_{\rm H}\right] .$$

Note that as in four-dimensions, $\mathcal{A}_{\rm H}$ is dimensionless. The above expression gives a series of power-law corrections to $S_{\rm BH}$.

3. Reissner Nordström black hole

For 6-dimensional Reissner Nordström black hole, we have

$$f(r) = g(r) = 1 - \left(\frac{r_0}{\tilde r}\right)^3 + \frac{\theta^2}{\tilde r^6} = 1 - \left(\frac{1}{\tilde r}\right)^3 + \frac{\chi}{r^6} \quad (85)$$

where the charge of the black hole is given by

$$Q = \frac{3}{2\pi}\frac{\theta^2}{G_6}\,;\, r = \frac{\tilde r}{r_0}\,;\, \chi = \frac{\theta^2}{r_0}\,. \quad (86)$$

As in 4-dimensions, this has two horizons – event and Cauchy horizon. The event horizon ($r_{\rm H}$) is the outer horizon while the inner horizon is the Cauchy horizon. Note that χ and r are dimensionless.

Substituting these in Eq. (75), we get,

$$S^{(6D)}_{\rm RN} = S_{\rm BH} + \frac{\pi}{252}\sqrt{\frac{31}{165}}\frac{r_0^2}{\ell^2_{\rm Pl}}\left[308r_{\rm H}^2 + \frac{72}{r_{\rm H}} - \frac{234\chi}{r_{\rm H}^4}\right] \quad (87)$$
$$+ \left[-\frac{23\chi^2}{280r_{\rm H}^{12}} + \frac{\chi}{140r_{\rm H}^9} + \frac{3-28\chi}{840r_{\rm H}^6} + \frac{1}{15r_{\rm H}^3}\right]\log\left[\frac{r_0^4}{\ell^4_{\rm Pl}}\mathcal{A}_{\rm H}\right] ,$$

where, again, $\mathcal{A}_{\rm H}$ is dimensionless. As can be seen, this also generates a series of power-law corrections to the Bekenstein-Hawking entropy.

4. Boulware-Deser black hole

Boulware-Deser black-hole [56] is an exact spherically symmetric solution of the Einstein action modified by the quadratic Gauss-Bonnet combination i. e.

$$I = \int d^6x\sqrt{-g}\left[R + \alpha_{\rm gb}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)\right] \quad (88)$$

where $\alpha_{\rm gb}$ is the Gauss-Bonnet coupling. The line-element is given by (4) where

$$f(r) = g(r) = 1 + \frac{\tilde r^2}{6\tilde\lambda}\left(1 - \sqrt{1 + \frac{12\omega\tilde\lambda}{\tilde r^5}}\right) \quad (89)$$
$$= 1 + \frac{r^2}{6\lambda}\left(1 - \sqrt{1 + \frac{12\lambda}{r^5}}\right)$$

where $\tilde\lambda = 6\alpha_{\rm gb}$, $r = \tilde r\,\omega^{-1/3}$, $\lambda = \tilde\lambda\,\omega^{-2/3}$ and ω is related to the ADM mass ($M_{\rm ADM}$) by the relation

$$M_{\rm ADM} = \frac{\omega}{4\pi}\,\mathcal{A}_{\rm H}\,. \quad (90)$$

Note that the rescaled variables r, λ are dimensionless. The horizon is given by the condition $f(r) = 0$ and occurs at $r = r_{\rm H}$ such that

$$r_{\rm H}^3 + 3\lambda r_{\rm H} - 1 = 0\,. \quad (91)$$

The existence of the horizon requires $\lambda > 0$ and which then gives $rH^3 < 1$. The surface gravity of the event horizon is given by

$$\kappa = \frac{\omega^{-1/3}}{2r_{\rm H}}\left(\frac{1 + 2r_{\rm H}^3}{2 - r_{\rm H}^3}\right) . \quad (92)$$

Substituting these in Eq. (75), we get,

$$\begin{aligned} S^{(6D)}_{\rm GB} &= S_{\rm BH} + \sqrt{\frac{148955}{523908}}\frac{\omega^{2/3}\pi}{\ell^2_{\rm Pl}}\left[r_{\rm H}^2 + \frac{7r_{\rm H}^2}{31(2-r_{\rm H}^3)} - \frac{25r_{\rm H}^2}{31(2-r_{\rm H}^3)^2}\right] \\ &+ \left[\frac{125}{216}\frac{1}{(2-r_{\rm H}^3)^6} - \frac{1175}{756}\frac{1}{(2-r_{\rm H}^3)^5} + \frac{725}{504}\frac{1}{(2-r_{\rm H}^3)^4} - \frac{649}{7560}\right. \\ &+ \left.\frac{355}{756}\frac{1}{(2-r_{\rm H}^3)^2} + \frac{17}{189}\frac{1}{2-r_{\rm H}^3} - \frac{655}{756}\frac{1}{(2-r_{\rm H}^3)^3}\right]\log\left[\frac{\omega^{4/3}}{\ell^4_{\rm Pl}}\mathcal{A}_{\rm H}\right] \end{aligned} \quad (93)$$

It is instructive to stress the implications of this result: (a) In the limit of $\lambda \to 0$ limit $S^{(6D)}_{\rm GB}$ reduces to $S^{(6D)}_{\rm Sch}$. (ii) The subleading corrections to the Bekenstein-Hawking entropy for the Boulware-Deser black-hole have been obtained earlier by other authors (see, for instance, Ref. [58]). Using the Noether charge, it was shown that

$$S^{\rm NC}_{\rm GB} = S_{\rm BH} + \frac{8\pi^2\,\omega^{-4/3}}{\ell^4_{\rm Pl}}\lambda r_{\rm H}^2 \quad (94)$$

Comparing the two results we see that the subleading corrections in the brick-wall approach, unlike the Noether charge approach, can be completely specified by the horizon properties.

VII. DISCUSSION

A. Summary

As we have pointed out repeatedly, the brick wall model has been a very popular approach that has been utilized to recover the Bekenstein-Hawking entropy $S_{\rm BH}$ in a multitude of situations [27–42]. In all these efforts, it is only the leading term in the WKB expansion (26) that has been taken into account in evaluating the density of states and the associated free energy and entropy of quantum fields around black holes. Also, the metric has almost always been assumed to be of the Rindler form near the event horizon.

In this work, we have extended the brick wall approach to the higher orders in the WKB approximation. Moreover, by expanding the metric functions $f(r)$ and $g(r)$ beyond the leading order near the event horizon, we have been able to evaluate the corrections to the Bekenstein-Hawking entropy for spherically symmetric black holes in four and six dimensions. To begin with, we have illustrated that, even the often considered zeroth order term in the WKB approximation leads to

corrections to the Bekenstein-Hawking entropy, provided the metric functions are expanded beyond the linear order near the horizon. Secondly, we have shown that all the higher order terms in the WKB approximation have the same form as the zeroth order term. Lastly, we find that, the higher order WKB terms actually contribute *more* to the entropy than the lower order terms.

