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Role of non-uniform slot injection (suction) model on the separation of a laminar boundary layer flow

S. Roy [a], P. Saikrishnan [b], R. Ravindran [c,*]

[a] Department of Mathematics, I.I.T. Madras, Chennai - 600036, India

[b] Department of Mathematics, National Institute of Technology, Tiruchirapalli, Tiruchirapalli - 620 015, India

[c] Center for Differential Equations, Continuum Mechanics and Applications, School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

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ABSTRACT

An analysis is performed to study the influence of non-uniform slot injection (suction) on a steady incompressible laminar boundary layer flow in a diverging channel with an exponentially decreasing free-stream velocity. The difficulties in obtaining the non-similar solutions at the starting point of the streamwise coordinate, at the edges of the slot and at the point of separation are overcome by applying an implicit finite difference scheme with the quasi-linearization technique and an appropriate selection of the finer step sizes along the streamwise direction. It is observed that the separation can be delayed by non-uniform slot suction and also by moving the slot downstream but the effect of non-uniform slot injection is just reverse.

1. Introduction

A detailed analysis of boundary layer flow problems taking non-similarity into account has become significantly important in recent past. In an earlier study, a review on the non-similarity solution methods along with the citations of some relevant publications is given by Dewey and Gross [1]. Subsequently, many attempts have been made to provide non-similar solutions of boundary layer flow problems using asymptotic methods [2,3], finite difference method [4,5] and more recently by an implicit finite difference method in combination with quasi-linearization technique [6–8].

In the presence of an adverse pressure gradient, the boundary layer grows in thickness and eventually breaks away from the solid surface. The point at which separation of the boundary layer occurs, for steady flow over a stationary surface, is generally taken as coinciding with or very near the point at which the skin friction vanishes. There are several studies on the phenomenon of separation, for example, by Brown and Stewartson [9], Williams [10] and Curle [11]. The work that is directly relevant to the present work is by Chiam [12] who has investigated the development of a steady two-dimensional laminar boundary layer flow with and without uniform suction. Mass transfer from a wall slot (i.e. mass transfer occurs in a small porous section of the body surface while there is no mass transfer in the remaining part of the body surface) into boundary layer is of interest for various potential applications including energizing the inner portion of boundary layer in adverse pressure gradient and skin friction reduction on control surfaces. In fact, mass transfer through a slot strongly influences the development of a boundary layer along a surface and in particular can prevent or at least delay separation of viscous region. In recent studies, several investigators [6–8] have investigated the effect of single slot injection (suction) into steady compressible and water boundary layer flows. Therefore, as a step towards the eventual development in the study of

* Corresponding author.
E-mail addresses: sjroy@iitm.ac.in (S. Roy), Ravindran.Ramalingam@wits.ac.za (R. Ravindran).

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Nomenclature

Roman letters

A	surface mass transfer parameter
c	characteristic length
f	dimensionless stream function
F	dimensionless velocity along surface
m	pressure gradient parameter
Re	Reynolds number
u, v	velocity components along x and y, respectively
x, y	Cartesian coordinates along and normal to surface, respectively

Greek letters

α	small parameter
η	transformed variable
ν	kinematic viscosity
$\xi (= \frac{x}{c})$	a scaled streamwise coordinate
ξ_0	slot location parameter
ψ	dimensional stream function
ω^*	slot length parameter

Subscripts

e, w	denote conditions at the edge of the boundary layer and on the surface, respectively
ξ, η	denote the partial derivatives w.r.t these variables, respectively

mass transfer into the boundary layer flows, it is interesting as well as useful to investigate the effect of single slot injection (suction) into a boundary layer flow with exponentially decreasing free-stream velocity distribution.

The effect of non-uniform slot injection (suction) on the steady laminar non-similar boundary layer flow with an exponentially decreasing free-stream velocity distribution is considered in the present investigation. The non-similar solutions have been obtained starting from the origin of the streamwise coordinate to the point of separation (zero skin friction in the streamwise direction) using quasi-linearization technique with an implicit finite difference scheme. There are two free parameters in this problem, one measures the length of the slot (i.e. the part of the body surface in which there is a mass transfer) and another parameter fixes the position of the slot. Thus, these two parameters help to vary the slot length and to move the slot location. The application of non-uniform slot injection or suction is helpful in suppressing recirculating bubbles and controlling transition and/or delaying the boundary layer separation over control surfaces. Present results without mass transfer are compared with the results of a recent study by Chiam [12] and found them in good agreement.

