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Bull. Sci. math. 137 (2013) 485–497

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Shift invariant spaces on compact groups

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Received 12 August 2012

Available online 8 November 2012

Abstract

We study the theory of shift invariant spaces in $L^2(G)$, where G is a compact group. We define a range function and show that a relation between an H-invariant space and the range function is valid as in the case of abelian group setting. Here H is assumed to be a closed normal subgroup of G. We also obtain a decomposition for an H-invariant space in terms of principle H-invariant spaces whose generators give rise to "generalized Parseval frames" and use this result to study H-preserving operators.

Keywords: Compact groups; H-invariant spaces; H-preserving operators; Parseval frame; Range functions

1. Introduction

Let $\mathbb{T}$ be the unit circle. A closed subspace $\mathcal{M}$ of $L^2(\mathbb{T})$ is called invariant if $\chi f \in \mathcal{M}$ $\forall f \in \mathcal{M}$, where $\chi(e^{ix}) = e^{ix}$. It is said to be doubly invariant if χf and $\chi^{-1} f \in \mathcal{M}$ $\forall f \in \mathcal{M}$. Let $L^2(\mathbb{T}, \mathcal{H})$ denote the class of $\mathcal{H}$-valued square integrable functions on $\mathbb{T}$, where $\mathcal{H}$ is a separable Hilbert space. A range function J is defined to be a function from $\mathbb{T}$ into a family of closed subspaces of $\mathcal{H}$. Let M_J be the set of all functions F in $L^2(\mathbb{T}, \mathcal{H})$ such that $F(e^{ix})$ lies in $J(e^{ix})$ a.e. Then it is well known that the doubly invariant subspaces of $L^2(\mathbb{T}, \mathcal{H})$ are precisely the subspaces M_J. We refer to Helson [8] for more details.

In [2], Bownik obtained a characterization of shift invariant spaces in $L^2(\mathbb{R}^n)$ through range function as in [8]. In [2], Bownik also looked at several interesting problems such as characterization of frames and Riesz families, decomposition of a shift invariant space into an orthogonal sum of subspaces having tight frame generators, characterization of shift preserving operators in terms of range operators.

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0007-4497/$ – see front matter
http://dx.doi.org/10.1016/j.bulsci.2012.11.003

Shift invariant spaces play an important role in modern analysis for the past two decades because of their rich underlying theory and their applications in various fields such as wavelets, theory of frames, approximation theory, nonuniform sampling problems and so on. We refer to [1] for more information about shift invariant spaces. The theory of shift invariant spaces for locally compact abelian groups was studied by Cabrelli and Paternostro in [3], where they extended many results of [2] in a locally compact abelian group setting. In [9], Kamyabi Gol and Raisi Tousi took a different approach of obtaining a decomposition of a shift invariant space of $L^2(G)$, where G is a locally compact abelian group. They showed that every shift invariant space can be decomposed as an orthogonal sum of spaces each of which is generated by a single function whose shifts form a Parseval frame.

In this paper, our aim is to extend the theory of shift invariant spaces to the setting of a compact group G, not necessarily abelian. We organize our paper as follows. In Section 2, we give the necessary background as preliminaries. In Section 3, we define a range function and prove some properties of it. In Section 4, we characterize the shift invariant subspaces of $L^2(G)$ in terms of the range function. In Section 5, we use the approach of [9], to obtain a decomposition for a shift invariant space in $L^2(G)$. Finally in the last section, we study H-preserving operators as in [2], using the decomposition discussed in Section 5.

2. Preliminaries

Let G be a compact group. Then it is well known that G possesses a unique Haar measure dx such that $\int_G dx = 1$. Further, an irreducible unitary representation of G is always finite-dimensional. Let $\widehat{G}$ denote the set of all irreducible unitary representations of G. Then $\widehat{G}$ is called the unitary dual of G and $\widehat{G}$ is given the discrete topology.

Recall that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces then the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$, is also a Hilbert space with the inner product

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \langle u_1, v_1 \rangle \langle u_2, v_2 \rangle,$$

for $u_1, v_1 \in \mathcal{H}_1$ and $u_2, v_2 \in \mathcal{H}_2$. This fact helps us to have the following definition. If π and σ are unitary representations of compact groups G_1 and G_2 respectively, then the map, denoted by $\pi \otimes \sigma$,

$$(g_1, g_2) \mapsto \pi(g_1) \otimes \sigma(g_2)$$

defines a unitary representation of $G_1 \times G_2$.

The following theorem will be useful in the sequel.

Theorem 2.1. *If G_1 and G_2 are compact groups, then $\widehat{G_1} \times \widehat{G_2} = \widehat{G_1 \times G_2}$ and the correspondence is given by $(\pi, \sigma) \mapsto \pi \otimes \sigma$.*

Given a representation π and $u, v \in \mathcal{H}_\pi$, the mapping $x \mapsto \langle \pi(x)u, v \rangle$ is called the coefficient function of π. Let $\mathcal{M}_\pi$ denote the space of all coefficient functions of the representation π. Then we have the following

Theorem 2.2. *Let G be a compact group.*

(i) *The coefficient function arising out of an irreducible unitary representation belongs to $L^2(G)$.*

(ii) (Schur's orthogonality relations). *If* $\pi, \sigma \in \widehat{G}$ *and* $\pi \neq \sigma$, *then the spaces* $\mathcal{M}_\pi$ *and* $\mathcal{M}_\sigma$ *are mutually orthogonal subspaces of* $L^2(G)$.