Specifically, we have shown that, upto the second order in the WKB approximation, the brick wall entropy of four dimensional black holes can be expressed as

$$S_{\rm BW}^{(4{\rm D})} = S_{\rm BH} + \mathcal{F}^{(4{\rm D})}(\mathcal{A}_{\rm H})\, \log\left(\frac{\mathcal{A}_{\rm H}}{\ell_{\rm Pl}^2}\right) ,$$

where $\mathcal{F}^{(4{\rm D})}(\mathcal{A}_{\rm H}) \propto \mathcal{A}_{\rm H}^n$ with $n < 1$. Whereas, in six dimensions, we find that the brick wall entropy up to the second order has the form

$$S_{\rm BW}^{(6{\rm D})} = S_{\rm BH} + \mathcal{G}(\mathcal{A}_{\rm H}) + \mathcal{F}^{(6{\rm D})}(\mathcal{A}_{\rm H})\, \log\left(\frac{\mathcal{A}_{\rm H}}{\ell_{\rm Pl}^2}\right) ,$$

where $\mathcal{G}(\mathcal{A}_{\rm H}) \propto \mathcal{A}_{\rm H}^n$ and $\mathcal{F}^{(6{\rm D})}(\mathcal{A}_{\rm H}) \propto \mathcal{A}_{\rm H}^m$ with $(n, m) < 1$. Note that, while the brick wall entropy in four dimensions depends only on the first and the second derivatives of the metric at the horizon, in six dimensions, it depends on the third derivative as well. It is tempting to propose that, at least in even dimensions, the brick wall entropy will depend on as many as derivatives of the metric as half the number of spacetime dimensions! However, the black hole entropy is a coordinate invariant concept. If the brick wall entropy depends on arbitrary derivatives of the metric functions at the horizon, then it is not apriori evident that the resulting entropy will be coordinate invariant. We believe that this is an issue that needs to be addressed satisfactorily.

B. Comparison with results from other approaches

Power law and logarithmic corrections to the Bekenstein-Hawking entropy $S_{\rm BH}$ that we have obtained in the brick wall approach has been encountered earlier in a few other approaches to black hole entropy. For instance, the Noether charge approach predicts a generic power law correction to the Bekenstein-Hawking entropy [15]. However, unlike our approach wherein the brick wall entropy can be completely expressed in terms of the metric and its first few derivatives at the event horizon, the Noether charge entropy can not be mapped to the horizon properties. It is also interesting to note that, in the case of the four dimensional Reissner-Nordström black hole, for large horizon area, i. e. when $M \gg \ell_{\rm Pl}$, the brick wall entropy $S_{\rm RN}^{(4{\rm D})}$ [cf. Eq. (74)] reduces to

$$S_{\rm RN}^{(4{\rm D})} \simeq S_{\rm BH} - \left(\frac{2\pi^{1/2}}{15}\right)\, \left(\frac{1}{\mathcal{A}_{\rm H}^{1/2}} - \frac{\ell_{\rm Pl}^2\, \mathcal{A}_{\rm H}^{3/2}}{M^2}\right) . \qquad (95)$$

Similar power law corrections arise on evaluating the entanglement entropy of such black holes [24]. This behavior seem to suggest a possible relationship between the brick wall model and the approach due to entanglement entropy. Another interesting feature is the absence of power law corrections in case of four dimensional Schwarzschild black hole. It seems to indicate that power law corrections to the Bekenstein-Hawking entropy are related with the presence of matter. The logarithmic corrections that we have obtained as in Eq. (69) for the case of the four dimensional Schwarzschild black hole has also been arrived at in other methods such as the approach through conformal field theory [19], statistical fluctuations around thermal equilibrium [23] and spin foam models [18]. However, it should be pointed out that the prefactor to the logarithmic term that we obtain turns out to be different from the one that arises in the other approaches.

C. A few words on the divergences

The divergence that results in the need of a brick wall cut-off arises even at the leading order in the WKB approximation, and is, obviously, well-known. So, it is not all suprising that such a divergence occurs at the higher orders terms as well. However, in addition to the brick wall divergence, we seem to encounter three more types of divergences at the higher orders. The first is the divergence that occurs at the upper limit when integrating over ℓ in the higher orders and the second is an infra-red divergence that arises at a sufficiently high order when integrating over E (as in the case of the sixth order term in four dimensions). As we mentioned above, the higher order terms contribute more to the brick wall entropy than the lower orders. Therefore, the third and last is a divergence that can arise if we get around to summing over all the terms n.

The first of these additional divergences is associated with the turning points. Such divergences are known to occur at the higher orders in the WKB approximation even in non-relativistic quantum mechanics. Evidently, these divergences are not a feature of the field theory, but a feature of the approximation. The procedure we have adopted to isolate and discard these divergences effectively deals with them.

In contrast, the remaining two divergences that occur at the higher orders are field theoretic divergences. The infra-red divergence is clearly one as it arises when integrating over all the modes. In the context of the leading order results, it has been argued that the brick wall divergence can be absorbed into the renormalization of the Newton's constant [30, 41]. Clearly, if the higher order WKB terms continue to contribute more to the brick wall entropy than the lower order ones, then a divergence will arise when the contributions from all the orders are summed over. We believe that, when working at the higher orders in the WKB approximation, these additional divergences need to be accommodated in a renormalization procedure, along with the brick wall divergence itself.

D. Outlook

Since the odd dimensional cases are analytically more involved, after first working in four dimensions, we had jumped to consider six-dimensional black holes. Needless to add, it will be interesting to extend the current analysis to black holes in odd dimensional spacetimes. The Banados-Teitelboim-Zanelli black hole in three dimensions [54, 55] and the five-dimensional Boulware-Deser black hole [56] are interesting cases that are to be studied. The canonical entropy has been calculated *exactly* around the the Banados-Teitelboim-Zanelli black hole (see, for example, Ref. [57]), and the entropy of the Boulware-Deser black hole is expected to contain a power-law correction ($\propto \mathcal{A}_{\rm H}^{1/3}$) to the Bekenstein-Hawking entropy (see, for instance, Ref. [58]). It will be worthwhile to investigate

as to how the brick wall entropy compares with these results. We hope to consider these cases in a future publication.

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APPENDIX A: RECOVERING $S_{\rm BH}$ FROM BRICK-WALL

In the first part of the appendix, we provide the steps leading to the $S_{\rm BH}$ in the Schwarzschild-like coordinate system i. e. $f(r) = g(r)$ in the line-element (4). In the second part, we obtain the same in the tortoise coordinate (5). As mentioned in Sec. (III), the brick-wall model assumes that the the WKB mode functions are a good approximation for the radial modes near the horizon. In the case of Schwarzschild-like coordinate system, it is not apparent whether such an approximation is valid.

