SHARP BOHR TYPE INEQUALITY

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Abstract. This article is devoted to the sharp improvement of the classical Bohr inequality for bounded analytic functions defined on the unit disk. We also prove two other sharp versions of the Bohr inequality by replacing the constant term by the absolute of the function and the square of the absolute of the function, respectively.

1. Introduction and Main Results

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Bohr's theorem (after subsequent improvements due to M. Riesz, I. Schur and F. Wiener) states that if $\sum_{k=0}^{\infty} a_k z^k$ is a bounded analytic function in $\mathbb{D}$ such that $|f(z)| \leq 1$ in $\mathbb{D}$, then [12]

$$B_f(r) := \sum_{k=0}^{\infty} |a_k| r^k \leq 1 \quad \text{for any } r \leq 1/3, \tag{1}$$

and the value 1/3 is sharp. There are many proofs of this inequality (cf. [37] and [38]). Also, we would to recall that Bombieri [13] proved the following which gives an upper bound on the growth of $B_f(r)$:

$$\sup B_f(r) = \frac{3-\sqrt{8(1-r^2)}}{r}, \quad 1/3 \leq r \leq 1/\sqrt{2}.$$

Bombieri and Bourgain [14] investigated asymptotical behaviour of Bohr's sums as $r \to 1$. In fact, they constructed a_n, and by a delicate analysis of exponential sums, proved that when $r \to 1$,

$$B_f(r) \geq (1-r^2)^{-1/2} - \left(c \log \frac{1}{1-r}\right)^{3/2+\epsilon},$$

where $c = c(\epsilon)$ depends on ϵ.

This result has created enormous interest on Bohr's inequality in various settings. See for example, [8–11, 14–20, 25–27, 30, 33, 35] and the recent survey on this topic by Abu-Muhanna et al. [7], [24], [23, Chapter 8] and the references therein. Some of these articles use various methods from complex analysis, functional analysis, number theory, and probability, and furthermore they provide new theory and applications of Bohr's results on his work on Dirichlet series. For example, several multidimensional generalizations of this result are obtained in [3–6, 11, 21]. Moreover, in [28], the authors answer the open problem about the powered Bohr radius posed by Djakov and Ramanujan in 2000.

For the case $a_0 = 0$, Tomić [38] showed that (1) holds for $0 \leq r \leq 1/2$. Later Ricci [36] improved it by showing that (1) holds for $0 \leq r \leq 3/5$, and the largest value of r for which

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(1) holds in this case would be in the interval $(3/5, 1/\sqrt{2}]$. Later in 1962, Bombieri [13] obtained the sharp upper estimate for $\sum_{k=0}^{\infty}|a_k|r^k$ in the case $r \in (1/3, 1/\sqrt{2}]$. See [25,26,28] for new proofs of it in a general form.

If f has a higher order zero at the origin, then we have only a partial answer about the range for r (see [32, Remark 2] and also [34] for some related investigation on this problem).

It is worth pointing out that the notion of Bohr's radius, initially defined for analytic functions from the unit disk $\mathbb{D}$ into $\mathbb{D}$, was generalized by authors to include mappings from $\mathbb{D}$ to some other domains Ω in $\mathbb{D}$ ([1,2,5]).

Unless otherwise stated, throughout this paper $S_r(f)$ denotes the area of the image of the subdisk $|z| < r$ under the mapping f and when there is no confusion, we let for brevity S_r for $S_r(f)$. More recently, Kayumov and Ponnusamy [29] improved Bohr's inequality (1) in various forms. For example, the following inequality is obtained in [27] (see also [29]).

Theorem A. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$|a_0| + \sum_{k=1}^{\infty}|a_k|r^k + \frac{16}{9}\left(\frac{S_r}{\pi}\right) \le 1 \quad \text{for} \quad r \le \frac{1}{3}, \tag{2}$$

and the numbers $1/3$ and $16/9$ cannot be improved.

Also, it is worth recalling that if the first term $|a_0|$ in (2) is replaced by $|a_0|^2$, then $1/3$ and $16/9$ could be replaced by $1/2$ and $9/8$, respectively (cf. [27]). For a harmonic analog of Bohr's inequality (1) and that of Theorem A, we refer to [22] and [27], respectively. More recently, the authors in [24] improved Theorem A by replacing the quantity S_r/π in the inequality (2) by $S_r/(\pi - S_r)$. Moreover, in [29] (see also [24, Corollary 3]), the following generalizations of Bohr's results were obtained. The idea used in these two articles [24], [25] were used for further investigation on Bohr's inequality by Liu et al. [31].