(iii) (Peter–Weyl theorem). *The space* $L^2(G)$ *is equal to the closure of the direct sum of coefficient spaces of the irreducible unitary representations of* G, *i.e.,*

$$L^2(G) = \overline{\bigoplus_{\pi \in \widehat{G}} \mathcal{M}_\pi}.$$

Definition 2.3. Let $f \in L^1(G)$. Then the Fourier transform of f is defined as

$$\hat{f}(\pi) = \sqrt{dim(\pi)} \int_G f(x)\pi(x)\,dx, \quad \pi \in \widehat{G}.$$

Let $\mathcal{B}_2(\mathcal{H})$ denote the Hilbert space of all Hilbert–Schmidt operators on a Hilbert space $\mathcal{H}$, with the inner product defined by

$$\langle T, S\rangle = tr\big(TS^*\big), \quad T, S \in \mathcal{B}_2(\mathcal{H}).$$

As a consequence of the Peter–Weyl theorem we have the following theorems.

Theorem 2.4 *(Plancheral theorem). The Fourier transform is a unitary map from* $L^2(G)$ *onto* $\bigoplus_{\pi \in \widehat{G}} \mathcal{B}_2(\mathcal{H}_\pi)$ *and*

$$\|f\|_2^2 = \sum_{\pi \in \widehat{G}} \|\hat{f}(\pi)\|^2_{\mathcal{B}_2(\mathcal{H}_\pi)}, \quad \textit{for } f \in L^2(G).$$

On polarization, one has

$$\langle f, g\rangle_{L^2(G)} = \sum_{\pi \in \widehat{G}} \big\langle \hat{f}(\pi), \hat{g}(\pi)\big\rangle_{\mathcal{B}_2(\mathcal{H}_\pi)}, \quad \textit{for } f, g \in L^2(G).$$

Theorem 2.5 *(Fourier inversion formula). Let* $f \in L^2(G)$. *Then the following inversion formula holds*:

$$f(x) = \sum_{\pi \in \widehat{G}} tr\big(\hat{f}(\pi)\pi(x)\big),$$

in the $L^2(G)$ *norm.*

We refer to [4,6] for more details on compact groups.

Let G be a locally compact group and H a closed subgroup of G.

Definition 2.6. A section of G/H is a set of representatives of the quotient, i.e., a subset $\mathcal{S}$ of G containing exactly one element of each coset.

It follows, by definition of the section of a quotient, that each element $x \in G$ has a unique expression of the form $x = s.h$ with $s \in \mathcal{S}$ and $h \in H$. The following theorem on the existence of a Borel section is from [5].

Theorem 2.7. *There always exists a Borel section of the quotient* G/H.

Also note that there is a one–one correspondence between G/H and the section $\mathcal{S} \subset G$ of G/H, given by

$$[x] \mapsto [x] \cap \mathcal{S},$$

which enables us to identify the Hilbert space $L^2(G)$ with $L^2(H \times G/H)$. More specifically we have the following.

Theorem 2.8. *The map*

$$f \mapsto \widetilde{f}$$

where $\widetilde{f}(h, [x]) = f(([x] \cap \mathcal{S}).h)$, *is a Hilbert space isomorphism between* $L^2(G)$ *and* $L^2(H \times G/H)$.

Hence if $f \in L^2(G)$ we denote by $\widetilde{f}$ the corresponding function in $L^2(H \times G/H)$.

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{x_n: n \in \mathbb{N}\} \subset \mathcal{H}$ is called a frame if there exist positive constants $A, B > 0$ with $B < \infty$ such that $\forall x \in \mathcal{H}$,

$$A\|x\|^2 \leqslant \sum_{n \in \mathbb{N}} \left|\langle x, x_n \rangle\right|^2 \leqslant B\|x\|^2.$$

If $A = B = 1$, then such a sequence is called a Parseval frame.

We refer to [7] for a study on frames.

Throughout this paper, G from now on will denote a compact group and H will denote a closed normal subgroup of G. Further, any subgroup of G under consideration is taken to be a closed normal subgroup of G.

3. Range function and its properties

In this section, we define the notion of range function and derive some of its properties. We begin with the definition of range function.

Definition 3.1. A range function is a function J mapping $\widehat{H}$ into a family of closed subspaces of $l^2(\widehat{G/H}, \mathcal{B}_2(L^2(H \times G/H)))$ such that $J(\pi) \subseteq \bigoplus_{\sigma \in \widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi \otimes \sigma})$ for all $\pi \in \widehat{H}$ and $J(\pi)$ is a $\mathcal{B}_2(\mathcal{H}_\pi)$-module.