2. Analysis

Consider a two-dimensional steady laminar incompressible boundary layer flow in a diverging channel with exponentially decreasing free-stream velocity distribution when mass transfer (suction/injection) occurs in a slot along the surface. Let x, y be the Cartesian coordinates along and normal to the surface, respectively, and u, v, the corresponding velocity components (see Fig. 1). The blowing rate of the fluid is assumed to be small and it does not affect the inviscid flow at the edge of boundary layer and also assumed that the injected fluid possesses the same physical properties as the boundary layer fluid [13]. Under the above assumptions, the boundary layer equations governing the flow are [12,13]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\mathrm{d}u_e}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial^2 y} \tag{2}$$

and the boundary conditions are given by

$$u = 0, \qquad v = v_w(x) \quad \text{at } y = 0 \tag{3}$$

$$u = u_e(x) \quad \text{at } y \longrightarrow \infty. \tag{4}$$

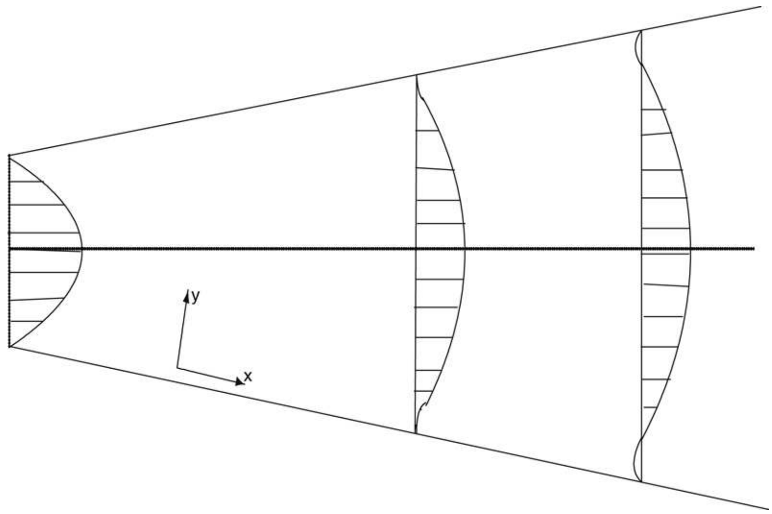

Fig. 1. Flow geometry.

Using the following transformations

$$\psi(x, y) = \sqrt{(u_e \nu x)} f(\xi, \eta), \qquad \eta = \sqrt{\frac{u_e}{\nu x}}\, y, \qquad \xi = \frac{x}{c},$$
$$f_\eta(\xi, \eta) = F(\xi, \eta), \qquad u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{5}$$

the boundary layer Eq. (2), where continuity equation (1) is identically satisfied by the above transformation, is reduced to

$$F_{\eta\eta} + \left(\frac{m+1}{2}\right) f F_\eta + m(1 - F^2) = \xi (F F_\xi - F_\eta f_\xi) \tag{6}$$

where m is a dimensionless pressure gradient parameter defined by $m = \frac{\xi}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}\xi}$. Here u_e is the external flow velocity at the edge of the boundary layer and is given by [12] $u_e = u_\infty(1 - \alpha \mathrm{e}^{\xi})$, $0 < \alpha < 1$ where u_∞ is constant, α is the small parameter and ξ, a scaled streamwise coordinate is defined by $\xi = \frac{x}{c}$, where c is the characteristic length. The transformed boundary conditions are

$$F(\xi, 0) = 0, \qquad F(\xi, \infty) = 1 \tag{7}$$

where $f = \int_o^\eta F \mathrm{d}\eta + f_w$.

Now

$$v = -\frac{\partial \psi}{\partial x} = -\frac{1}{2}(u_e \nu x)^{-1/2}\left[u_e \nu + \frac{\mathrm{d}u_e}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x}\right] f - (u_e \nu x)^{1/2}\left[\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}\right],$$

where $\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{\partial \xi}{\partial x} = \frac{1}{c}$ and $\frac{\partial \eta}{\partial x} = -\frac{\eta}{2x}\left[\frac{1+\alpha(\xi-1)\mathrm{e}^{\xi}}{1-\alpha \mathrm{e}^{\xi}}\right]$.