Theorem B. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then*

$$|f(z)| + \sum_{k=1}^{\infty}|a_k|r^k \le 1 \quad \text{for} \quad |z| = r \le \sqrt{5} - 2 \approx 0.236068$$

and no larger radius than $\sqrt{5} - 2$ will do. Moreover,

$$|f(z)|^2 + \sum_{k=1}^{\infty}|a_k|r^k \le 1 \quad \text{for} \quad |z| = r \le 1/3$$

and no larger radius than $1/3$ will do.

Our main goal of this article is to derive sharp version of Theorems A and B, and hence certain results from the survey article [24]. It is important to point out that for individual functions the Bohr radius is always greater than 1/3.

We now state our main results and their proofs will be presented in the next section. Moreover, in the interesting results which are presented below, there is an extremal function such that 1/3 cannot be increased. The price for such important fact is essentially due to nonlinearity of the obtained functionals.

Theorem 1. *Suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is analytic in* $\mathbb{D}$ *and* $|f(z)| \leq 1$ *in* $\mathbb{D}$*. Then*

$$M(r) := \sum_{k=0}^{\infty} |a_k| r^k + \frac{16}{9}\left(\frac{S_r}{\pi}\right) + \lambda \left(\frac{S_r}{\pi}\right)^2 \leq 1 \quad \text{for} \quad r \leq \frac{1}{3} \tag{3}$$

where

$$\lambda = \frac{4(486 - 261a - 324a^2 + 2a^3 + 30a^4 + 3a^5)}{81(1+a)^3(3-5a)} = 18.6095\ldots$$

and $a \approx 0.567284$*, is the unique positive root of the equation* $\psi(t) = 0$ *in the interval* $(0,1)$*, where*

$$\psi(t) = -405 + 473t + 402t^2 + 38t^3 + 3t^4 + t^5.$$

The equality is achieved for the function

$$f(z) = \frac{a - z}{1 - az}.$$

Remark 1. *It is evident that* λ *is an algebraic number. Moreover, it can be shown that* λ *is indeed the positive root of the algebraic equation*

$$285212672 + 6268596224x + 37178714880x^2 + 87178893840x^3 +$$

$$+ 97745285925x^4 - 5509980288x^5 = 0.$$

Remark 2. *From the proof of Theorem 1, the following observation is clear: for any function* $F : [0,\infty) \to [0,\infty)$ *such that* $F(t) > 0$ *for* $t > 0$*, there exist analytic functions* $f : \mathbb{D} \to \mathbb{D}$ *for which the inequality*

$$\sum_{k=0}^{\infty} |a_k| r^k + \frac{16}{9}\left(\frac{S_r}{\pi}\right) + \lambda \left(\frac{S_r}{\pi}\right)^2 + F(S_r) \leq 1 \quad \text{for} \quad r \leq \frac{1}{3}$$

is false, where λ *is as in Theorem 1.*

This result follows from the fact that there is a concrete function for which the equality in (3) holds so that one can add no strictly positive terms.

Theorem 2. *Suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is analytic in* $\mathbb{D}$ *and* $|f(z)| \leq 1$ *in* $\mathbb{D}$*. Then*

$$M(z,r) := |f(z)|^2 + \sum_{k=1}^{\infty} |a_k| r^k + \frac{16}{9}\left(\frac{S_r}{\pi}\right) + \lambda \left(\frac{S_r}{\pi}\right)^2 \leq 1 \quad \text{for} \quad |z| = r \leq \frac{1}{3}, \tag{4}$$

where

$$\lambda = \frac{-81 + 1044a + 54a^2 - 116a^3 - 5a^4}{162(a+1)^2(2a-1)} = 16.4618\ldots$$

and $a \approx 0.537869$ *is the unique positive root of the equation*

$$-513 + 910t + 80t^2 + 2t^3 + t^4 = 0$$

in the interval $(0,1)$*.*

The equality is achieved for the function

$$f(z) = \frac{a - z}{1 - az}.$$

Remark 3. *One can check that actually λ is the positive root of the algebraic equation*

$$575930368 + 4437874624x + 11353360788x^2 + 10868034060x^3 - 703096443x^4 = 0.$$

One can replace $|f(z)|^2$ by $|f(z)|$ in Theorem 2 but this will decrease the Bohr radius. Namely, the following theorem is valid.