For a given range function J, we associate to each $\pi \in \widehat{H}$, the orthogonal projection onto $J(\pi)$,

$$P_\pi : \bigoplus_{\sigma \in \widehat{G/K}} \mathcal{B}_2(\mathcal{H}_{\pi \otimes \sigma}) \to J(\pi).$$

Given a range function J, we define the subset M_J of $\bigoplus_{\pi \in \widehat{H}} \bigoplus_{\sigma \in \widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi \otimes \sigma}) \subseteq l^2(\widehat{H}, l^2(\widehat{G/H}, \mathcal{B}_2(L^2(H \times G/H))))$ as

$$M_J := \left\{\Phi \in \bigoplus_{\pi \in \widehat{H}} \bigoplus_{\sigma \in \widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi \otimes \sigma}) \colon \Phi(\pi) \in J(\pi)\ \forall \pi \in \widehat{H}\right\}.$$

Lemma 3.2. *The set* M_J *is a closed subset of* $\bigoplus_{\pi \in \widehat{H}} \bigoplus_{\sigma \in \widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi \otimes \sigma})$.

Proof. Let $\{\Phi_j\}_{j\in\mathbb{N}} \subset M_J$ be such that $\Phi_j \to \Phi$ in $\bigoplus_{\pi\in\widehat{H}} \bigoplus_{\sigma\in\widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})$. Consider the functions $g_j : \widehat{H} \to \mathbb{R}$ defined as $g_j(\pi) := \|\Phi_j(\pi) - \Phi(\pi)\|$. Then it is clear that g_j converges to 0 pointwise as $j \to \infty$ and hence $\Phi_j(\pi)$ converges to $\Phi(\pi)$ in $\bigoplus_{\sigma\in\widehat{G/H}} \mathcal{B}_2(\mathcal{H}_\sigma)$. Since $\Phi_j(\pi) \in J(\pi)$ for all $\pi \in \widehat{H}$ and $J(\pi)$ is closed, $\Phi(\pi) \in J(\pi)$ for all $\pi \in \widehat{H}$. Therefore $\Phi \in M_J$. $\square$

Let $\mathcal{Q} : \bigoplus_{\pi\in\widehat{H}} \bigoplus_{\sigma\in\widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma}) \to \bigoplus_{\pi\in\widehat{H}} \bigoplus_{\sigma\in\widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})$ be the linear mapping defined by $(\mathcal{Q}\Phi)(\pi) = P_\pi(\Phi(\pi))$.

Proposition 3.3. *Let J be a range function and P_π, $\pi \in \widehat{H}$, be the associated orthogonal projection. Denote by $\mathcal{P}$ the orthogonal projection onto M_J. Then*

$$(\mathcal{P}\Phi)(\pi) = P_\pi\big(\Phi(\pi)\big) \quad \forall\pi \in \widehat{H},\ \Phi \in \bigoplus_{\pi\in\widehat{H}} \bigoplus_{\sigma\in\widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma}).$$

Proof. The proof is complete once we prove that $\mathcal{P} = \mathcal{Q}$. But this follows exactly as in the case of abelian group setting. See [3]. $\square$

Lemma 3.4. *If J_1 and J_2 are two range functions such that $M_{J_1} = M_{J_2}$, then $J_1(\pi) = J_2(\pi)$ $\forall\pi \in \widehat{H}$, i.e., J_1 and J_2 are equal.*

Proof. If P_π and Q_π are the orthogonal projections associated to J_1 and J_2 respectively, then one can show that $P_\pi(e_{ij}^{\sigma'}) = Q_\pi(e_{ij}^{\sigma'})$ for all $\sigma' \in \widehat{G/H}$, $1 \leqslant i, j \leqslant dim(\pi\otimes\sigma')$, where $e_{ij}^{\sigma'} \in \bigoplus_{\sigma\in\widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})$ is defined by

$$e_{ij}^{\sigma'}(\sigma) = \begin{cases} 1_{\pi\otimes\sigma'}^{ij} & \text{if } \sigma = \sigma', \\ 0 & \text{otherwise.} \end{cases}$$

The remaining proof will follow similar lines as in [3]. $\square$

4. Characterization of H-invariant spaces

Definition 4.1. We say that a closed subspace $V \subseteq L^2(G)$ is H-invariant if $f \in V \Rightarrow L_h f \in V$ $\forall h \in H$, where $L_h f(x) = f(h^{-1}x)$.

For a subset $\mathcal{A} \subseteq L^2(G)$, define

$$E_H(\mathcal{A}) = \{L_h\phi\colon\ h \in H,\ \phi \in \mathcal{A}\} \quad \text{and}$$
$$\mathcal{S}(\mathcal{A}) = \overline{span}\, E_H(\mathcal{A}).$$

If $\mathcal{A} = \{\varphi\}$, then we denote by $\mathcal{S}(\varphi)$ the space $\mathcal{S}(\mathcal{A})$. We call it a principle H-invariant space.

Lemma 4.2. *The mapping $\mathcal{T} : L^2(G) \to \bigoplus_{\pi\in\widehat{H}} \bigoplus_{\sigma\in\widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})$ defined by*

$$\mathcal{T}f = \big\{\big\{\hat{\hat{f}}(\pi\otimes\sigma)\big\}_{\sigma\in\widehat{G/H}}\big\}_{\pi\in\widehat{H}}$$

is an isometric isomorphism.