1. Schwarzschild-like coordinate

In the case of massless scalar field, the leading order WKB modes are given by

$$P_0 = \pm\frac{1}{g(r)}\left[E^2 - g(r)\frac{L^2}{r^2}\right]^{1/2} . \tag{A1}$$

Substituting the above expression in Eq. (27), we get

$$\Gamma_0(E) = \frac{2E^3}{3\hbar^3}\int\limits_{r_{\rm H}+h}^{L}\frac{r^2\,dr}{g^2(r)} . \tag{A2}$$

Substituting the above expression in (15) and integrating over E, the free energy F now reads

$$F_0 = -\frac{2\pi^3}{45\hbar^3}\frac{1}{\beta^4}\int\limits_{r_{\rm H}+h}^{L}\frac{r^2}{g^2(r)}dr , \tag{A3}$$

and the entropy is

$$S_0 = \frac{8\pi^3}{45\hbar^3}\frac{1}{\beta^3}\int\limits_{r_{\rm H}+h}^{L}\frac{r^2}{g^2(r)}dr . \tag{A4}$$

On expanding the metric near the horizon up to the first-order, we recover the following standard result [12]

$$S_0^{(\rm Std)} = \frac{r_{\rm H}^2}{90h_c^2} . \tag{A5}$$

However, if we expand the metric to higher orders (11), we get

$$S_0 = \frac{r_{\rm H}^2}{90h_c^2} + \left[\frac{\kappa r_{\rm H}}{90} - \frac{g''(r_{\rm H})r_{\rm H}^2}{360}\right]\log\left(\frac{r_{\rm H}^2}{h_c^2}\right) . \tag{A6}$$

It is interesting to note that the form of the brick-wall entropy in the lowest-order WKB is same as the one obtained in the higher-order WKB. See Sec. (IV) for details.

2. Tortoise coordinates

For the case $f(r) = g(r)$ in (4), the tortoise coordinate simplifies to

$$x = \int\frac{dr}{g(r)} . \tag{A7}$$

As mentioned in Sec. (II), in the Rindler approximation (10) the new radial coordinate x takes the form $x = \log(r - r_{\rm H})/g'(r_{\rm H})$. Thus, the horizon $r = r_{\rm H}$ corresponds to $x \to -\infty$. Hence the tortoise coordinate cover the range $(-\infty, \infty)$. The massless scalar field propagating in the background (5) is nothing but the Regge-Wheeler equation given by

$$\frac{d^2\widehat{R}}{dx^2} + \left[\omega^2 - V_{\rm RW}[r(x)]\right]\widehat{R} = 0 \tag{A8}$$

where

$$V_{\rm RW}[r(x)] = \frac{l(l+1)}{r^2}\;g(r) + \frac{g(r)\,g'(r)}{r} \tag{A9}$$

The Regge-Wheeler potential decays exponentially near the event-horizon and as a power-law near spatial infinity (for asymptotically flat spacetimes), i.e.,

$$V[r(x)] \overset{x\to-\infty}{\simeq} \exp(2\kappa_h x)\;;V[r(x)] \overset{x\to\infty}{\simeq} \frac{1}{x^2} . \tag{A10}$$

Thus, the general solution to Eq. (A8) as $x \to \mp\infty$ can be written as a superposition of plane-waves:

$$\widehat{R}[x] \overset{x\to\pm\infty}{\sim} C_1^{\pm}\exp(i\omega x) + C_2^{\pm}\exp(-i\omega x) , \tag{A11}$$

where $C_1^{\pm}, C_2^{\pm}$ are the constants determined by the choice of the boundary conditions. Thus, in other words, the WKB modes is a good approximation for the radial modes close to the horizon.

Using the procedure discussed in Sec. (III), the leading order density of states is given by

$$\Gamma_0^{(x)}(E) = -\frac{2}{3\pi\hbar^3}\int\limits_{-H}^{L}dx\;\frac{r^2}{g(r)}\left[E^2 - \frac{g(r)g'(r)}{r}\right]^{3/2}, \tag{A12}$$

where H is a large positive number corresponding to the cutoff h. At the linear order in the near horizon approximation we finally obtain,

$$\begin{aligned}\Gamma_0^{(x)}(E) &= -\frac{2}{3\pi\hbar^3}\int\limits_{-H}^{L}dx\frac{E^3 r_{\rm H}^2}{g'(r_{\rm H})}\exp(-2\kappa x) \\ &= -\frac{E^3}{6\pi\hbar^3}\,\frac{r_{\rm H}^2}{\kappa^2}\,\exp(2\kappa H) .\end{aligned} \tag{A13}$$

Following the procedure discussed above, the leading order free energy and the entropy are given by

$$F_0^{(x)} = -\frac{\pi^3}{90\hbar^3\beta^4}\,\frac{r_{\rm H}^2}{\kappa^2}\,\exp\left(2\kappa H\right) \tag{A14}$$

$$S_0^{(x)} = \frac{r_{\rm H}^2}{180}\,\kappa\,\exp\left(2\kappa H\right). \tag{A15}$$

Now, we want to express this entropy in terms of invariant cutoff H_c defined as,

$$H_c = \int_{-\infty}^{-H} dr_*\sqrt{g(r)} = \sqrt{\frac{2}{\kappa}}\exp\left(-H\kappa\right). \tag{A16}$$

Using this invariant cutoff, the final expression of entropy is given by,

$$S_0^{(x)} = \frac{r_{\rm H}^2}{90H_c^2}\,. \tag{A17}$$

The entropy obtained in the tortoise coordinate is same as obtained in the Schwarzschild coordinate.

APPENDIX B: ISOLATING THE FINITE CONTRIBUTION

In this appendix, we shall outline as to how we isolate the finite part of the integrals using the Leibniz rule (35).

The first integral in the RHS of Eq. (38) does not lead to any divergent term. Using the relation (35) — with $a(\mathcal{E}) = 0, b(\mathcal{E}) = \mathcal{E}$ — we get

$$\int_0^{\mathcal{E}} d\lambda\; P_2^{(0)}(r)\;\frac{\partial\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}} = \frac{\partial}{\partial\mathcal{E}}\int_0^{\mathcal{E}} d\lambda\; P_2^{(0)}(r)\;\mathcal{G}(\mathcal{E},r)\,. \tag{B1}$$

The second and third integral in the RHS of Eq. (38) lead to divergent terms. The origin of the divergent terms can be associated to the break-down of WKB approximation at the turning point ($\mathcal{E}$). In order to see this, we evaluate the two integrals explicitly.

Using the relation (35) in the second integral of Eq. (38), we get,

$$\begin{aligned}
&\int_0^{\mathcal{E}} \lambda\frac{\partial^2\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^2}d\lambda \qquad\qquad (B2)\\
&= \frac{\partial}{\partial\mathcal{E}}\int_0^{\mathcal{E}}\lambda\frac{\partial\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}}d\lambda - \mathcal{E}\,\frac{\partial\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}}\Big|_{\text{at }\mathcal{E}=\lambda}\\
&= \frac{\partial^2}{\partial\mathcal{E}^2}\int_0^{\mathcal{E}}\lambda(r)\mathcal{G}(\mathcal{E},r)d\lambda - \frac{\mathcal{E}}{(\mathcal{E}-\lambda)^{1/2}}\Big|_{\text{at }\mathcal{E}=\lambda},
\end{aligned}$$

where we have used Eq. (B1) to obtain the second equation.