Theorem 3. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$|f(z)| + \sum_{k=1}^{\infty} |a_k| r^k + p\left(\frac{S_r}{\pi}\right) \le 1 \quad \text{for} \quad |z| = r \le \sqrt{5} - 2 \approx 0.236068, \tag{5}$$

where the constants $r_0 = \sqrt{5} - 2$ and $p = 2(\sqrt{5} - 1)$ are sharp.

2. Proofs of Theorem 1, 2 and 3

For the proof of our results, we need the following lemmas.

Lemma C. [25, Lemma 2] *Let $|b_0| < 1$ and $0 < r \le 1$. If $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is analytic and satisfies the inequality $|g(z)| \le 1$ in $\mathbb{D}$, then the following sharp inequality holds:*

$$\sum_{k=1}^{\infty} |b_k|^2 r^{pk} \le r^p \frac{(1 - |b_0|^2)^2}{1 - |b_0|^2 r^p}. \tag{6}$$

Lemma D. [29, Lemma 1] *Suppose that $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is analytic in $\mathbb{D}$, $|g(z)| < 1$ in $\mathbb{D}$ and $S_r(g)$ denotes the area of the image of the subdisk $|z| < r$ under the mapping g. Then the following sharp inequality holds:*

$$\frac{S_r(g)}{\pi} := \sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \le r^2 \frac{(1 - |b_0|^2)^2}{(1 - |b_0|^2 r^2)^2} \quad \text{for } 0 < r \le 1/\sqrt{2}. \tag{7}$$

Remark 4. *Lemma C is not true for $r > 1/\sqrt{2}$ unless g is univalent in $\mathbb{D}$. For instance, consider $g(z) = z^n$ where $n \ge 2$.*

Lemma E. *Let $p \in \mathbb{N}$, $0 \le m \le p$ and $f(z) = \sum_{k=0}^{\infty} a_{pk+m} z^{pk+m}$ be analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then the following inequalities hold:*

$$\sum_{k=1}^{\infty} |a_{pk+m}| r^{pk} \le \begin{cases} r^p \dfrac{1 - |a_m|^2}{1 - r^p |a_m|} & \text{for } |a_m| \ge r \\[2ex] r^p \dfrac{\sqrt{1 - |a_m|^2}}{\sqrt{1 - r^{2p}}} & \text{for } |a_m| < r. \end{cases} \tag{8}$$

Proof. Proof of this lemma follows from the proof of Theorem 1 in [26] (see also [24], and [25, Proof of Theorem 1] for the case $m = 0$ and [26]). The proof uses the classical Cauchy-Schwarz inequality and (6). However, for the sake of completeness, we supply here some details. At first, we write $f(z) = z^m g(z^p)$, where $|g(z)| \le 1$ in $\mathbb{D}$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is analytic in $\mathbb{D}$ with $b_k = a_{pk+m}$. Let $|b_0| = a$. Choose any $\rho > 1$ such that $\rho r \leq 1$. Then it follows from the classical Cauchy-Schwarz inequality and (6) with ρr in place of r that

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{pk+m}| r^{pk} &= \sum_{k=1}^{\infty} |b_k| r^{pk} \\ &\leq \sqrt{\sum_{k=1}^{\infty} |b_k|^2 \rho^{pk} r^{pk}} \sqrt{\sum_{k=1}^{\infty} \rho^{-pk} r^{pk}} \\ &\leq \sqrt{r^p \rho^p \frac{(1-a^2)^2}{1-a^2 r^p \rho^p}} \sqrt{\frac{\rho^{-p} r^p}{1-\rho^{-p} r^p}} \quad \text{(by Lemma C)} \\ &= \frac{r^p (1-a^2)}{\sqrt{1-a^2 r^p \rho^p}} \frac{1}{\sqrt{1-\rho^{-p} r^p}}.\end{aligned}$$

We need to consider the cases $a \geq r^p$ and $a < r^p$, independently. If $a \geq r^p$, then we may set $\rho = 1/\sqrt[p]{a}$. On the other hand, if $a < r^p$, then we set $\rho = 1/r$. As a result of these substitutions, we easily obtain that

$$\sum_{k=1}^{\infty} |b_k| r^{pk} \leq \begin{cases} r^p \dfrac{(1-a^2)}{1-r^p a} & \text{for } a \geq r^p \\ r^p \dfrac{\sqrt{1-a^2}}{\sqrt{1-r^{2p}}} & \text{for } a < r^p \end{cases}$$

and the desired inequalities (8) follow by setting $b_k = a_{pk+m}$ and $a = |a_m|$. □

In the proof of our main results, we just need Lemma D and the case $m = 0$ and $p = 1$ of Lemma E.