Proof. Notice that the map is well-defined. Further,

$$\begin{aligned}
\|\mathcal{T}f\|^2 &= \sum_{\pi\in\widehat{H}}\sum_{\sigma\in\widehat{G/H}}\big\|\hat{\tilde{f}}(\pi\otimes\sigma)\big\|^2_{\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})}\\
&= \sum_{\pi\otimes\sigma\in\widehat{H}\times\widehat{G/H}}\big\|\hat{\tilde{f}}(\pi\otimes\sigma)\big\|^2_{\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})}\\
&= \sum_{\chi\in\widehat{H\times G/H}}\big\|\hat{\tilde{f}}(\chi)\big\|^2_{\mathcal{B}_2(\mathcal{H}_\chi)}\\
&= \|\tilde{f}\|_{L^2(H\times G/H)} = \|f\|_{L^2(G)}.
\end{aligned}$$

It remains to show that $\mathcal{T}$ is onto. Let $\Phi\in\bigoplus_{\pi\in\widehat{H}}\bigoplus_{\sigma\in\widehat{G/H}}\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})$. Define $F\in\bigoplus_{\chi\in\widehat{G}}\mathcal{B}_2(\mathcal{H}_\chi)$ as

$$F(\chi)=\Phi(\pi\otimes\sigma),\quad \text{where } \chi=\pi\otimes\sigma.$$

Take $f\in L^2(G)$ such that $\hat{\tilde{f}}=F$, which will be the requirement. □

Theorem 4.3. *Let $V\subseteq L^2(G)$ be a closed subspace. Then V is H-invariant if and only if there exists a range function J such that*

$$V=\big\{f\in L^2(G)\colon \mathcal{T}f(\pi)\in J(\pi)\ \forall\pi\in\widehat{H}\big\}.$$

Proof. Suppose V is H-invariant. Since $L^2(G)$ is separable, $V=\mathcal{S}(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$.

Define J as

$$J(\pi)=\overline{span}\big\{\mathcal{T}\varphi(\pi)\colon \varphi\in\mathcal{A}\big\},\quad \pi\in\widehat{H}.$$

We claim that $V=\{f\in L^2(G)\colon \mathcal{T}f(\pi)\in J(\pi)\ \forall\pi\in\widehat{H}\}$. In order to prove our claim, it is enough to show that $M=M_J$ where $M:=\mathcal{T}V$. Then as in Theorem 3.10 of [3], we can show that $M\subset M_J$.

In order to show that $M_J\subset M$, let $\Psi\in M_J$ be such that $M\perp\Psi$. Then for each $\Phi\in M$, $\langle\Phi,\Psi\rangle=0$. In particular, if $\Phi\in\mathcal{T}(\mathcal{A})\subseteq\mathcal{T}(V)=M$ and $h\in H$, we have $\mathcal{T}(L_h\mathcal{T}^{-1}\Phi)\in M$ and hence

$$\begin{aligned}
0&=\big\langle\mathcal{T}\big(L_h\mathcal{T}^{-1}\Phi\big),\Psi\big\rangle\\
&=\sum_{\pi\in\widehat{H}}\big\langle\pi(h)\Phi(\pi),\Psi(\pi)\big\rangle\\
&=\sum_{\pi\in\widehat{H}}\sum_{\sigma\in\widehat{G/H}}\big\langle\pi(h)\Phi(\pi)(\sigma),\Psi(\pi)(\sigma)\big\rangle\\
&=\sum_{\pi\in\widehat{H}}\sum_{\sigma\in\widehat{G/H}}tr\big(\pi(h)\Phi(\pi)(\sigma)\Psi(\pi)(\sigma)^*\big)\\
&=\sum_{\pi\in\widehat{H}}\sum_{\sigma\in\widehat{G/H}}tr\big(\Phi(\pi)(\sigma)\Psi(\pi)(\sigma)^*\pi\big(h^{-1}\big)\big).
\end{aligned}$$

Define a function $g \in L^2(H)$ as $\hat{g}(\pi) := \sum_{\sigma \in \widehat{G/H}} \Phi(\pi)(\sigma)\Psi(\pi)(\sigma)^*$. As $g \equiv 0$, $\hat{g}(\pi) = 0$ for all $\pi \in \widehat{H}$. Thus

$$\begin{aligned}\langle \Phi(\pi), \Psi(\pi) \rangle &= \sum_{\sigma \in \widehat{G/H}} \langle \Phi(\pi)(\sigma), \Psi(\pi)(\sigma) \rangle \\ &= \sum_{\sigma \in \widehat{G/H}} tr\big(\Phi(\pi)(\sigma)\Psi(\pi)(\sigma)^*\big) \\ &= tr\Bigg(\sum_{\sigma \in \widehat{G/H}} \Phi(\pi)(\sigma)\Psi(\pi)(\sigma)^*\Bigg) = 0 \quad \forall \pi \in \widehat{\pi}.\end{aligned}$$

Since this holds for all $\Phi \in \mathcal{T}(\mathcal{A})$, we have $\Psi(\pi) \in J(\pi)^{\perp}$ $\forall \pi \in \widehat{H}$. Also, since $\Psi \in M_J$, $\Psi \in J(\pi)$ for all $\pi \in \widehat{H}$. Thus $\Psi(\pi) = 0$ for all $\pi \in \widehat{H}$ and hence $\Psi \equiv 0$. Thus $M = M_J$.

