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## Stochastic Creep Damage Growth due to Random Thermal Fluctuations Using Continuum Damage Mechanics

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### Abstract

The focus of this study is on the development of a CDM based methodology for estimating the stochastic creep damage growth in structural components subjected to randomly varying thermal loads. Following available frameworks in the literature, the damage growth equations are derived from thermodynamic principles. The fluctuations are first modeled as Gaussian white noise processes. However this leads to physically unrealizable negative damage growth. Alternative non-Gaussian models for the noise, such as the geometric Brownian motion, have been subsequently used to study the growth of creep damage. Numerical results have been presented to illustrate the developments.

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### 1. Introduction

The degradation of engineering materials due to fatigue, creep, corrosion and other aging mechanisms lead to loss of structural integrity, making them susceptible to failures. The degradation mechanism usually involves the formation of voids, cavities and micro-cracks, their propagation along the material medium and eventual coalescence to form macro-cracks. The formation and growth of these damaging processes occur intrinsically in the material, depend on the service loading and the environmental conditions and essentially have random temporal and spatial variations. A detailed study in the macro-mechanical scale is essential to gain an understanding on the growth of these damage mechanisms. However, if the interest is on quantifying the risk of the structural component as a whole, one needs to focus on characterizing macro-scale damage.

The subject of continuum damage mechanics (CDM) enables one to develop a relationship between macroscopic manifestations of damage and microscopic defects and discontinuities present within a material [1]. This permits safety assessment of the structure. A CDM based approach to overall damage growth in the

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pre-localization stage can accommodate its intrinsic randomness in a natural way and enhance the service life predictions of degrading structural systems and their components.

Using principles of thermodynamics, damage growth equations can be formulated in terms of the Helmholtz free energy potential. Spatial and temporal fluctuations in the state variables, caused by the intrinsic variations in the material micro structure, as well as due to environmental and loading conditions, can be modeled by treating Helmholtz free energy as a random process. This leads to a stochastic differential equation (SDE) for the random damage growth, the solution of which describes the evolution of time-dependent damage growth mechanism [2-4]. The studies in the literature have focused on modeling the noise associated with the damage growth equations as Brownian motion leading to the possibilities of physically unrealizable negative damage increments.

The present study focuses on the development of creep damage growth equations for structural materials subjected to high temperatures for extended periods of time. For example, structural components in nuclear power plants carrying high temperature liquids would be susceptible to such creep damage effects. Complicating effects such as modeling the noise processes as non-Gaussian processes to ensure that the damage growth increments are always positive have been studied. Numerical examples which illustrate the developments made in this paper have been presented.

## 2. CDM-based damage growth equations

Following the methodology adopted in [3-4], the CDM based damage growth equations are first developed. Introducing the Helmholtz free energy potential  $\psi = \psi(T, \varepsilon, D)$  for a closed system  $R$  in diathermal contact with a heat reservoir and applying the second law of thermodynamics, it can be shown that [3]

$$\dot{K}_E - \frac{\partial \psi}{\partial \varepsilon} \cdot \dot{\varepsilon} - \frac{\partial \psi}{\partial D} \cdot \dot{D} - \dot{W} \geq 0. \quad (1)$$

Here,  $T$  is the absolute temperature,  $\varepsilon$  are state variables comprising the strain tensor and  $D$  is the damage tensor which is of order one if the material is assumed to be isotropic,  $W$  is the work done on  $R$ ,  $K_E$  is the kinetic energy of  $R$  and the dots represent the time derivatives with respect to time  $t$ . In writing Eq. (1) it is assumed that  $\dot{T} = 0$ . This is not strictly true for systems which are exposed to temperature fluctuations. However, in this study, we assume that the temperature fluctuations are gradual and very slow and the system can be assumed to be in quasi-static thermodynamic equilibrium at constant temperature. Compactly, Eq. (1) can be written as  $\Lambda \geq 0$ , where  $\Lambda$  is the heat dissipation rate. If  $\Lambda = 0$ , then the thermodynamic process becomes reversible.