2.1. **Proof of Theorem 1.** Consider the function $M(r)$ given by (3). Since $M(r)$ is an increasing function of r, we have $M(r) \leq M(1/3)$ for $0 \leq r \leq 1/3$ and thus, it suffices to prove the inequality (3) for $r = 1/3$. Moreover, for $m = 0$ and $p = 1$, Lemma E gives

$$\sum_{k=1}^{\infty} |a_k| r^k \leq \begin{cases} A(r) := r\dfrac{1-|a_0|^2}{1-r|a_0|} & \text{for } |a_0| \geq r \\ B(r) := r\dfrac{\sqrt{1-|a_0|^2}}{\sqrt{1-r^2}} & \text{for } |a_0| < r. \end{cases} \tag{9}$$

At first we consider the case $|a_0| \geq 1/3$. In this case, using (9) (with $r = 1/3$) and Lemma D, we have

$$\begin{aligned} M(1/3) &\leq |a_0| + A(1/3) + \frac{16}{9\pi} S_{1/3} + \lambda \left(\frac{S_{1/3}}{\pi}\right)^2 \\ &\leq |a_0| + \frac{1-|a_0|^2}{3-|a_0|} + 16\frac{(1-|a_0|^2)^2}{(9-|a_0|^2)^2} + 81\lambda \frac{(1-|a_0|^2)^4}{(9-|a_0|^2)^4} \\ &= 1 - \frac{(1-|a_0|)^3}{(9-|a_0|^2)^4} \Phi(|a_0|), \end{aligned}$$

where

$$\begin{aligned}\Phi(t) &= 3078+1944t-522t^2-432t^3+2t^4+24t^5+2t^6\\ &\quad +\lambda(-81-243t-162t^2+162t^3+243t^4+81t^5).\end{aligned}$$

One can verify that the function $\Phi(t)$ in the interval $[1/3,1]$ has exactly one stationary point $t_0=0.567284\ldots$ which is the positive root of the equation $\Phi'(t)=0$.

Let us show that $t_0=a$ and that $\Phi(t_0)=0$. The equation $\Phi'(a)=0$ is fulfilled automatically (in fact, λ was chosen in such the way at the beginning). A little algebra gives

$$\Phi(a)=2\frac{a^2-9}{3-5a}\psi(a).$$

Consequently, $\Phi(a)=0$. Besides this observation, we have $\Phi(1/3)>0$ and $\Phi(1)>0$. Thus, $\Phi(t)\geq 0$ in the interval $[1/3,1]$ which proves that $M(r)\leq 1$ for $|a_0|\in[1/3,1]$ and for $r\leq 1/3$

Next we consider the case $|a_0|<1/3$. Again, using (9) (with $r=1/3$) and Lemma D, we deduce that

$$\begin{aligned}M(1/3) &\leq |a_0|+B(1/3)+16\frac{(1-|a_0|^2)^2}{(9-|a_0|^2)^2}+81\lambda\frac{(1-|a_0|^2)^4}{(9-|a_0|^2)^4}\\ &\leq |a_0|+\frac{\sqrt{1-|a_0|^2}}{\sqrt{8}}+16\frac{(1-|a_0|^2)^2}{(9-|a_0|^2)^2}+81\lambda\frac{(1-|a_0|^2)^4}{(9-|a_0|^2)^4}\\ &= \Psi(|a_0|).\end{aligned}$$

Routine and straightforward calculations show that the last expression for $\Psi(t)$ is an increasing function of t and so its maximum is achieved at the point $t_0=|a_0|=1/3$, and the maximum value of $\Psi(t_0)$ is seen to be less than 1 (in fact it is ≤ 0.98) so that the desired inequality (3) follows for $|a_0|\in[0,1/3)$. This proves that $M(r)\leq 1$ for $|a_0|\in[0,1/3)$ and for $r\leq 1/3$.