The converse is similar to the proof of Theorem 3.10 in [3], since $\mathcal{T}(L_h f)(\pi) = \pi(h)Tf(\pi)$ for all $\pi \in \widehat{H}$. □

Now as in the abelian case [3], we can conclude the following corollary.

Corollary 4.4. *The correspondence between the H-invariant spaces and the range function is one-to-one and onto.*

We also have the following results.

Corollary 4.5. *Let V be an H-invariant subspace of $L^2(G)$ and J the corresponding range function. If P and P_π are the orthogonal projections onto V and $J(\pi)$ respectively, then for every $f \in L^2(G)$, $\mathcal{T}(Pf)(\pi) = P_\pi(\mathcal{T}f(\pi))$.*

Proof. This follows from the definitions of J, P and P_π. □

Corollary 4.6. *Let V be an H-invariant subspace of $L^2(G)$ such that $V = \bigoplus_{\alpha \in I} V_\alpha$, where V_α is also an H-invariant subspace of $L^2(G)$. Let J and J_α, $\alpha \in I$, be the corresponding range functions associated with V and V_α, $\alpha \in I$, respectively. Then $J(\pi) = \bigoplus_{\alpha \in I} J_\alpha(\pi)$, $\forall \pi \in \widehat{H}$.*

Proof. This follows from Corollary 4.4. □

5. Decomposition

The aim of this section is to decompose an H-invariant space into principle H-invariant spaces whose generators give rise to "generalized Parseval frames." By a generalized Parseval frame we mean the following.

Definition 5.1. Let $(\wedge, d\alpha)$ be a measure space and $\mathcal{H}$ be a Hilbert space. A family $\{x_\alpha\colon \alpha \in \wedge\}$ of $\mathcal{H}$ is called a generalized frame if there exist two numbers $0 < A \leqslant B < \infty$, so that

$$A\|x\|^2 \leqslant \int_{\wedge} \big|\langle x, x_\alpha \rangle\big|^2 \, d\alpha \leqslant B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

If $A = B = 1$, then $\{x_\alpha\colon \alpha \in \wedge\}$ is called a generalized Parseval frame.

Let $\varphi \in L^2(G)$. Let $\bigoplus_{\pi\in\widehat{H}}(\mathcal{B}_2(\mathcal{H}_\pi),\varphi)$ denote the space consisting of functions $r:\widehat{H}\to\mathcal{B}_2(L^2(H\times G/H))$ satisfying

$$\sum_{\pi\in\widehat{H}}\sum_{\sigma\in\widehat{G/H}}\left\|r(\pi)\hat{\tilde{\varphi}}(\pi\otimes\sigma)\right\|^2_{\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})}<\infty$$

or equivalently

$$\sum_{\pi\in\widehat{H}}\left\|r(\pi)\sum_{\sigma\in\widehat{G/H}}\hat{\tilde{\varphi}}(\pi\otimes\sigma)\right\|^2_{\mathcal{B}_2(\mathcal{H}_\pi)}<\infty.$$

We endow this space with the following norm:

$$\|r\|^2:=\sum_{\pi\in\widehat{H}}\sum_{\sigma\in\widehat{G/H}}\left\|r(\pi)\hat{\tilde{\varphi}}(\pi\otimes\sigma)\right\|^2_{\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})}.$$

Proposition 5.2. *Let $\varphi\in L^2(G)$. Then $f\in\mathcal{S}(\varphi)$ if and only if $\hat{f}(\pi)=r(\pi)\hat{\tilde{\varphi}}(\pi)$ for some $r\in\bigoplus_{\pi\in\widehat{H}}(\mathcal{B}_2(\mathcal{H}_\pi),\varphi)$ and $\|f\|_2^2=\|r\|^2$.*

Proof. Let $f\in E_H(\varphi)$. Then $f(x)=\sum_{i=1}^n a_i\varphi(h_i^{-1}x), a_i\in\mathbb{C}, h_i\in H, 1\leqslant i\leqslant n$. Then we have

$$\hat{\tilde{f}}(\pi)=\sum_{i=1}^n a_i\pi(h_i)\hat{\tilde{\varphi}}(\pi)=\left(\sum_{i=1}^n a_i\pi(h_i)\right)\hat{\tilde{\varphi}}(\pi).$$

Let $r(\pi)=\sum_{i=1}^n a_i\pi(h_i)$. Then it is clear that $r\in\bigoplus_{\pi\in\widehat{H}}(\mathcal{B}_2(\mathcal{H}_\pi),\varphi)$. Also any expression of the above form gives rise to an $f\in E_H(\varphi)$. Thus $f\in E_H(\varphi)$ if and only if $\hat{\tilde{f}}(\pi)=r(\pi)\hat{\tilde{\varphi}}(\pi)$, where r is as mentioned above. Now, let $\mathscr{P}$ denote the set of all such elements. Define a mapping

$$U:E_H(\varphi)\to\mathscr{P}$$

given by

$$U(f)=r.$$

Clearly U is onto. Further

$$\begin{aligned}\|f\|_2^2&=\|\tilde{f}\|_2^2\\&=\sum_{\pi\in\widehat{H}}\sum_{\sigma\in\widehat{G/H}}\left\|\hat{\tilde{f}}(\pi\otimes\sigma)\right\|^2_{\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})}\\&=\sum_{\pi\in\widehat{H}}\sum_{\sigma\in\widehat{G/H}}\left\|r(\pi)\hat{\tilde{\varphi}}(\pi\otimes\sigma)\right\|^2_{\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})}=\|r\|^2_{\bigoplus_{\pi\in\widehat{H}}(\mathcal{B}_2(\mathcal{H}_\pi),\varphi)}.\end{aligned}$$