Thus at wall ($y = 0$), $\eta = 0$, $\frac{\partial \eta}{\partial x} = 0$ and v can be written at wall as

$$v_w(\xi) = -\frac{u_\infty}{Re^{1/2}}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left([\xi(1 - \alpha \mathrm{e}^{\xi})]^{1/2}\right) f(\xi, 0) + [\xi(1 - \alpha \mathrm{e}^{\xi})]^{1/2} f_\xi(\xi, 0)\right]$$

i.e., $v_w(\xi) = -\frac{u_\infty}{Re^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left([\xi(1 - \alpha \mathrm{e}^{\xi})]^{1/2} f(\xi, 0)\right)$.

Integrating with respect to ξ from 0 to ξ, we get

$$f(\xi, 0) = f_w(\xi) = -[\xi(1 - \alpha \mathrm{e}^{\xi})]^{-1/2}\frac{Re^{1/2}}{u_\infty}\int_o^\xi v_w(\xi)\mathrm{d}\xi$$

where the Reynolds number $Re = \frac{u_\infty c}{\nu}$.

The boundary condition $v_w(\xi)$ is considered in terms of transformed coordinate ξ and $v_w(\xi)$ is taken as sinusoidal function given by

$$v_w(\xi) = \begin{cases} -u_\infty (Re)^{-\frac{1}{2}} A\omega^* \sin(\omega^*(\xi - \xi_o)), & \xi_o \le \xi \le \xi_o^* \\ 0, & \xi \le \xi_o \quad \text{and} \quad \xi \ge \xi_o^* \end{cases}$$

where ω^* and ξ_o are the two free parameters which determine the slot length and slot location, respectively. The function $v_w(\xi)$ is continuous for all values of ξ and it has a nonzero value only in the interval $[\xi_o,\ \xi_o^*]$. The reason for taking such a function is that it allows the mass transfer to change slowly in the neighbourhood of leading and trailing edges of the slot. The surface mass transfer parameter $A > 0$ or $A < 0$ according to whether there is a suction or an injection.

Using the above $v_w(\xi)$, the expression for $f_w(\xi)$ is

$$f_w = \begin{cases} 0, & \xi \le \xi_o \\ A(P_1)^{-1/2} C(\xi, \xi_o), & \xi_o \le \xi \le \xi_o^* \\ A(P_1)^{-1/2} C(\xi_o^*, \xi_o), & \xi \ge \xi_o^* \end{cases} \tag{8}$$

where $C(\xi, \xi_o) = 1 - \cos(\omega^*(\xi - \xi_o))$ and $P_1 = \xi(1 - \alpha\, e^{\xi})$.

3. Numerical method

The boundary value problem represented by Eqs. (6) and (7) is solved by implicit finite difference scheme in combination with the quasi-linearization technique. Quasi-linearization technique can be viewed as a generalization of the Newton–Raphson approximation technique in functional space. An iterative sequence of linear equations are carefully constructed to approximate the nonlinear equation (6) for achieving quadratic convergence and monotonicity. The efficiency and accuracy of the method have been illustrated through its applications to many boundary value problems in the book by Bellman and Kalaba [14].

Applying the quasi-linearization technique [14,15], the nonlinear partial differential equation (6) reduce to the following linear partial differential equation

$$F_{\eta\eta}^{i+1} + X_1^i F_\eta^{i+1} + X_2^i F^{i+1} + X_3^i F_\xi^{i+1} = X_4^i. \tag{9}$$

The boundary conditions become

$$\begin{aligned} F^{i+1} &= 0, \quad \text{at } \eta = 0 \\ F^{i+1} &= 1, \quad \text{at } \eta = \eta_\infty \end{aligned} \tag{10}$$

where η_∞ is the edge of the boundary layer. The coefficient functions with iterative index i are known where the functions with iterative index $(i + 1)$ are to be determined and the coefficients are given by

$$\begin{aligned} X_1^i &= \left(\frac{m+1}{2}\right) f + \xi f_\xi \\ X_2^i &= -2mF - \xi F_\xi \\ X_3^i &= -\xi F \\ X_4^i &= -m(1 + F^2) - \xi F F_\xi. \end{aligned}$$