Now the system  $R$  is said to be in equilibrium when it is in diathermal contact with a heat reservoir if the first variation of Helmholtz free energy is zero *i.e.*,  $\delta\psi(t) = 0$ . The equilibrium system undergoes rapid changes of its micro-state and causes random fluctuations in state variables; thus the evaluation of free energy can be described as random process [5]. Using variational principles, one can write

$$\delta\psi(t) = \delta \int_{t_0}^t (\dot{W} - \dot{K}) dt - \int_{t_0}^t \Lambda dt + \delta S(t) \approx 0 \quad (2)$$

Here,  $t_0$  is the initial equilibrium state and  $S(t)$  is a random process representing the random fluctuations in the free energy. These fluctuations are due to the uncertainties arising from modeling uncertainties, micro structural variations and environmental conditions. Eq. (2) can be further expressed as

$$\delta\psi(t) = \int_{t_0}^t \delta I_1(t) dt - \int_{t_0}^t \delta I_2(t) dt \approx 0, \quad (3)$$

the terms  $I_1$  and  $I_2$  in Eq. (3) are defined as

$$I_1 = \dot{W} - \dot{K}_E + \frac{\partial \psi}{\partial D} \dot{D} + \dot{S}, \quad I_2 = \dot{W} - \dot{K}_E + \frac{\partial \psi}{\partial \varepsilon} \dot{\varepsilon}. \quad (4)$$

Here,  $\dot{S}(t)$  is defined in the mean-square sense. In a deformable body where damage accumulates close to thermodynamic equilibrium within and along its boundary,  $\delta I_1(t) = 0$  at every instant  $t$ , it can be shown that for a deformable body undergoing isotropic damage accumulation caused by uniaxial loading, the damage growth equation can be represented as

$$\sigma_\infty + \psi_D \frac{dD}{d\varepsilon} + S_b = 0. \tag{5}$$

Here,  $\sigma_\infty$  is the far-field stress acting normal to the surface,  $\psi_D$  is the partial derivative of the free energy per unit volume with respect to the damage variable  $D$  and  $S_b$  is the rate of change of fluctuations in  $\psi$  per unit strain.

It is reasonable to assume that the source of the fluctuations in the free energy  $\psi$  is due to unknown modeling errors and hence can be modeled as random process. Additionally, it has been assumed in [5] that  $S_b$  is a zero mean process with equal probability of having positive and negative values, indicating that the probability density function is symmetric about zero and that the mean-square value of the fluctuations do not change with strain or time. It can be shown that the above conditions can be satisfied if  $S_b$  follows Langevin equation given by

$$\frac{dS_b}{d\varepsilon} = -k_1 S_b + \sqrt{k_2} \xi(\varepsilon), \tag{6}$$

where,  $\xi(\varepsilon)$  is a Gaussian white noise indexed with strain and  $k_1, k_2$  are appropriate positive constants of the Langevin equation defined in Eq. (6) such that,  $k_1$  is very large; see [5] for details. Now, Eqs. (5) and (6) describe two processes with very different scales of strain (or time). A large number of fluctuations occur in  $S_b$  in a strain during which the damage growth is insignificant. Therefore, on a scale where damage growth is assumed to be constant, from Eq. (5), we get  $dS_b/d\varepsilon = 0$ . Thus, from Eq. (6), we can approximate it as  $S_b(\varepsilon) \approx \sqrt{k_2/k_1} \xi(\varepsilon)$ . Eq. (5) can be subsequently written as follows;

$$dD(\varepsilon) = \frac{-\sigma_\infty}{\psi_D} d\varepsilon - \frac{\sqrt{k_2/k_1}}{\psi_D} dW(\varepsilon), \tag{7}$$

where,  $dW$  are the Weiner increments. If  $t$  is the indexing parameter rather than  $\varepsilon$  then Eq. (7) can be written in terms of the strain rate  $\dot{\varepsilon}$

$$dD(t) = \left\{ \frac{-\sigma_\infty}{\psi_D} \dot{\varepsilon}(D, t) \right\} dt - \left\{ \frac{\sqrt{k_4/k_3}}{\psi_D} \dot{\varepsilon}(D, t) \right\} dW(t). \tag{8}$$

Here,  $k_3$  and  $k_4$  are positive constants and are different from  $k_1$  and  $k_2$  due to Eq. (8) being now indexed with time  $t$ , such that,  $S_b(t) \approx \sqrt{k_4/k_3} \xi(t)$ . Eq. (8) represents a stochastic differential equation (SDE), where the coefficient for  $dt$  is known as the drift coefficient and the coefficient for  $dW$  is known as the diffusion coefficient. It has been shown in [5] that  $\psi_D \approx 3/4\sigma_f$ , where  $\sigma_f$  is the failure stress of the material at temperature  $T$ .