Therefore U is unitary. Thus there is a unique isometry $\widetilde{U}:\mathcal{S}(\varphi)\to\overline{\mathscr{P}}$ which extends U from $\mathcal{S}$ to $\overline{\mathscr{P}}$. By Theorem 2.4 $\overline{\mathscr{P}}$ is $\bigoplus_{\pi\in\widehat{H}}(\mathcal{B}_2(\mathcal{H}_\pi),\varphi)$. □

For $\varphi\in L^2(G)$, define

$$w_\varphi(\pi):=\sum_{\sigma\in\widehat{G/H}}\hat{\tilde{\varphi}}(\pi\otimes\sigma)\hat{\tilde{\varphi}}(\pi\otimes\sigma)^*$$

and let

$$\Omega = \{\pi \in \widehat{H}\colon\ w_\varphi(\pi) \text{ is invertible}\}.$$

Theorem 5.3. *Let $\varphi \in L^2(G)$. The shifts of φ (w.r.t. H) form a generalized Parseval frame for the space $\mathcal{S}(\varphi)$ if*

$$w_\varphi(\pi) = \begin{cases} I & \text{if } \pi \in \Omega, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let $\varphi \in L^2(G)$. By Proposition 5.2, for every $f \in \mathcal{S}(\varphi)$, we have $\hat{\tilde{f}}(\pi) = r(\pi)\hat{\tilde{\varphi}}(\pi)$. So

$$\begin{aligned}
\langle f, L_h\varphi\rangle &= \langle \tilde{f}, \widetilde{L_h\varphi}\rangle \\
&= \langle \hat{\tilde{f}}, \widehat{\widetilde{L_h\varphi}}\rangle \\
&= \sum_{\chi \in \widehat{H\times G/H}} \big\langle \hat{\tilde{f}}(\chi), \widehat{\widetilde{L_h\varphi}}(\chi)\big\rangle \\
&= \sum_{\pi\in\widehat{H}} \sum_{\sigma\in\widehat{G/H}} \big\langle r(\pi)\hat{\tilde{\varphi}}(\pi\otimes\sigma), \pi(h)\hat{\tilde{\varphi}}(\pi\otimes\sigma)\big\rangle \\
&= \sum_{\pi\in\widehat{H}} \sum_{\sigma\in\widehat{G/H}} tr\big(r(\pi)\hat{\tilde{\varphi}}(\pi\otimes\sigma)\hat{\tilde{\varphi}}(\pi\otimes\sigma)^*\pi\big(h^{-1}\big)\big) \\
&= \sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)\pi\big(h^{-1}\big)\big).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_H \big|\langle f, L_h\varphi\rangle\big|^2\,dh &= \int_H \langle f, L_h\varphi\rangle\overline{\langle f, L_h\varphi\rangle}\,dh \\
&= \int_H \sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)\pi\big(h^{-1}\big)\big) \times \overline{\sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)\pi\big(h^{-1}\big)\big)}\,dh \\
&= \sum_{\pi\in\widehat{H}} \big\langle r(\pi)w_\varphi(\pi), r(\pi)w_\varphi(\pi)\big\rangle \\
&= \sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)w_\varphi(\pi)^*r(\pi)^*\big).
\end{aligned}$$

Thus

$$\int_H \big|\langle f, L_h\varphi\rangle\big|^2\,dh = \sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)w_\varphi(\pi)^*r(\pi)\big). \tag{1}$$

Using (1), it follows from the definition of generalized Parseval frame that $\{L_h\varphi\colon\ h \in H\}$ is a generalized Parseval frame if and only if

$$\sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)w_\varphi(\pi)^*r(\pi)\big) = \|f\|_2^2.$$

This essentially means

$$\sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)w_\varphi(\pi)^* r(\pi)\big) = \sum_{\pi\in\widehat{H}} \bigg\| r(\pi) \sum_{\sigma\in\widehat{G/H}} \hat{\hat{\varphi}}(\pi\otimes\sigma)\bigg\|^2_{\mathcal{B}_2(\mathcal{H}_\pi)}$$

by Proposition 5.2. But

$$\sum_{\pi\in\widehat{H}} \bigg\| r(\pi) \sum_{\sigma\in\widehat{G/H}} \hat{\hat{\varphi}}(\pi\otimes\sigma)\bigg\|^2_{\mathcal{B}_2(\mathcal{H}_\pi)} = \sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)r(\pi)^*\big).$$

Thus we can conclude that $\{L_h\varphi\colon h\in H\}$ forms a generalized Parseval frame if and only if

$$\sum_{\pi\in\widehat{H}} tr\big(r(\pi)w_\varphi(\pi)\big(w_\varphi(\pi)-I\big)^* r(\pi)^*\big) = 0,$$

from which the required result follows. □

Corollary 5.4. *Let $\varphi\in L^2(G)$. Define $\psi\in L^2(G)$ as*

$$\hat{\hat{\psi}}(\pi) = \begin{cases} w_\varphi(\pi)^{-1/2}\hat{\hat{\varphi}}(\pi) & \text{if } \pi\in\Omega,\\ 0 & \text{otherwise}.\end{cases}$$

Then $\{L_h\psi\colon h\in H\}$ is a generalized Parseval frame for $\mathcal{S}(\varphi)$.

