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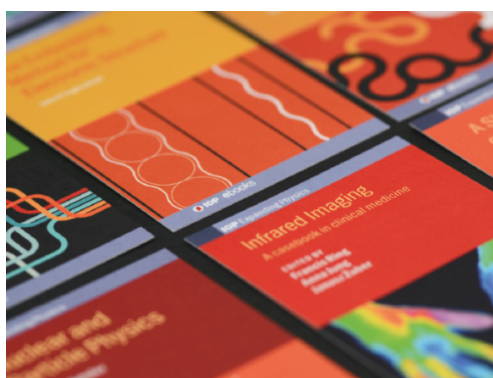
Small scale structure of spacetime in presence of a minimal length

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Small scale structure of spacetime in presence of a minimal length

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Abstract. I present some recent results which yields some specific consequences of having a minimal spacetime length L_0 while maintaining Lorentz covariance. In particular: (i) the resultant description of geometry must be in terms of a non-local bi-tensor $q_{ab}(P, p, L_0^2)$ instead of the conventional metric tensor $g_{ab}(p)$, (ii) q_{ab} and g_{ab} are related disformally through the Synge world function, (iii) the resultant non-local d'Alembartian yields Green's functions with a natural covariant short distance cut-off, (iv) the Ricci bi-scalar $R(P, p, L_0^2)$ corresponding to q_{ab} yields a local scalar $\mathcal{S}(P, L_0^2) = R(P, P, L_0^2)$ in the coincidence such that

$$\lim_{L \to 0} \mathcal{S}(P; L) \propto R_{ab} t^a t^b$$

where t^a are arbitrary normalized vectors representing gauge degrees of freedom at each spacetime event; this limit turns out to be independent of the precise details of the short distance modification, (v) the Gibbons-Hawking-York surface term for q_{ab} yields a residual entropy for each spacetime event.

1. Introduction

If the space-time possesses a minimal length, say L_0, which does not violate Lorentz invariance, it has widely been believed that this will result in space-time having some structure at small scales, and also that such a structure would go hand in hand with the short distance regularisation of the propagators which arises from having such a minimal length scale. In recent work, we have arrived at such a characterisation of small scale structure of space-time in terms of the Synge distance function $d(P, p) = \sqrt{\epsilon \sigma^2(P, p)}$. In particular, we have derived a description of geometry in terms of a second rank bi-tensor $q_{ab}(P, p, L_0^2)$ which arises as a consequence of excising a geodesic neighbourhood of size L_0 around any arbitrary space-time event P.

In this talk, I summarise the key results of this work, and in particular highlight those features which are expected to be generic and independent of the precise model of quantum gravity; in fact, I would clarify that the non-perturbative nature of some of these results make them qualitatively similarly to the various anomalies that arise in quantum field theory in curved space-time, and would therefore be independent of how small L_0 is. Here is the plan of the talk:

(i) Minimal length and the small scale structure of space-time

- UV behaviour of propagators and the Synge world function bi-scalar $\Omega(p, P)$.

- Non-local, disformal coupling of the metric to $\Omega(p, P)$: the *qmetric*.

- Anomalous behaviour of space-time curvature – the Cheshire Grin.

$$\lim_{L_0 \to 0} \lim_{p \to P} \neq \lim_{p \to P} \lim_{L_0 \to 0}$$

(ii) Entropy density of space-time from zero point length

- geodesic structure of spacetime
- GHY term & zero-point entropy of space-time

(iii) Discussion and Outlook

Note: All mathematical details of the results presented here can be found in the references in the bibliography. For brevity, the article contains no explicit citations, and all additional references concerning previous related work on this topic can be found in the references listed below, particularly in [1]. The metric signature is $(-, +, +, +, \ldots)$.

2. Minimal length and the small scale structure of space-time
2.1. Background
For a smooth differentiable manifold, the metric near any event has a taylor expansion in the Riemann normal coordinates, given by

$$g_{ab}(x; X) \approx \eta_{ab} - \frac{1}{3} R_{acbd}(X) \, (x - X)^c (x - X)^d + \text{higher order terms}$$

However, such an expansion is questionable at least in 2 cases:

- $d \sim L_0$, when continuous structures like R_{abcd} may not make much sense, and
- near spacetime singularities.