### 2.1. Stochastic creep damage

Next we modify Eq. (8) for thermal creep damage. Here, we use the Bailey-Norton creep strain rate model, given by [6]

$$\dot{\varepsilon}_c = \frac{A(T)}{(1-D)^m} \phi \sigma_\infty^m t^{\phi-1}, \tag{9}$$

where,  $A(T)$  is a temperature dependent constant,  $m$  and  $\phi$  are material parameters and the remaining parameters have the meanings as defined earlier. Assuming  $\phi = 1$  (for the steady state creep condition) and substituting in Eq. (8), the stochastic creep damage growth equations can be expressed as

$$dD(t) = \frac{A_1}{(1-D(t))^m} dt + \frac{B_1}{(1-D(t))^m} dW(t). \quad (10)$$

Here,  $A_1$  and  $B_1$  are defined as

$$A_1 = \frac{4A\sigma_\infty^{m+1}}{3\sigma_f}, \quad B_1 = \frac{4A\sigma_\infty^{m+1}(\sqrt{k_4}/k_3)}{3\sigma_f}. \quad (11)$$

The right hand side of Eq. (10) denotes the increments in creep damage growth. An inspection of Eq. (10) reveals that depending on the magnitude of the Weiner increments, which takes values from  $[-\infty, \infty]$ , there is a non-zero probability of having negative damage increments irrespective of the values of  $A_1$  and  $B_1$ . This is physically impossible. There is a need to investigate the use of non-Gaussian models for the random fluctuations in the free energy. This is discussed in the following section.

### 3. Non-Gaussian model for creep damage growth

The model for  $dW$  in Eq. (10) has been defined following the assumptions that the fluctuations in the free energy is zero-mean and that there is an equal probability of having positive and negative fluctuations about the mean value. Alternative models for the damage growth may be obtained instead if one makes the assumption that damage growth increments are always positive. A more general form of Eq. (10) would be to replace  $dW(t)$  by  $dS(t)$  where,  $dS(t) \cong S_b(t)dt$  is always greater than zero. A simple model for  $dS(t)$  would be to assume the increments to follow the Geometric Brownian motion (GBM). A stochastic process  $S(t)$  is said to follow a GBM if it satisfies the following SDE [7]

$$dS(t) = \mu S(t)dt + \lambda S(t)dW(t), \quad (12)$$

where,  $W(t)$  is a Weiner process,  $\mu$  and  $\lambda$  are respectively the drift and diffusion coefficients for the GBM and

$$S(t) = S_0 \exp \left[ \left( \mu - \frac{\lambda^2}{2} \right) t + \lambda W(t) \right]. \quad (13)$$

Assuming  $\mu = 0$ , the corresponding creep damage growth equation can be written as

$$dD(t) = \frac{A_1}{(1-D(t))^m} dt + \frac{B_1}{(1-D(t))^m} \lambda S(t) dW(t). \quad (14)$$

Closed form solutions for either Eq. (10) or Eq. (14) are not possible and we have resort to numerical integration. This is discussed in the following section.

### 4. Numerical examples and discussions

The stochastic creep damage growth in Type-A36 stainless steel stressed to 83 MPa at 1000°F (or 538°C) is considered. The material parameters of A36 stainless steel are taken from the literature [5] and are:  $A(T) = 2.967 \times 10^{-13}$ ,  $m = 4.8$ ,  $\sigma_f = 373$  MPa,  $\sigma_\infty = 83$  MPa and  $\sqrt{k_4}/k_3 = 69$  MPa $\sqrt{\text{hr}}$ . The material is initially assumed to be defect free, *i.e.*,  $D_0 = 0$  and the material parameters are considered to be deterministic quantities.

Since the creep damage growth is random, different realizations would lead to different damage growth trajectories. To characterize the creep damage growth, Monte Carlo simulations are carried out.  $N = 1 \times 10^4$  sample realizations of the trajectories are simulated in the computer. Two cases have been studied: (a) Case 1-

where the fluctuations have been modeled as Brownian motion, and (b) Case 2- where the fluctuations are modeled as geometric Brownian motion. This involves generating trajectories of the Brownian motion and geometric Brownian motion. Eqs. (10) and (14) are numerically integrated using the principles of stochastic calculus. Two algorithms- the simpler Euler-Maruyama method of order 0.5 and Milstein’s approximation of order 1.0 have been used to generate trajectories of damage growth. Subsequently, probabilistic parameters such as mean, variance and the probability density functions (pdf) have been obtained through statistical processing.