To prove that the constant λ is sharp, we consider the function f given by

$$f(z)=\frac{a-z}{1-az}=a-(1-a^2)\sum_{k=1}^{\infty}a^{k-1}z^k,\quad z\in\mathbb{D}, \tag{10}$$

where $a\in(0,1)$, and compute the value of $M(r)$ for this function. Indeed, we may let

$$M_1(1/3)=\sum_{k=0}^{\infty}|a_k|3^{-k}+\frac{16}{9\pi}S_{1/3}+\lambda_1\left(\frac{S_{1/3}}{\pi}\right)^2,$$

where $a_0=a$ and $a_k=-(1-a^2)a^{k-1}$. Straightforward calculations show that

$$M_1(1/3) = a+\frac{1-a^2}{3-a}+16\frac{(1-a^2)^2}{(9-a^2)^2}+81\lambda\frac{(1-a^2)^4}{(9-a^2)^4}+81(\lambda_1-\lambda)\frac{(1-a^2)^4}{(9-a^2)^4}$$

Choose a as the positive root t_0 of the equation $\psi(t)=0$. As a consequence, we see that

$$M_1(1/3)=1+81(\lambda_1-\lambda)\frac{(1-a^2)^4}{(9-a^2)^4}$$

which is bigger than 1 in case $\lambda_1>\lambda$. This proves the sharpness assertion and the proof is complete.

2.2. **Proof of Theorem 2.** Let $M(z,r)$ be defined by (4). As $M(z,r)$ is an increasing function of r, it suffices to prove the inequality (4) for $r = 1/3$. Moreover, by the assumption and the Schwarz-Pick lemma applied to the function f show that

$$|f(z)| \leq \frac{r+|a_0|}{1+r|a_0|} =: D(r), \quad |z| \leq r.$$

At first we consider the case $|a_0| \geq 1/3$. In this case, using (9) (with $r = 1/3$), Lemma D and the last inequality, we have for $r \leq 1/3$ that

$$\begin{aligned}
M(z,r) &\leq M(z,1/3) \\
&\leq (D(1/3))^2 + A(1/3) + \frac{16}{9\pi}S_{1/3} + \lambda\left(\frac{S_{1/3}}{\pi}\right)^2 \\
&\leq 1 - \left[1 - \left(\frac{1+3|a_0|}{3+|a_0|}\right)^2 - \frac{1-|a_0|^2}{3-|a_0|} - 16\frac{(1-|a_0|^2)^2}{(9-|a_0|^2)^2} - 81\lambda\frac{(1-|a_0|^2)^4}{(9-|a_0|^2)^4}\right] \\
&= 1 - (1-|a_0|)^2(1+|a_0|)\left[\frac{|a_0|+15}{(3+|a_0|)^2(3-|a_0|)} - 16\frac{(1+|a_0|)}{(9-|a_0|^2)^2}\right. \\
&\qquad \left. -81\lambda\frac{(1-|a_0|^2)^2(1+|a_0|)}{(9-|a_0|^2)^4}\right] \\
&= 1 - (1-|a_0|)^3(1+|a_0|)\left[\frac{|a_0|+29}{(9-|a_0|^2)^2} - 81\lambda\frac{(1-|a_0|^2)(1+|a_0|)^2}{(9-|a_0|^2)^4}\right] \\
&= 1 - \frac{(1-|a_0|)^3(1+|a_0|)}{(9-|a_0|^2)^4}\Phi(|a_0|),
\end{aligned}$$

where

$$\Phi(t) = 2349 + 81t - 522t^2 - 18t^3 + 29t^4 + t^5 + \lambda(-81 - 162t + 162t^3 + 81t^4), \ 1/3 \leq t \leq 1.$$

We see that the function $\Phi(t)$ in the interval $[1/3, 1]$ has exactly one stationary point $t_0 = 0.537869\ldots$ which is the positive root of the equation $\Phi'(t) = 0$.

Let us show that $t_0 = a$ and that $\Phi(t_0) = 0$. The equation $\Phi'(a) = 0$ is fulfilled automatically. A little algebra gives

$$\Phi(a) = \frac{9-a^2}{2(2a-1)}(-513 + 910a + 80a^2 + 2a^3 + a^4)$$

so that $\Phi(t_0) = 0$.

Besides this observation, we find that $\Phi(1/3) > 0$ and $\Phi(1) > 0$. Thus, $\Phi(t) \geq 0$ in the interval $[1/3, 1]$ which proves that $M(z,r) \leq 1$ for $|a_0| \in [1/3, 1]$ and $r \leq 1/3$. Thus, the desired inequality (4) follows for $|a_0| \in [1/3, 1]$.