The relevant deformation of geometry which incorporates the fact that space-time might have some structure at small scales can then become non-local. In fact, in a recent paper, it was shown that, if geodesic distances are modified as $\sigma^2 \to \sigma^2 + L_0^2$, then the required deformation of geometry which can yield the modified geodesic function is given by

$$q_{ab}(P, p, L_0^2) \;=\; \mathcal{A} g_{ab}(p) - \epsilon \left(\mathcal{A} - \mathcal{A}^{-1} \right) t_a t_b$$

$$\mathcal{A}[\sigma; L_0] \;=\; 1 + \frac{L_0^2}{\sigma_{\boldsymbol{g}}^2}$$

which is:

- disformally coupled to g_{ab}
- a non-local bi-tensor – a 2nd rank tensor at p and a scalar at P
- is singular in the limit $p \to P$

An important consequence of this last characteristic is the fact that, although $\lim_{L_0 \to 0} q_{ab} = g_{ab}$, the *local scalars* constructed out of q_{ab} need not reduce, in the limit $L_0 \to 0$, to their corresponding form in g_{ab}; in fact, this is just what happens! i will now discuss the structure of the two crucial terms in the gravitational action (in units of $\hbar$).

Let $L_0^2 \to 0$ with $p \neq P$ (no surprises here!)	The Strategy	Let $p \to P$ with $L_0^2 \neq 0$ (leads to entropy density functional)
$\sigma^2 \to \sigma^2(P,p)$	(1) Start with the geodesic interval $\sigma^2(P,p)$ for a metric g_{ab}	$\sigma^2 \to 0$
$\sigma_{(q)}^2 \to \sigma^2(P,p)$	(2) Incorporate some QG effects by the ansatz $\sigma_{(q)}^2(P,p,L_0^2) = \sigma^2(P,p) + L_0^2$	$\sigma_{(q)}^2 \to L_0^2$
$q_{ab}(P,p,L_0^2) \to g_{ab}(P)$	(3) Find the qmetric $q_{ab}(P,p,L_0^2)$ related to $\sigma_{(q)}^2(P,p,L_0^2)$	Diverges as $\frac{L_0^2}{\sigma^2}$
$R(P,p,L_0^2) \to R(P)$	(4) Compute the Ricci biscalar $R(P,p,L_0^2)$ for the $q_{ab}(P,p,L_0^2)$	$\boxed{R(P,p,L_0^2) \to O(1)R_{ab}t^a t^b + O(L_0^2)}$

Table 1. Summary of the results. The strategy adopted in this work is described in the middle column with the logical flow being from top to bottom. We take the coincidence limit to obtain local quantities from bi-tensors; these limits are shown in the right column. As a crosscheck, we have included a left column showing what happens when we set $L_P^2 = 0$. (The results in this column are trivial and as expected.)

$$\left(16\pi L_p^2\right) S_{\mathrm{grav}} = \int\limits_{\mathcal{V}} R[g] \;+\; 2\int\limits_{\partial\mathcal{V}} K[h]$$

and show that

- $\lim\limits_{L_0 \to 0} \lim\limits_{\sigma^2 \to 0} R[q] \neq R[g]$.

- the surface term $K\sqrt{h}$ is finite in the limit $\sigma^2 \to 0$, and yields *entropy density of spacetime* with a *zero-point term.*

- these results turn out to be independent of precise nature of how geodesic distances are modified, hence presumably also of any specific framework of quantum gravity.

The final results are summarised in Table 1.

The exact form of the Ricci scalar, and many more features of the disformal geometry including the extrinsic curvature on the boundaries of the *equi-geodesic* geodesic neighbourhood, can be found in the references. The exact expressions turn out to be:

$$\frac{1}{16\pi L_P^2}\,\mathcal{R}(P,L_0) = \alpha\,\mathcal{S}_g \;-\; \frac{\mu^2}{240\pi}\left[\frac{1}{3}\mathcal{S}_{ab}\mathcal{S}^{ab} + \frac{3}{2}\ddot{\mathcal{S}} + \frac{5}{3}\mathcal{S}^2\right]$$

$$\frac{1}{8\pi L_P^2}\lim\limits_{|\sigma^2| \to 0}\left[K\sqrt{h}\right]_q = \mathcal{S}_0 - \frac{\mu^4}{24\pi}\mathcal{S}_g$$

where

- $\mu = L_0/L_P = O(1)$
- $\mathcal{S}_{ab} = R_{aibj} t^i t^j$
- $\mathcal{S}_0 = (3/8\pi)\,\mu^2$
- $\mathcal{S}_g = R_{ab} t^a t^b$

Some important comments on the result:

- Note that there are no terms of the form $1/L_0^2$ in curvature, which one often encounters in *naive* arguments involving vacuum fluctuations of geometry.
- Apart from a cubic divergence which is regularizable in standard point-splitting scheme, there are no divergences despite the manifestly divergent form of the qmetric in the coincidence limit.
- The above two points are a consequence of some miraculous cancellations in the intermediate steps, which remove terms like $1/\sigma^2$ and $1/L_0^2$; this in turn is a consequence of the disformal structure of the qmetric; such terms do appear, for e.g., in quantised conformal fluctuations.