Figure (1) shows sample paths of  $D(t)$  with time-step of  $\Delta t = 1$  hr. The scatter in the sample paths of damage is found to increase with higher noise intensity. The deterministic solution for the same condition of stress and temperature agrees closely with predicted mean damage. This is expected as the noise is Gaussian and the fluctuations about the mean value are equal on the average.

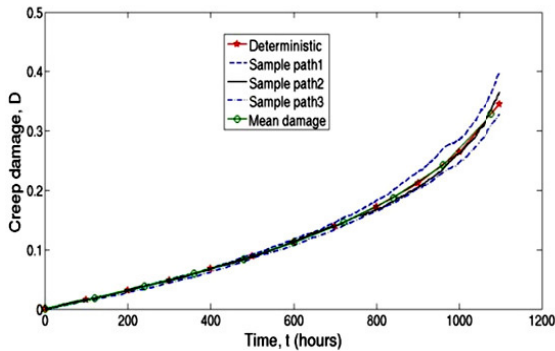


Fig. 1. Case 1: Creep damage growth trajectories.

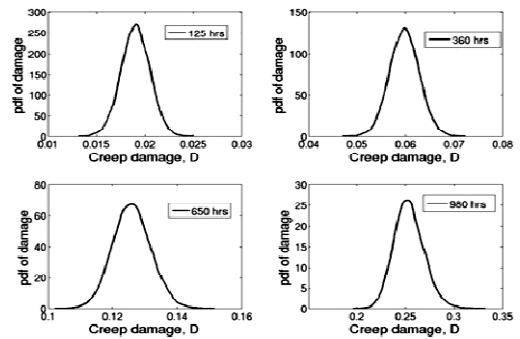


Fig. 2. Case 1: The pdf of creep damage at different time instants.

Figure (2) illustrates the probability density functions (pdf) of damage at different time instants. When these pdfs are plotted in the same figure (see Fig. 3), it can be clearly seen that not only the mean damage increases with time, but also the variance of the damage increases. The mean damage growth trajectory corresponding to case 2 where the noise is modeled as geometric Brownian motion (GBM) is shown in Figure (4) for different values of the volatility parameter  $\lambda$  and is compared with the mean growth trajectory when the noise is modeled as Gaussian. It can be seen that the mean of the damage growth of GBM model almost overlaps with mean damage growth of Gaussian model for all values of  $\lambda$  except  $\lambda = 0.04$ . Figures (5-6) illustrate the pdfs of damage corresponding to case 2 at various time instants.

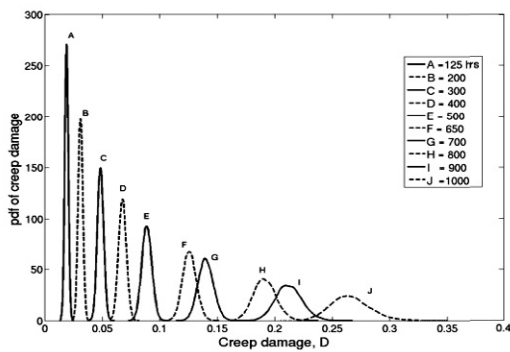


Fig. 3. Case 1: Variation of the pdf of creep damage at different time instants.

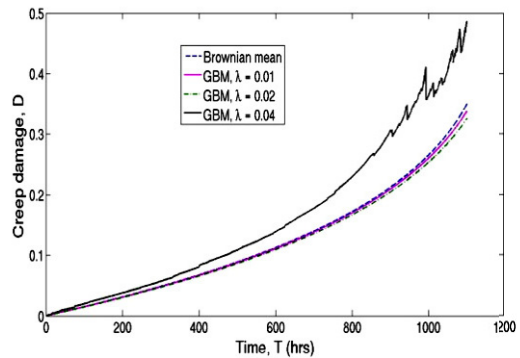


Fig. 4. Comparison of the mean of creep damage growth for case 1 and case 2.