Using the relation (35) in the third integral of Eq. (38), we get

$$\begin{aligned}
\int_0^{\mathcal{E}}\lambda^2\frac{\partial^3\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^3}d\lambda &= \frac{\partial^3}{\partial\mathcal{E}^3}\int_0^{\mathcal{E}} d\lambda\;\lambda^2\,\mathcal{G}(\mathcal{E},r) \qquad (B3)\\
&- \left[\frac{\partial}{\partial\mathcal{E}}\left[\frac{\mathcal{E}^2}{2\mathcal{G}(\mathcal{E},r)}\right] - \frac{\mathcal{E}^2}{4\mathcal{G}^3(\mathcal{E},r)}\right]\Big|_{\mathcal{E}=\lambda}.
\end{aligned}$$

From Eqs. (B2, B3), it is clear that both the integrals have a finite and divergent part. The divergence occurs at the turning point $\mathcal{E} = \lambda$. This is not a physical divergences, this is occurring due to the fact that the WKB approximation is not valid close to the turning points. However, it can be shown that by introducing a cutoff close to the turning point the results are independent of the cutoff. (For details, see Sec. (10.7) in Ref. [48].)

APPENDIX C: SECOND ORDER CONTRIBUTION WHEN $\mathbf{f(r) \neq g(r)}$

Earlier, in Subsection IV A, we had computed the brick wall entropy at the second order in the WKB approximation for the specific case wherein $f(r) = g(r)$ in the line-element (4). In this appendix, we shall obtain the corresponding result for the more general case of $f(r) \neq g(r)$. For simplicity, we shall again consider a massless field.

When $f(r) \neq g(r)$, we find that, the 'momentum' at the second order $P_2(r)$ [cf. Eq. (25b)] can be written as

$$P_2^{(G)}(r) = \left(\frac{P_{2G}^{(0)}(r)}{\mathcal{G}(\mathcal{E},r)}\right) + \lambda(r)\,\left(\frac{P_{2G}^{(1)}(r)}{\mathcal{G}^3(\mathcal{E},r)}\right) + \lambda^2(r)\,\left(\frac{P_{2G}^{(2)}(r)}{\mathcal{G}^5(\mathcal{E},r)}\right), \tag{C1}$$

where $\mathcal{G}(\mathcal{E},r)$ is given by Eq. (32). We have defined the functions $P_{2G}^{(0)}(r)$, $P_{2G}^{(1)}(r)$ and $P_{2G}^{(2)}(r)$ to be

$$\begin{aligned}
P_{2G}^{(0)}(r) &= -\left(\frac{G(r)\,f'(r)}{4\,r\,f(r)}\right) - \left(\frac{G(r)\,g'(r)}{4\,r\,g(r)}\right),\\
P_{2G}^{(1)}(r) &= \left(\frac{3\,G(r)\,f(r)}{4r^4}\right) - \left(\frac{5\,G(r)\,f'(r)}{8\,r^3}\right) + \left(\frac{G(r)\,f'(r)^2}{16\,r^2\,f(r)}\right) - \left(\frac{G(r)\,f(r)\,g'(r)}{8r^3g(r)}\right) + \left(\frac{G(r)\,f'(r)\,g'(r)}{16\,r^2\,g(r)}\right) + \left(\frac{G(r)\,f''(r)}{8\,r^2}\right),\\
P_{2G}^{(2)}(r) &= \left(\frac{5\,G(r)\,f(r)^2}{8\,r^6}\right) - \left(\frac{5\,G(r)\,f(r)\,f'(r)}{8\,r^5}\right) + \left(\frac{5\,G(r)\,f'(r)^2}{32\,r^4}\right), \qquad (C2)
\end{aligned}$$

with the quantity $G(r)$ given by Eq. (19). On we find that the number of states at the second order $\Gamma_2^{(G)}(E)$ can be expressed

as follows:

$$\Gamma_2^{(G)}(E) = \left(\frac{1}{\pi\,\hbar}\right) \int\limits_{r_{\rm H}+h}^{\mathcal{L}} dr\ \left(2\,P_{2G}^{(0)}(r)\ \frac{\partial}{\partial\mathcal{E}} \int\limits_0^{\mathcal{E}} d\lambda\ \mathcal{G}(\mathcal{E},r) - 4\,P_{2G}^{(1)}(r)\ \frac{\partial^2}{\partial\mathcal{E}^2} \int\limits_0^{\mathcal{E}} d\lambda\ \lambda\ \mathcal{G}(\mathcal{E},r) + \left(\frac{8}{3}\right)\ P_{2G}^{(2)}(r)\ \frac{\partial^3}{\partial\mathcal{E}^3} \int\limits_0^{\mathcal{E}} d\lambda\ \lambda^2\ \mathcal{G}(\mathcal{E},r)\right), \tag{C3}$$

Repeating the procedure discussed in the last appendix to identify and ignore the divergences in the above expression and substituting this expression in the integral (15) for the free energy, we obtain that

$$\begin{aligned} F_2^{(G)} \ &= \ \left(\frac{\pi}{36\,\beta\,\hbar}\right) \int\limits_{r_{\rm H}+h}^{\mathcal{L}} \frac{dr\,G(r)}{g(r)\,f^3(r)}\ \Big[4\,f^2(r)\,g(r) - 16\,r\,f(r)\,f'(r)\,g(r) + 7\,r^2\,f'^2(r)\,g(r) \\ &\qquad\qquad - 3\,r^2\,f(r)\,f'(r)\,g'(r) - 6\,r^2\,f(r)\,g(r)\,f''(r)\Big]\,. \end{aligned} \tag{C4}$$

Using the relation (16) and expanding the functions $f(r)$ an $g(r)$ about the event horizon as in Eq. (11), we obtain the the expression for the entropy to be

$$S_2^{(G)} = \left(\frac{r_{\rm H}^2}{9\,h_c^2}\right) - \left[\left(\frac{g''(r_{\rm H})}{144}\right) + \left(\frac{f''(r_{\rm H})\,g'(r_{\rm H})}{48\,f'(r_{\rm H})}\right) + \left(\frac{4\,\pi\,r_{\rm H}}{9\,\beta\,\hbar}\right)\,\sqrt{\frac{g'(r_{\rm H})}{f'(r_{\rm H})}}\right] \log\left(\frac{r_{\rm H}^2}{h_c^2}\right). \tag{C5}$$

APPENDIX D: SECOND ORDER CONTRIBUTION FOR A MASSIVE FIELD

In Subsection IV A, we had evaluated the brick wall entropy at the second order for a massless field in four spacetime dimensions. We had considered the specific case wherein $f(r) = g(r)$ in the line-element (4). In this appendix, we shall discuss the corresponding result for a massive field.