Next we consider the case $|a_0| < 1/3$. Again, using (9) (with $r = 1/3$) and Lemma D, we deduce that

$$\begin{aligned} M(z,r) &\leq (D(1/3))^2 + B(1/3) + \frac{16}{9\pi}S_{1/3} + \lambda\left(\frac{S_{1/3}}{\pi}\right)^2 \\ &\leq \left(\frac{1+3|a_0|}{3+|a_0|}\right)^2 + \frac{\sqrt{1-|a_0|^2}}{\sqrt{8}} + 16\frac{(1-|a_0|^2)^2}{(9-|a_0|^2)^2} + 81\lambda\frac{(1-|a_0|^2)^4}{(9-|a_0|^2)^4} \\ &= \Psi(|a_0|). \end{aligned}$$

Routine and straightforward calculations show that the last expression for $\Psi(t)$ is an increasing function of t in the interval $[0, 1/3]$, and so its maximum is achieved at the point $t_0 = |a_0| = 1/3$, and the maximum value of $\Psi(t_0)$ is seen to be less than 1 (in fact it is ≤ 0.987) so that the desired inequality (4) follows for $|a_0| \in [0, 1/3)$. This proves the inequality (4).

Finally, to prove that the constant λ is sharp, as in the previous theorem, we consider the function f given by (10) and compute the value of $M(z,r)$ for this function. Indeed, we may let

$$M_1(z,r) = \left(\frac{1+3a}{3+a}\right)^2 + \sum_{k=1}^{\infty}|a_k|3^{-k} + \frac{16}{9\pi}S_{1/3} + \lambda_1\left(\frac{S_{1/3}}{\pi}\right)^2,$$

where $a_0 = a$ and $a_k = -(1-a^2)a^{k-1}$. Straightforward calculations show that

$$M_1(z,r) = \left(\frac{1+3a}{3+a}\right)^2 + \frac{1-a^2}{3-a} + 16\frac{(1-a^2)^2}{(9-a^2)^2} + 81\lambda\frac{(1-a^2)^4}{(9-a^2)^4} + 81(\lambda_1-\lambda)\frac{(1-a^2)^4}{(9-a^2)^4}.$$

Choose a as the positive root t_0 of the equation $-513 + 910t + 80t^2 + 2t^3 + t^4 = 0$. As a consequence, we see that

$$M_1(z,r)) = 1 + 81(\lambda_1-\lambda)\frac{(1-a^2)^4}{(9-a^2)^4}$$

which is bigger than 1 when $\lambda_1 > \lambda$. The proof of the theorem is complete.

2.3. **Proof of Theorem 3.** Let

$$M(r) := \sup_{|z|=r}|f(z)| + \sum_{k=1}^{\infty}|a_k|r^k + p\left(\frac{S_r}{\pi}\right).$$

In the case $|a_0| = a \geq r$ from (9) it follows that

$$M(r) \leq M_1(r) = \frac{r+a}{1+ar} + r\frac{1-a^2}{1-ra} + p\frac{(1-a^2)r^2}{(1-a^2r^2)^2}.$$

Clearly, it suffices to show that

$$M_1(\sqrt{5}-2) = \frac{a+\sqrt{5}-2}{\left(\sqrt{5}-2\right)a+1} + \frac{\left(\sqrt{5}-2\right)\left(a^2-1\right)}{\left(\sqrt{5}-2\right)a-1} + p\frac{\left(9-4\sqrt{5}\right)\left(a^2-1\right)^2}{\left(\left(4\sqrt{5}-9\right)a^2+1\right)^2} \leq 1$$

which is equivalent to the following inequality

$$\frac{(1-a)^3(7(-9+4\sqrt{5})+4(-47+21\sqrt{5})a+(-161+72\sqrt{5})a^2)}{\left(\left(4\sqrt{5}-9\right)a^2+1\right)^2}\leq 0.$$

The last inequality is clearly valid in the unit interval $[0,1]$ because $-9+4\sqrt{5}\leq 0$, $-47+21\sqrt{5}\leq 0$ and $-161+72\sqrt{5}\leq 0$.

It means that the desired inequality (5) is correct in the case $|a_0|\geq r$ and $r\leq\sqrt{5}-2$.