As in case 1, both the mean and the variance increase with time. However, interestingly, as can be seen from Figure 6, the pdf of damage at different time instants show multiple peaks when the noise is modeled as a GBM process. This implies that the growth of damage occurs in jumps with multiple modes.

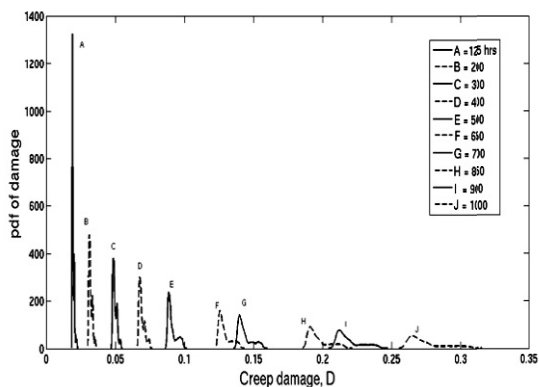


Fig. 5. Case 2: Variation of the pdf of creep damage at different time instants.

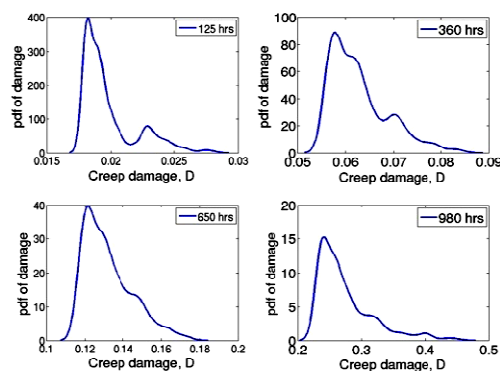


Fig. 6. Case2: The pdf of creep damage at different time instants.

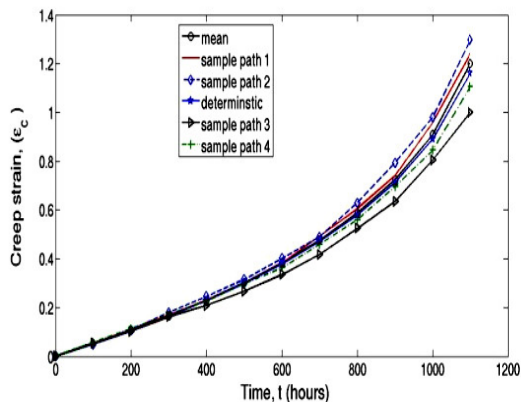


Fig. 7. Response of creep strain variation with time.

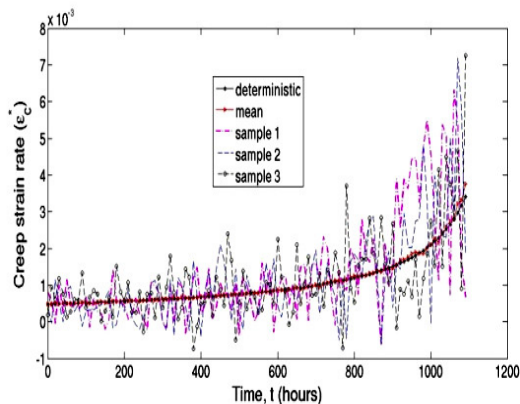


Fig. 8. Variation of creep strain rate with time.

The behavior of creep strain and creep strain rate with respect to time obtained from Eq. 9 is illustrated in Figs. 7 and 8 respectively. We observe that the damage growth and creep strain have a similar shape. This is expected as the damage growth is proportional to the creep strain. For a given time, the variability in the creep strain is observed to be significant underlying the importance of modeling the uncertainties. The creep strain rate in Fig. 8 is observed to be almost flat initially indicating steady state conditions. However, the sample fluctuations reveal significant variabilities from each sample realizations.

### 5. Conclusion

Investigations have been carried out for modeling the stochastic creep damage growth in materials subjected to high temperature environments. The damage growth equations have been derived in CDM framework and by applying thermodynamic principles. Uncertainties arising due to modeling assumptions and fluctuations in environmental conditions have been modeled as white noise processes. Research has been carried out to study the effects of modeling the noise as Gaussian and non-Gaussian. It is observed that the choice of models for noise can significantly affect the damage predictions. Comparisons with experimental results are necessary to arrive at appropriate models for the uncertain variations.

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