For a massive field, the quantity $P_2(r)$ as given by Eq. (25b) can be expressed as follows:

$$P_2^{(m)}(r) = \left(\frac{P_2^{(0)}(r)}{\mathcal{G}_{(m)}(\mathcal{E},r)}\right) + \left(\frac{\lambda(r)\,P_2^{(1)}(r) + P_{2m}^{(0)}(r)}{{\mathcal{G}_m}^3(\mathcal{E},r)}\right) + \left(\frac{\lambda^2(r)\,P_2^{(2)}(r) + \lambda(r)\ \left[P_{2m}^{(1)}(r) + P_{2m}^{(2)}(r)\right]}{{\mathcal{G}_m}^5(\mathcal{E},r)}\right), \tag{D1}$$

where we have defined the functions $P_{2m}^{(0)}(r)$, $P_{2m}^{(1)}(r)$ and $P_{2m}^{(2)}(r)$ to be

$$P_{2m}^{(0)}(r) = \left(\frac{m^2}{8}\right)\Big[g'^2(r) + g(r)\,g''(r)\Big],\ \ P_{2m}^{(1)}(r) = \left(\frac{5\,m^2\,g(r)\,g'(r)}{16\,r^3}\right)\Big[2\,g(r) + r\,g'(r)\Big] \ \text{ and } \ P_{2m}^{(2)}(r) = \left(\frac{5}{32}\right)\Big[m^4\,g(r)\,g'^2(r)\Big], \tag{D2}$$

with the quantity $\mathcal{G}_m(\mathcal{E},r)$ being given by

$$\mathcal{G}_m(\mathcal{E},r) = \big[\mathcal{E} - \lambda(r) - m^2 g(r)\big]^{1/2}. \tag{D3}$$

Carrying out the procedure we had discussed in Appendix B to identify and discard the divergences, we obtain the second order density of states $\Gamma_2^{(m)}(E)$ for the massive field to be

$$\begin{aligned} \Gamma_2^{(m)}(E) \ &= \ \left(\frac{1}{2\,\hbar\,\pi}\right) \int\limits_{r_{\rm H}+h}^{L} dr\,r^2\ \Bigg[P_2^{(0)}(r)\ \frac{\partial}{\partial\mathcal{E}} \int\limits_0^{\mathcal{E}_m} d\lambda\ \mathcal{G}_{(m)}(\mathcal{E},r) - 2\,P_2^{(1)}(r)\ \frac{\partial^2}{\partial\mathcal{E}^2} \int\limits_0^{\mathcal{E}_m} d\lambda\ \lambda\ \mathcal{G}_{(m)}(\mathcal{E},r) \\ &\qquad + 4\,P_{2m}^{(0)}(r)\ \frac{\partial^2}{\partial\mathcal{E}^2} \int\limits_0^{\mathcal{E}_m} d\lambda\ \mathcal{G}_{(m)}(\mathcal{E},r) + \left(\frac{4}{3}\right)\ P_2^{(2)}(r)\ \frac{\partial^3}{\partial\mathcal{E}^3} \int\limits_0^{\mathcal{E}_m} d\lambda\ \lambda^2\ \mathcal{G}_{(m)}(\mathcal{E},r) \\ &\qquad + \left(\frac{8}{3}\right)\ \Big[P_{2m}^{(1)}(r) + P_{2m}^{(2)}(r)\Big]\ \frac{\partial^3}{\partial\mathcal{E}^3} \int\limits_0^{\mathcal{E}_m} d\lambda\ \lambda\ \mathcal{G}_{(m)}(\mathcal{E},r)\Bigg], \end{aligned} \tag{D4}$$

where $\mathcal{E}_m = \big[\mathcal{E} - m^2\,g(r)\big]$. As in the massless case, on expanding the function $g(r)$ near the event horizon as in Eq. (11), we find that $\Gamma_2^{(m)}(E)$ can be expressed as

$$\Gamma_2^{(m)}(E) \ = \ \left(\frac{1}{\pi\,\hbar\pi}\right) \int\limits_{r_{\rm H}+h}^{L} dr\ \left[-\left(\frac{2\,E\,r_{\rm H}}{3\,(r - r_{\rm H})}\right) + \left(\frac{E\,r_{\rm H}^2}{3\,(r - r_{\rm H})^2}\right) - \left(\frac{E\,r_{\rm H}^2\,g''(r_{\rm H})}{12\,(r - r_{\rm H})\,\kappa}\right) + \left(\frac{m^2\,r_{\rm H}^2\,\kappa}{E\,(r - r_{\rm H})}\right)\right]. \tag{D5}$$

Note that the last term containing the mass m of the field is inversely proportional to E. Recall that, in Subsection IV B, we had encountered such a behavior at the fourth order for the massless field [cf. Eq. (47)]. As we had then pointed out, such a dependence on the energy E in the number of states leads to a free energy that turns out to be independent of the inverse temperature β and, hence, the term does not contribute to the entropy. Therefore, the massless and the massive fields lead to the same entropy at the second order in the WKB approximation.

APPENDIX E: EXPLICIT FORMS OF $\mathbf{P_4^{(i)}(r)}$

The functions $P_4^{(i)}(r)$ [where $i = 0, \cdots, 4$] are given by,

$$\begin{aligned} P_4^{(0)}(r) &= -\frac{5}{2}g(r)P_2^{(0)}(r)^2 - \frac{g(r)}{r}\,g'(r)\,P_2^{(0)}(r) - \frac{1}{4}g'(r)^2P_2^{(0)}(r) \\ &\quad -\frac{3}{4}g(r)g'(r)P_2'^{(0)}(r) - \frac{1}{4}g(r)P_2^{(0)}(r)g''(r) - \frac{1}{4}g(r)^2P_2''^{(0)}(r)\,, \end{aligned} \tag{E1}$$

$$\begin{aligned} P_4^{(1)}(r) &= -5g(r)P_2^{(0)}(r)P_2^{(1)}(r) + \frac{5}{4r^3}\left[g(r)^2\,g'(r)\,P_2^{(0)}(r) + g(r)^3P_2'^{(0)}(r)\right] \\ &\quad -\frac{5}{8r^2}\left[g(r)P_2^{(0)}(r)g'(r)^2 + g(r)^2g'(r)P_2'^{(0)}(r)\right] - \frac{1}{r}g(r)g'(r)P_2^{(1)}(r) \\ &\quad -\frac{1}{4}\left[g'(r)^2P_2^{(1)}(r) + 3g(r)g'(r)P_2'^{(1)}(r) + g(r)g''(r)P_2^{(1)}(r) + g(r)^2P_2''^{(1)}(r)\right]\,, \end{aligned} \tag{E2}$$

$$\begin{aligned} P_4^{(2)}(r) &= -5g(r)P_2^{(0)}(r)P_2^{(2)}(r) - \frac{5}{2}g(r)P_2^{(1)}(r)^2 - \frac{3}{16}g(r)g'(r)P_2'^{(2)}(r) \\ &\quad -\frac{1}{4}\left[g(r)g''(r)P_2^{(2)}(r) + g(r)^2P_2''^{(2)}(r) + g'(r)^2P_2^{(2)}(r)\right] - \frac{5}{4r^6}g(r)^4P_2^{(0)}(r) \\ &\quad +\frac{5}{4r^5}g(r)^3g'(r)P_2^{(0)}(r) - \frac{1}{2r^4}\left[3g(r)^3P_2^{(1)}(r) - \frac{5}{8}g(r)^2g'(r)^2P_2^{(0)}(r)\right] \\ &\quad +\frac{1}{4r^3}\left[15g(r)^2g'(r)P_2^{(1)}(r) + 9g(r)^3P_2'^{(1)}(r)\right] - \frac{1}{r}g(r)g'(r)P_2^{(2)}(r) \\ &\quad -\frac{1}{8r^2}\left[11g'(r)^2g(r)P_2^{(1)}(r) + 9g(r)^2g'(r)P_2'^{(1)}(r) + 2g(r)^2g''(r)P_2^{(1)}(r)\right] \end{aligned} \tag{E3}$$