Next we consider the $|a_0|=a\leq r$. In this case, from (9) we obtain that

$$M(r)\leq M_2(r)=\frac{r+a}{1+ar}+r\frac{\sqrt{1-a^2}}{\sqrt{1-r^2}}+p\frac{(1-a^2)r^2}{(1-a^2r^2)^2}.$$

Again routine computations show that the last expression is an increasing function of r for $a\in[0,r]$ and hence its maximum is achieved at the point $a=r$. This case was settled in the previous considerations.

Let us finally show that the estimate for p cannot be improved. We set

$$f(z)=\frac{z+a}{1+az}$$

and choose $z=r=\sqrt{5}-2$. One can check that

$$|f(z)|+\sum_{k=1}^{\infty}|a_k|r^k+(2(\sqrt{5}-1)+\varepsilon)\frac{S_r}{\pi}=$$

$$\frac{a+\sqrt{5}-2}{\left(\sqrt{5}-2\right)a+1}+\frac{\left(\sqrt{5}-2\right)\left(a^2-1\right)}{\left(\sqrt{5}-2\right)a-1}+(2(\sqrt{5}-1)+\varepsilon)\frac{\left(9-4\sqrt{5}\right)\left(a^2-1\right)^2}{\left(\left(4\sqrt{5}-9\right)a^2+1\right)^2}=$$

$$1+\frac{(1-a)^3\left(a\left(\left(72\sqrt{5}-161\right)a+84\sqrt{5}-188\right)+7\left(4\sqrt{5}-9\right)\right)}{\left(\left(4\sqrt{5}-9\right)a^2+1\right)^2}+\varepsilon\frac{\left(9-4\sqrt{5}\right)\left(a^2-1\right)^2}{\left(\left(4\sqrt{5}-9\right)a^2+1\right)^2}.$$

From here we see that in the case when $a\to 1$ this expression behaves like $1+C(1-a)^2\varepsilon>1$. This concludes the proof.

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References

1. Y. Abu-Muhanna, Bohr's phenomenon in subordination and bounded harmonic classes, *Complex Var. Elliptic Equ.* **55**(11) (2010), 1071–1078.
2. Y. Abu-Muhanna and R. M. Ali, Bohr's phenomenon for analytic functions into the exterior of a compact convex body, *J. Math. Anal. Appl.* **379**(2) (2011), 512–517.
3. L. Aizenberg, Multidimensional analogues of Bohr's theorem on power series, *Proc. Amer. Math. Soc.* **128**(4) (1999), 1147–1155.