3. Discussion and Future Outlook

In this last section, I discuss the implications of the result and some open issue which need to be addressed by future research.

(i) Relics of the space-time minimal length:

- Our result suggests that non-local and non-analytic effects of a minimal length might leave residues which are independent of L_0.

- In our case, such effects leave their imprints in the form of

  - $\mathcal{S}_0 = 3\mu^2/8\pi$ zero-point entropy density of space-time
  - $\mathcal{S}_g = R_{ab} t^a t^b$ gravitational entropy density of space-time

- Such quantum relics are not unfamiliar in physics; e.g. effects of Lorentz violating regulators at higher energies can generically get dragged to lower energies due to radiative corrections, leaving $O(1)$ residual effects [Collins et. al., Polchinski]; conformal anomaly, $D \rightarrow 4$ limit in dimensional regularization [see, for e.g., Birrell & Davies]; non-relativistic relic of the $O(1/c)$ expansion of the relativistic point particle wave function [Padmanabhan et. al.]; nonlocal quantum residue of discreteness of the causal set type [Sorkin].

(ii) Implications for space-time thermodynamics: The above analysis also leads to an unexpected and non-trivial connection between **small scale structure of spacetime** and **spacetime thermodynamics and gravity**. In particular:

- The object $\mathcal{S}_g = R_{ab} t^a t^b$ is well-known to be connected with the Noether charge of diffeomorphism invariance, and hence to entropy of local Rindler horizons; this in turn has been used to derive gravitational dynamics from space-time thermodynamics [Jacobson, Padmanabhan].

- Our result for modified Ricci scalar and surface term in gravitational action suggests that *even classically*, the correct variational principle for gravity must be based on $R_{ab} t^a t^b$ rather than the conventional Einstein-Hilbert lagrangian R.

- Such a thermodynamic variational principle for gravity based on an *entropy functional* was already proposed few years back, in which the degrees of freedom to be varied are arbitrary normalised vectors t^a [Padmanabhan et.al.].

The figure below summarises this connection:
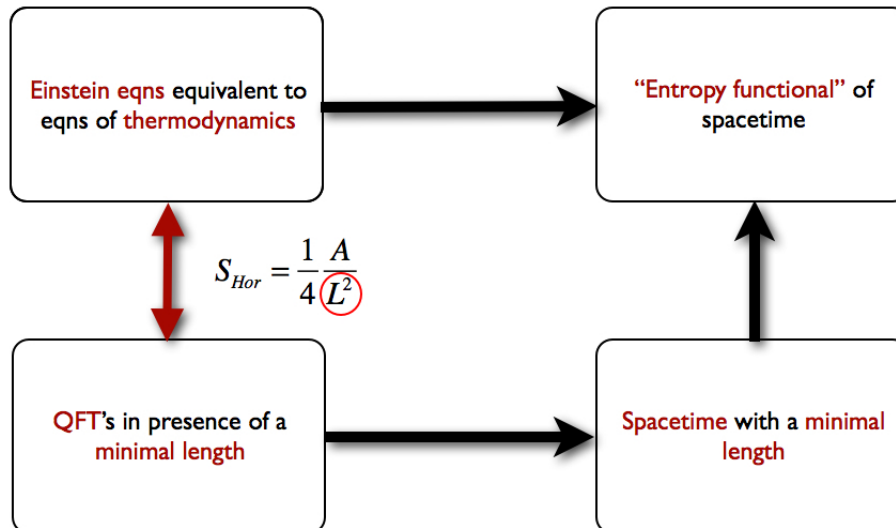
Future Outlook: Some of the future directions which can be explored based on the constructs and ideas described here are as follows:

- Implications for the cosmological constant problem.

  our approach seems to connect the notion of the cosmological constant being a non-local relic of quantum gravity and it's role in the emergent gravity paradigm

- Implications of the non-locality for the high frequency behaviour of Hawking radiation; the trans-planckian issue.

- Short distance structure of the two-point functions & implications for QFT.

- Implications for spacetime singularities

  how does a minimal length scale affect focussing and de-focussing of geodesics?

4. Acknowledgments

References
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[3] Dawood Kothawala Gen. Rel. Grav. **46**, 1836 (2014) [arXiv:1406.2672].