$$\begin{aligned} P_4^{(3)}(r) &= -5g(r)P_2^{(1)}(r)P_2^{(2)}(r) - \frac{23}{4r^6}g(r)^4P_2^{(1)}(r) + \frac{23g(r)^3P_2^{(1)}(r)g'(r)}{4r^5} \\ &\quad -\frac{1}{r^4}\left[3g(r)^3P_2^{(2)}(r) - \frac{23}{16}g(r)^2g'(r)^2P_2^{(1)}(r)\right] \\ &\quad +\frac{1}{4r^3}\left[25g(r)^2P_2^{(2)}(r)g'(r) + 13g(r)^3P_2'^{(2)}(r)\right] \\ &\quad -\frac{1}{8r^2}\left[17g(r)P_2^{(2)}(r)g'(r)^2 + 13g(r)^2g'(r)P_2'^{(2)}(r) + 4g(r)^2P_2^{(2)}(r)g''(r)\right]\,, \end{aligned} \tag{E4}$$

$$P_4^{(4)}(r) = -\frac{49}{4r^6}g(r)^4P_2^{(2)}(r) + \frac{49}{4r^5}g(r)^3g'(r)P_2^{(2)}(r) - \frac{5}{2}g(r)P_2^{(2)}(r)^2 - \frac{49}{16r^4}g(r)^2g'(r)^2P_2^{(2)}(r)\,, \tag{E5}$$

$$\begin{aligned} \Sigma^{(4)}(r) &= \frac{323}{10080}\,\frac{r^2g'(r)^4}{g(r)^2} + \frac{101r^2g'(r)^2g''(r) - 631\,r\,g'(r)^3}{1680\,g(r)} \\ &\quad +\frac{7r^2}{840}\left[g''(r)^2 + 7g'(r)g^{(3)}(r) + 5g^{(4)}(r)g(r)\right] + \frac{r}{840}\left[155g'(r)g''(r) + 252g^{(3)}(r)g(r)\right] \\ &\quad +\frac{467g'(r)^2 + 150g''(r)g(r)}{420} + \frac{17}{630}\,\frac{g(r)^2}{r^2} - \frac{1223}{2520}\frac{g'(r)g(r)}{r} \end{aligned} \tag{E6}$$

APPENDIX F: EXPLICIT FORMS OF $\mathbf{P_6^{(i)}(r)}$

The functions $P_6^{(i)}(r)$ [where $i = 0, \cdots, 6$] are given by,

$$\begin{aligned}
P_6^{(0)}(r) &= -2g(r)^2 P_2^{(0)}(r)^3 - 5g(r)P_2^{(0)}(r)P_4^{(0)}(r) - \frac{g(r)g'(r)}{2r}\left[g(r)P_2^{(0)}(r)^2 + P_4^{(0)}(r)\right] \\
&\quad -\frac{g(r)}{8}\left[2g(r)^2 P_2^{(0)}(r)P_2''^{(0)}(r) - g'(r)^2 P_2^{(0)}(r)^2 - 3g(r)^2 P_2'^{(0)}(r)^2 + 6g'(r)P_4'^{(0)}(r)\right. \\
&\quad \left. +2g(r)g''(r)P_2^{(0)}(r)^2 + 2g(r)P_4''^{(0)}(r) + 2g''(r)P_4^{(0)}(r)\right] - \frac{1}{4}P_4^{(0)}(r)g'(r)^2 \qquad \text{(F1)}
\end{aligned}$$

$$\begin{aligned}
P_6^{(1)}(r) &= -\frac{3}{4r^4}g(r)^3\left[g(r)P_2^{(0)}(r)^2 + 2P_4^{(0)}(r)\right] - \frac{g(r)}{r}\left[g(r)g'(r)P_2^{(0)}(r)P_2^{(1)}(r) + g'(r)P_4^{(1)}(r)\right] \\
&\quad -\frac{g(r)^2}{4r^3}\left[2g(r)P_2^{(0)}(r)^2 g'(r) - 15g'(r)P_4^{(0)}(r) + g(r)^2 P_2^{(0)}(r)P_2'^{(0)}(r) - 9g(r)P_4'^{(0)}(r)\right] \\
&\quad +\frac{g(r)}{8r^2}\left[g(r)^2 g'(r)P_2^{(0)}(r)P_2'^{(0)}(r) - 11g'(r)^2 P_4^{(0)}(r) - 9g(r)g'(r)P_4'^{(0)}(r) - g(r)^2 g''(r)P_2^{(0)}(r)^2\right] \\
&\quad -\frac{1}{4r^2}g(r)^3 g''(r)P_4^{(0)}(r) - 6g(r)^2 P_2^{(0)}(r)^2 P_2^{(1)}(r) - 5g(r)P_2^{(1)}(r)P_4^{(0)}(r) - 5g(r)P_2^{(0)}(r)P_4^{(1)}(r) \\
&\quad +\frac{g(r)}{4}\left[g'(r)^2 P_2^{(0)}(r)P_2^{(1)}(r) + 3g(r)^2 P_2'^{(0)}(r)P_2'^{(1)}(r) - g(r)^2 P_2^{(1)}(r)P_2''^{(0)}(r) - g(r)^2 P_2^{(0)}(r)P_2''^{(1)}(r)\right. \\
&\quad \left. -2g(r)g''(r)P_2^{(0)}(r)P_2^{(1)}(r)\right] - \frac{1}{4}\left[g'(r)^2 P_4^{(1)}(r) + g(r)^2 P_4''^{(1)}(r) + g(r)P_4^{(1)}(r)g''(r) + 3g(r)g'(r)P_4'^{(1)}(r)\right] \qquad \text{(F2)}
\end{aligned}$$