Proof. By Proposition 5.2, $\psi\in\mathcal{S}(\varphi)$. Also it is clear that ψ satisfies Theorem 5.3 and hence the proof is complete. □

Theorem 5.5. *If V is an H-invariant subspace of $L^2(G)$, then there exists a family of functions $\{\varphi_\alpha\colon \alpha\in I\}$ in $L^2(G)$ (where I is an index set), such that $V=\bigoplus_{\alpha\in I}\mathcal{S}(\varphi_\alpha)$ and $\{L_h\varphi_\alpha\colon h\in H\}$ is a generalized Parseval frame for $\mathcal{S}(\varphi_\alpha)$.*

The proof is exactly the same as in the abelian group case. See [9].

6. Range operators and H-preserving operators

Using the decomposition obtained in the previous section, we shall study H-preserving operators and range operators in this section.

Definition 6.1. A bounded operator $T: V\to L^2(G)$ defined on an H-invariant space V is H-preserving if $TL_h = L_hT$ $\forall h\in H$.

Definition 6.2. Let V be a shift invariant subspace of $L^2(G)$ with J the range function given in Theorem 4.3 and P its associated projection. A range operator on J is a mapping $R:\widehat{H}\to\{$bounded operators on closed subspaces of $l^2(\widehat{G/H},\mathcal{B}_2(L^2(H\times G/H)))\}$ so that the domain of $R(\pi)$ equals $J(\pi)$ and such that $R(\pi)$ is a bounded operator from a closed subspace of $\bigoplus_{\sigma\in\widehat{G/H}}\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})$ to $\bigoplus_{\sigma\in\widehat{G/H}}\mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})$ and $R(\pi)(\pi(h)a)=\pi(h)R(\pi)(a)$.

Theorem 6.3. *Suppose $V \subset L^2(G)$ is an H-invariant space and J is its range function.*

(i) *For every H-preserving operator $T : V \to L^2(G)$ there exists a range operator R on J such that*

$$(\mathcal{T} \circ T) f(\pi) = R(\pi)\big(\mathcal{T} f(\pi)\big) \quad \forall \pi \in \widehat{H},\ f \in V.$$

The resulting R satisfies

$$\sup_{\pi \in \widehat{G}} \big\| R(\pi) \big\| < \infty.$$

(ii) *Conversely, given a range operator R on J with $\sup_{\pi \in \widehat{H}} \|R(\pi)\| < \infty$ there is a bounded H-preserving operator $T : V \to L^2(G)$ such that $(\mathcal{T} \circ T) f(\pi) = R(\pi)(\mathcal{T} f(\pi))$ $\forall \pi \in \widehat{H}$, $f \in V$ holds.*

Proof. (i) Decompose V into principal H-invariant spaces indexed by some set, say I. Hence

$$V = \bigoplus_{\alpha \in I} \mathcal{S}(\varphi_\alpha). \tag{2}$$

Let J_α be the corresponding range function. Define, for each $\pi \in \widehat{H}$, $\alpha \in I$,

$$R_\alpha(\pi) : J_\alpha(\pi) \to \bigoplus_{\sigma \in \widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi \otimes \sigma})$$

by

$$R_\alpha(\pi)\big(\mathcal{T}(L_h \varphi_\alpha)\big) = \pi(h)\big(\mathcal{T} \circ T \circ \mathcal{T}^{-1}\big)(\mathcal{T}\varphi_\alpha)(\pi).$$

Note that for each α, R_α is a bounded operator. For a given $f \in \mathcal{S}(\varphi_\alpha)$, let $\{g_n\} \subset span\, E_H(\varphi)$ be such that $g_n \to f$. Let $\Phi_n = \mathcal{T}(g_n)$. Then

$$\begin{aligned}(\mathcal{T} \circ T) f(\pi) &= \mathcal{T}(Tf)(\pi) = \mathcal{T}\Big(T\Big(\lim_n g_n\Big)\Big)(\pi) = \lim_n \mathcal{T}\big(T\mathcal{T}^{-1}(\Phi_n)\big)(\pi) \\ &= \lim_n \big(\mathcal{T} \circ T \circ \mathcal{T}^{-1}\big)\Phi_n(\pi) = \big(\mathcal{T} \circ T \circ \mathcal{T}^{-1}\big) \lim_n (\mathcal{T} g_n)(\pi) \\ &= \big(\mathcal{T} \circ T \circ \mathcal{T}^{-1}\big)(\mathcal{T} f)(\pi) = \big(R_\alpha(\pi) \circ \mathcal{T}\big)(f)(\pi).\end{aligned}$$

Now define $R(\pi)(f) = R(\pi)(\sum_{\alpha \in I} f_\alpha) = \sum_{\alpha \in I} R_\alpha(\pi)(f_\alpha)$. The operator $R(\pi)$ maps $J(\pi)$ into $\bigoplus_{\sigma \in \widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi \otimes \sigma})$. Also $R(\pi)$ is bounded. Further $R(\pi)$ commutes with $\pi(h)$ for all $h \in H$.