4. L. Aizenberg, Generalization of Carathéodory's inequality and the Bohr radius for multidimensional power series, in *Selected topics in complex analysis*, pp. 87–94. *Oper. Theory Adv. Appl.*, **158** Birkhäuser, Basel (2005).
5. L. Aizenberg, Generalization of results about the Bohr radius for power series, *Stud. Math.* **180** (2007), 161–168.
6. L. Aizenberg, A. Aytuna, P. Djakov, Generalization of a theorem of Bohr for basis in spaces of holomorphic functions of several complex variables, J. Math. Anal. Appl. **258**(2) (2001), 429–447.
7. R. M. Ali, Y. Abu-Muhanna and S. Ponnusamy, On the Bohr inequality, In "Progress in Approximation Theory and Applicable Complex Analysis" (Edited by N.K. Govil et al.), Springer Optimization and Its Applications **117** (2016), 265–295.
8. S. A. Alkhaleefah, I. R. Kayumov and S. Ponnusamy, On the Bohr inequality with a fixed zero coefficient, *Proc. Amer. Math. Soc.* **147**(12) (2019), 5263–5274.
9. C. Bénéteau, A. Dahlner and D. Khavinson, Remarks on the Bohr phenomenon, *Comput. Methods Funct. Theory* **4**(1) (2004), 1–19.
10. B. Bhowmik and N. Das, Bohr phenomenon for subordinating families of certain univalent functions, *J. Math. Anal. Appl.* **462**(2) (2018), 1087–1098.
11. H. P. Boas and D. Khavinson, Bohr's power series theorem in several variables, *Proc. Amer. Math. Soc.* **125**(10) (1997), 2975–2979.
12. H. Bohr, A theorem concerning power series, *Proc. London Math. Soc.* **13**(2) (1914), 1–5.
13. E. Bombieri, Sopra un teorema di H. Bohr e G. Ricci sulle funzioni maggioranti delle serie di potenze, *Boll. Un. Mat. Ital.* **17** (3)(1962), 276–282.
14. E. Bombieri and J. Bourgain, A remark on Bohr's inequality, *Int. Math. Res. Not.* **80** (2004), 4307–4330.
15. A. Defant and L. Frerick, A logarithmic lower bound for multi-dimensional bohr radii, Israel J. Math. **152**(1) (2006), 17–28.
16. A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, K. Seip, The Bohnenblust-Hille inequality for homogenous polynomials is hypercontractive, Ann. of Math. (2) **174** (2011), 512–517.
17. A. Defant, D. García and M. Maestre, Bohr's power series theorem and local Banach space theory, J. reine angew. Math. **557** (2003), 173–197.
18. A. Defant, S. R. Garcia and M. Maestre, Asymptotic estimates for the first and second Bohr radii of Reinhardt domains, *J. Approx. Theory* **128** (2004), 53–68.
19. A. Defant, M. Maestre, and U. Schwarting, Bohr radii of vector valued holomorphic functions, Adv. Math. **231**(5) (2012), 2837–2857.
20. A. Defant and C. Prengel, Christopher Harald Bohr meets Stefan Banach. Methods in Banach space theory, London Math. Soc. Lecture Note Ser. **337**, Cambridge Univ. Press, Cambridge (2006), 317–339.
21. P. B. Djakov and M. S. Ramanujan, A remark on Bohr's theorems and its generalizations, *J. Analysis* **8** (2000), 65–77.
22. S. Evdoridis, S. Ponnusamy and A. Rasila, Improved Bohr's inequality for locally univalent harmonic mappings, *Indag. Math. (N.S.)*, **30** (2019), 201–213.
23. S. R. Garcia, J. Mashreghi and W. T. Ross, *Finite Blaschke products and their connections*, Springer, Cham, 2018.
24. A. Ismagilov, A. Kayumova, I. R. Kayumov and S. Ponnusamy, Bohr type inequalities in some classes of analytic functions, *Complex analysis, Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz.*, **153**, VINITI, Moscow, (2018), 69-83 (in Russian)
25. I. R. Kayumov and S. Ponnusamy, Bohr inequality for odd analytic functions, *Comput. Methods Funct. Theory* **17** (2017), 679–688.
26. I.R. Kayumov, S. Ponnusamy, Bohr's inequalities for the analytic functions with lacunary series and harmonic functions, *J. Math. Anal. and Appl.*, **465**(2)(2018), 857 – 871.
27. I.R. Kayumov, S. Ponnusamy, "Improved version of Bohr's inequality", *C. R. Math. Acad. Sci. Paris* **356**(3)(2018), 272–277.

28. I. R. Kayumov and S. Ponnusamy, On a powered Bohr inequality, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **44**(2019), 301–310.
29. I. R. Kayumov and S. Ponnusamy, Bohr–Rogosinski radius for analytic functions, Preprint. See `https://arxiv.org/abs/1708.05585`
30. I. R. Kayumov, S. Ponnusamy, and N. Shakirov, Bohr radius for locally univalent harmonic mappings, *Math. Nachr.* **291**(11-12)(2018), 1757–1768 .
31. M. S. Liu, Y. M. Shang, and J. F. Xu, Bohr-type inequalities of analytic functions, *J. Inequal. Appl.*, (2018), Paper No. **345**, 13 pp.
32. V. I. Paulsen, G. Popescu and D. Singh, On Bohr's inequality, *Proc. London Math. Soc.* **85**(2) (2002), 493–512.
33. V. I. Paulsen and D. Singh, Bohrs inequality for uniform algebras, *Proc. Amer. Math. Soc.* **132(12)**(2004), 3577–3579,
34. S. Ponnusamy and K.-J. Wirths, Bohr type inequalities for functions with a multiple zero at the origin, *Comput. Methods Funct. Theory* (2020), To appear.
35. G. Popescu, Multivariable Bohr inequalities, *Trans. Amer. Math. Soc.* **359**(11) (2007), 5283–5317.
36. G. Ricci, Complementi a un teorema di H. Bohr riguardante le serie di potenze, *Rev. Un.Mat. Argentina* **17** (1955/1956), 185–195.
37. S. Sidon, Über einen Satz von Herrn Bohr, *Math. Z.* **26**(1) (1927), 731–732.
38. M. Tomić, Sur un théorème de H. Bohr, *Math. Scand.* **11** (1962), 103–106.

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