$$\begin{aligned}
P_6^{(2)}(r) &= -\frac{g(r)^4}{8r^6}\left[3g(r)P_2^{(0)}(r)^2 + 23P_4^{(0)}(r)\right] + \frac{g(r)^3 g'(r)}{8r^5}\left[3g(r)P_2^{(0)}(r)^2 + 46P_4^{(0)}(r)\right] \\
&\quad -\frac{3g(r)^3}{r^4}\left[g(r)P_2^{(0)}(r)P_2^{(1)}(r) + P_4^{(1)}(r)\right] - \frac{g(r)^2 g'(r)^2}{32r^4}\left[3g(r)P_2^{(0)}(r)^2 + 46P_4^{(0)}(r)\right] \\
&\quad -\frac{g(r)g''(r)}{4}\left[P_4^{(2)}(r) + 2g(r)P_2^{(0)}(r)P_2^{(2)}(r) + g(r)g''(r)P_2^{(1)}(r)^2\right] + \frac{g(r)^2}{4r^3}\left[3P_2^{(0)}(r)P_2'^{(1)}(r) + 13g(r)P_4'^{(1)}(r)\right] \\
&\quad +\frac{g(r)^2}{4r^3}\left[8g(r)g'(r)P_2^{(0)}(r)P_2^{(1)}(r) + 25g'(r)P_4^{(1)}(r) - 7g(r)^2 P_2^{(1)}(r)P_2'^{(0)}(r)\right] \\
&\quad +\frac{g(r)g'(r)}{8r^2}\left[-17g'(r)P_4^{(1)}(r) + 7g(r)^2 P_2^{(1)}(r)P_2'^{(0)}(r) - 3g(r)^2 P_2^{(0)}(r)P_2'^{(1)}(r) - 13g(r)P_4'^{(1)}(r)\right] \\
&\quad -\frac{g(r)^2 g''(r)}{2r^2}\left[P_2^{(0)}(r)P_2^{(1)}(r) + P_4^{(1)}(r)\right] - \frac{g(r)g'(r)}{2r}\left[g(r)P_2^{(1)}(r)^2 + 2g(r)P_2^{(0)}(r)P_2^{(2)}(r) + P_4^{(2)}(r)\right] \\
&\quad -6g(r)^2\left[P_2^{(0)}(r)P_2^{(1)}(r)^2 + P_2^{(0)}(r)^2 P_2^{(2)}(r)\right] - 5g(r)\left[P_2^{(2)}(r)P_4^{(0)}(r) + P_2^{(1)}(r)P_4^{(1)}(r) + P_2^{(0)}(r)P_4^{(2)}(r)\right] \\
&\quad +\frac{g(r)g'(r)^2}{8}\left[P_2^{(1)}(r)^2 + P_2^{(0)}(r)P_2^{(2)}(r) - \frac{6}{g(r)}P_4'^{(2)}(r) - \frac{2}{g(r)}P_4^{(2)}(r)\right] \qquad \text{(F3)} \\
&\quad -\frac{1}{4}g(r)^3\left[P_2^{(0)}(r)P_2''^{(2)}(r) + P_2^{(1)}(r)P_2''^{(1)}(r) + \frac{P_4''^{(2)}(r)}{g(r)} + P_2^{(2)}(r)P_2''^{(0)}(r) + \frac{3}{2}P_2'^{(1)}(r)^2 + 3P_2'^{(0)}(r)P_2'^{(2)}(r)\right]
\end{aligned}$$

$$
\begin{aligned}
P_6^{(3)}(r) \;=\; & -\frac{g(r)^4}{4r^6}\left[9g(r)P_2^{(0)}(r)P_2^{(1)}(r)+49P_4^{(1)}(r)\right]+\frac{g'(r)g(r)^3}{4r^5}\left[9g(r)P_2^{(0)}(r)P_2^{(1)}(r)+49P_4^{(1)}(r)\right] \\
& -\frac{9}{4r^4}g(r)^4\left[P_2^{(1)}(r)^2+2P_2^{(0)}(r)P_2^{(2)}(r)+\frac{2}{g(r)}P_4^{(2)}(r)\right]-\frac{g(r)^2g'(r)^2}{16r^4}\left[9g(r)P_2^{(0)}(r)P_2^{(1)}(r)+49P_4^{(1)}(r)\right] \\
& -\frac{g(r)^4}{4r^3}\left[13P_2^{(2)}(r)P_2'^{(0)}(r)-7P_2^{(0)}(r)P_2'^{(2)}(r)+3g(r)^4P_2^{(1)}(r)P_2'^{(1)}(r)-\frac{17}{g(r)}P_4'^{(2)}(r)\right] \\
& +\frac{g(r)^3g'(r)}{4r^3}\left[\frac{35}{g(r)}P_4^{(2)}(r)+6P_2^{(1)}(r)^2+12P_2^{(0)}(r)P_2^{(2)}(r)\right]-\frac{g(r)}{4}\left[3g'(r)P_4'^{(3)}(r)+g''(r)P_4^{(3)}(r)\right] \\
& -\frac{3}{8r^2}g(r)^3g''(r)\left[P_2^{(1)}(r)^2+2P_2^{(0)}(r)P_2^{(2)}(r)-\frac{2}{g(r)}P_4^{(2)}(r)\right]-\frac{g(r)g'(r)^2}{8r^2}\left[23g'(r)P_4^{(2)}(r)+17g(r)P_4'^{(2)}(r)\right] \\
& +\frac{g(r)^3g'(r)}{8r^2}\left[13P_2^{(2)}(r)P_2'^{(0)}(r)+3P_2^{(1)}(r)P_2'^{(1)}(r)-7P_2^{(0)}(r)P_2'^{(2)}(r)\right] \\
& -\frac{g(r)g'(r)}{r}\left[g(r)P_2^{(1)}(r)P_2^{(2)}(r)+P_4^{(3)}(r)\right]-2g(r)^2P_2^{(1)}(r)^3-12g(r)^2P_2^{(0)}(r)P_2^{(1)}(r)P_2^{(2)}(r)-\frac{1}{4}g(r)^2P_4''^{(3)}(r) \\
& -5g(r)\left[P_2^{(2)}(r)P_4^{(1)}(r)+P_2^{(1)}(r)P_4^{(2)}(r)+P_2^{(0)}(r)P_4^{(3)}(r)\right]+\frac{g(r)g'(r)^2}{4}\left[g(r)P_2^{(1)}(r)P_2^{(2)}(r)-P_4^{(3)}(r)\right] \\
& +\frac{g(r)^3}{4}\left[3P_2'^{(1)}(r)P_2'^{(2)}(r)-2P_2^{(1)}(r)P_2''^{(2)}(r)-2P_2^{(2)}(r)P_2''^{(1)}(r)-P_2^{(1)}(r)P_2''^{(2)}(r)\right] \qquad \text{(F4)}
\end{aligned}
$$