By (2), any $f \in V$ is of the form $(\sum_\alpha f_\alpha)$, $f_\alpha \in \mathcal{S}(\varphi_\alpha)$. Hence,

$$\begin{aligned}(\mathcal{T} \circ T)(f)(\pi) &= (\mathcal{T} \circ T)\Big(\sum_\alpha f_\alpha\Big)(\pi) \\ &= \sum_\alpha (\mathcal{T} \circ T) f_\alpha(\pi) \\ &= \sum_\alpha \big(R_\alpha(\pi) \circ \mathcal{T}\big) f_\alpha(\pi) = \big(R(\pi) \circ \mathcal{T}\big)(f)(\pi).\end{aligned}$$

Now we shall show that $\sup_{\pi \in \widehat{H}} \|R(\pi)\| < \infty$. Let $\Phi \in M_J$ be such that $\|\Phi\| \leqslant 1$. We shall show that $\sup_{\pi \in \widehat{H}} \|R(\pi)\Phi(\pi)\| \leqslant \|T\|$. If it were not true, then there would exist an $\epsilon > 0$ and a $\theta \in \widehat{H}$ such that

$$\big\| R(\theta)\Phi(\theta)\big\| > \|T\| + \epsilon.$$

Consider $f \in V$ given by

$$\mathcal{T}f(\pi) = \begin{cases} id_{H_\theta}\Phi(\theta) & \text{if } \pi = \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\big\|(\mathcal{T}\circ T)f\big\| = \|Tf\| \leqslant \|T\|\|f\|.$$

Also,

$$\begin{aligned} \big\|(\mathcal{T}\circ T)f\big\|^2 &= \sum_{\pi\in\widehat{H}} \big\|(\mathcal{T}\circ T)f(\pi)\big\|^2 \\ &= \sum_{\pi\in\widehat{H}} \big\|R(\pi)\mathcal{T}f(\pi)\big\|^2 \\ &= \big\|R(\theta)\Phi(\theta)\big\|^2 \\ &\geqslant \big(\|T\| + \epsilon\big)^2 \big\|\Phi(\theta)\big\|^2 = \big(\|T\| + \epsilon\big)^2 \|f\|^2, \end{aligned}$$

which is a contradiction, thus proving our result.

(ii) For $f \in V$, define $\widetilde{F}(\pi) = R(\pi)(\mathcal{T}f(\pi))$. Define

$$T : V \to L^2(G)$$

as

$$Tf = \mathcal{T}^{-1}(\widetilde{F}).$$

Then T is linear. Since $\sup\|R(\pi)\| < \infty$, T is bounded. By a straightforward computation we can show that $T(L_h f) = L_h Tf$ for all $f \in V$. □

Theorem 6.4. *Suppose T is an H-preserving operator on V and R is its range operator on J. Then T is bounded below with constant $c > 0$ i.e.,*

$$\|Tf\| \geqslant c\|f\| \quad \forall f \in V$$

if and only if $\forall\pi \in \widehat{H}$ $R(\pi)$ is bounded below with the same constant c.

Proof. One part of the proof easily follows from the definitions. For the other part, let us assume that $\|Tf\| \geqslant c\|f\|$ for all $f \in V$. In order to prove that $R(\pi)$ is bounded below with the constant c, we mention that it is enough to prove the following:

$$\big\|R(\pi)P(\pi)a\big\| \geqslant \big\|P(\pi)a\big\| \quad \forall a \in \mathscr{A}_\pi,$$

where $\mathscr{A}_\pi$ is a dense subset of $\bigoplus_{\sigma\in\widehat{G/H}} \mathcal{B}_2(\mathcal{H}_{\pi\otimes\sigma})$. Now the proof follows similar steps as in [2]. □

We can also obtain the following result by proceeding in the same direction as in the proof given in [2].

Theorem 6.5. *Suppose $V \subseteq L^2(G)$ is an H-invariant space with its range function J and $T : V \to V$ is an H-preserving operator with the associated range operator R.*

(i) *The dual operator $T^* : V \to V$ is also H-preserving with its corresponding range operator R^* given by $R^*(\pi) := R(\pi)^*$ for all $\pi \in \widehat{H}$.*
(ii) *Let $A \leqslant B$ be two real numbers. T is self-adjoint with $\sigma(T) \subset [A, B]$ if and only if $R(\pi)$ is self-adjoint with $\sigma(R(\pi)) \subset [A, B]$ for all $\pi \in \widehat{H}$.*
(iii) *T is unitary if and only if $R(\pi)$ is unitary for all $\pi \in \widehat{H}$.*

Acknowledgements

We thank National Board for Higher Mathematics, Department of Atomic Energy, India for the project fund with the Project No. 2/48(10)/2009-R&D 11/1086.

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