$$
\begin{aligned}
P_6^{(4)}(r) \;=\; & -\frac{g(r)^5}{8r^6}\left[3P_2^{(1)}(r)^2+46P_2^{(0)}(r)P_2^{(2)}(r)+\frac{166}{g(r)}P_4^{(2)}(r)\right]-\frac{g(r)^2g''(r)}{r^2}\left[g(r)P_2^{(1)}(r)P_2^{(2)}(r)+P_4^{(3)}(r)\right] \\
& +\frac{g'(r)g(r)^4}{8r^5}\left[46P_2^{(0)}(r)P_2^{(2)}(r)+3P_2^{(1)}(r)^2+\frac{166}{g(r)}P_4^{(2)}(r)\right]-\frac{6g(r)^3}{r^4}\left[g(r)P_1^{(2)}(r)P_2^{(2)}(r)+P_4^{(3)}(r)\right] \\
& -\frac{g(r)^3g'(r)^2}{32r^4}\left[3P_2^{(1)}(r)^2+46P_2^{(0)}(r)P_2^{(2)}(r)+\frac{166}{g(r)}P_4^{(2)}(r)\right]-\frac{g'(r)g(r)}{2r}\left[g(r)P_2^{(2)}(r)^2+2P_4^{(4)}(r)\right] \\
& +\frac{g(r)^2}{4r^3}\left[16g(r)g'(r)P_2^{(1)}(r)P_2^{(2)}(r)+45g(r)P_4^{(3)}(r)+84g(r)P_4'^{(3)}(r)\right] \\
& -\frac{g(r)^4}{4r^3}\left[9P_2^{(2)}(r)P_2'^{(1)}(r)+g(r)^4P_2^{(1)}(r)P_2'^{(2)}(r)\right]-\frac{g(r)g''(r)}{4}\left[g(r)P_2^{(2)}(r)^2+P_4^{(4)}(r)\right] \\
& +\frac{g(r)^3g'(r)}{8r^2}\left[9P_2^{(2)}(r)P_2'^{(1)}(r)-P_2^{(1)}(r)P_2'^{(2)}(r)-\frac{21}{g(r)}P_4'^{(3)}(r)-29\frac{g'(r)}{g(r)}P_4^{(3)}(r)\right] \\
& -6g(r)^2\left[P_2^{(1)}(r)^2P_2^{(2)}(r)+P_2^{(0)}(r)P_2^{(2)}(r)^2\right]-5g(r)\left[P_2^{(2)}(r)P_4^{(2)}(r)+P_2^{(1)}(r)P_4^{(3)}(r)+P_2^{(0)}(r)P_4^{(4)}(r)\right] \qquad \text{(F5)} \\
& +\frac{1}{8}g(r)g'(r)\left[g'(r)P_2^{(2)}(r)^2-2\frac{g'(r)}{g(r)}P_4^{(4)}(r)-6P_4'^{(4)}(r)\right]+\frac{1}{8}g(r)^3\left[P_2'^{(2)}(r)^2-2P_2^{(2)}(r)P_2''^{(2)}(r)-\frac{2}{g(r)}P_4''^{(4)}(r)\right]
\end{aligned}
$$

$$
\begin{aligned}
P_6^{(5)}(r) \;=\; & -\frac{5g(r)^4}{4r^6}\left[g(r)P_2^{(1)}(r)P_2^{(2)}(r)+25P_4^{(3)}(r)\right]+\frac{5g'(r)g(r)^2}{4r^5}\left[g(r)P_2^{(1)}(r)P_2^{(2)}(r)+25P_4^{(3)}(r)\right] \\
& -\frac{15}{4r^4}g(r)^3\left[g(r)P_2^{(2)}(r)^2+2P_4^{(4)}(r)\right]-\frac{5}{16r^4}g(r)^2g'(r)^2\left[g(r)P_2^{(1)}(r)P_2^{(2)}(r)+25P_4^{(3)}(r)\right] \\
& +\frac{5g(r)^2g'(r)}{4r^3}\left[2g(r)P_2^{(2)}(r)^2+11g(r)^2P_4^{(4)}(r)\right]-\frac{5g(r)^3}{4r^3}\left[g(r)P_2^{(2)}(r)P_2'^{(2)}(r)+25P_4'^{(4)}(r)\right] \\
& -\frac{35g(r)g'(r)^2}{8r^2}P_4^{(4)}(r)+\frac{5g(r)^3g'(r)}{8r^2}\left[g(r)P_2^{(2)}(r)P_2'^{(2)}(r)-5P_4'^{(4)}(r)\right]-6g(r)^2P_2^{(1)}(r)P_2^{(2)}(r)^2 \\
& -\frac{5g''(r)g(r)^2}{8r^2}\left[g(r)P_2^{(2)}(r)^2-2P_4^{(4)}(r)\right]-5g(r)\left[P_2^{(2)}(r)P_4^{(3)}(r)+P_2^{(1)}(r)P_4^{(4)}(r)\right] \qquad \text{(F6)}
\end{aligned}
$$

$$
\begin{aligned}
P_6^{(6)}(r) \;=\; & \frac{5}{8r^6}g(r)^4\left[g(r)P_2^{(2)}(r)^2-70P_4^{(4)}(r)\right]-\frac{5g(r)^3g'(r)}{8r^5}\left[g(r)P_2^{(2)}(r)^2-70P_4^{(4)}(r)\right] \\
& +\frac{5g(r)^2g'(r)^2}{32r^4}\left[g(r)P_2^{(2)}(r)^2-70P_4^{(4)}(r)\right]-2g(r)^2P_2^{(2)}(r)^3-5g(r)P_2^{(2)}(r)P_4^{(4)}(r) \qquad \text{(F7)}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^{(6)}(r) \;=\; & \frac{9341 g(r)^4}{180180\, r^4} - \frac{4741 g'(r) g(r)^3}{16380\, r^3} + \frac{1308784\, g''(r) g(r)^3 + 3926504\, g'(r)^2 g(r)^2}{5765760\, r^2} \\
& - \frac{536120 g^{(3)}(r)\, g(r)^3 + 3719032\, g'(r)\, g''(r) g(r)^3 + 2869040\, g'(r)^3 g(r)}{5765760\, r} \\
& + \frac{213928 g^{(4)}(r) g(r)^3 + 761280 g''(r)^2 g(r)^2 + 1261208 g'(r) g^{(3)}(r) g(r)^2 + 1508748 g'(r)^2 g''(r) g(r) + 435674 g'(r)^4}{5765760} \\
& + \frac{r\left(137280 g^{(5)}(r) g(r)^3 + 947804 g''(r) g^{(3)}(r) g(r)^2 + 903188 g'(r) g^{(4)}(r) g(r)^2 + 462228 g'(r) g''(r)^2 g(r)\right)}{5765760} \\
& + \frac{r\left(971568 g'(r)^2 g^{(3)}(r) g(r)^2 - 4496 g'(r)^3 g''(r) g(r) - 24496 g'(r)^5\right)}{5765760\, g(r)} \\
& + \frac{r^2\left(3473 g'(r)^6 - 27895 g(r) g''(r) g'(r)^4 + 84032 g(r)^2 g^{(3)}(r) g'(r)^3 + 113100 g(r)^2 g''(r)^2 g'(r)^2\right)}{5765760 g(r)^2} \\
& + \frac{r^2\left(183898 g(r)^3 g^{(4)}(r) g'(r)^2 + 316316 g(r)^3 g''(r) g^{(3)}(r) g'(r) + 99528 g(r)^4 g^{(5)}(r) g'(r)\right)}{5765760 g(r)^2} \\
& + \frac{r^2\left(40040 g(r)^3 g''(r)^3 + 61776 g(r)^4 g^{(3)}(r)^2 + 123552 g(r)^4 g''(r) g^{(4)}(r) + 12012 g(r)^5 g^{(6)}(r)\right)}{5765760 g(r)^2}
\end{aligned}
\tag{F8}
$$

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