Since the method is described for ordinary differential equations by Inouye and Tate [15] and also explained for partial differential equations in a recent article by Singh and Roy [16], its detailed description is not presented here for the sake of brevity. In brief at each iteration step, the sequence of linear partial differential equation (9) under boundary conditions (10) were expressed in difference form using central difference in η-direction and backward difference in ξ-direction. Finally at each iteration step, the equation reduces to a system of algebraic equation in tri-diagonal form which is solved by using Thomas algorithm. The step size in the η-direction has been chosen as $\Delta\eta = 0.01$ throughout the computation, as it has been found that further decrease in $\Delta\eta$ does not change the results up to the fourth decimal place. In the ξ-direction, $\Delta\xi = 0.01$ has been used for small values of ξ and then it has been decreased to $\Delta\xi = 0.005$. There after the step size has been reduced further, ultimately choosing a value $\Delta\xi = 0.0001$ in the neighbourhood of the zero skin friction. This has been done because the convergence becomes slower when the point of vanishing skin friction is approached. A convergence criterion based on the relative difference between the current and the previous iterations has been used. The solution is assumed to have converged and the iterative process is terminated when

$$\text{Max}|(F_\eta)_w^{i+1} - (F_\eta)_w^i| < 10^{-4}. \tag{11}$$

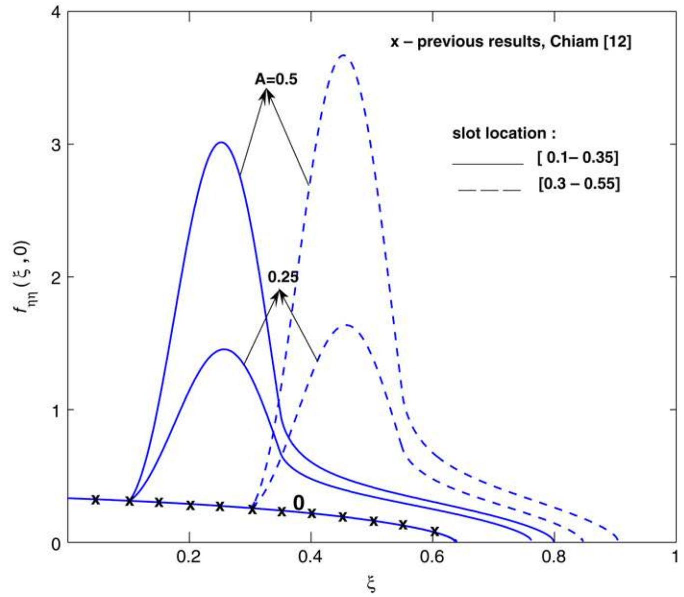

Fig. 2. Effect of suction ($A > 0$) on $f_{\eta\eta}(\xi, 0)$ for $\omega^* = 4\pi$ and $\alpha = 0.1$.

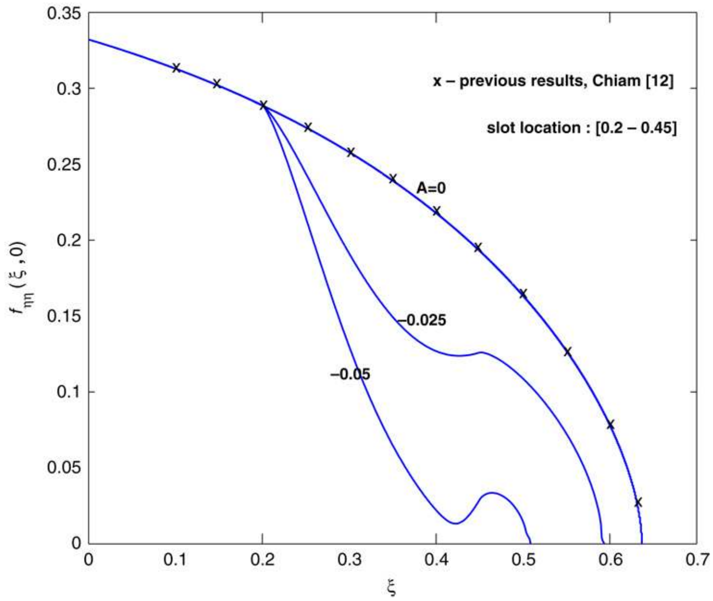

Fig. 3. Effect of injection ($A < 0$) on $f_{\eta\eta}(\xi, 0)$ for $\omega^* = 4\pi$ and $\alpha = 0.1$.

4. Results and discussion

Computations have been carried out for various values of α ($10^{-6} - 10^{-1}$) and mass transfer parameter A ($-0.1 \le A \le 0.5$). In all numerical computations the edge of the boundary layer η_∞ is taken as 9.0. In order to assess the accuracy of the procedure, solutions have been obtained for the incompressible flow cases with uniform mass transfer [12] and comparisons are included in Figs. 2–7. The present results are found to be in good agreement.

The effects of non-uniform slot suction (or injection) parameter ($A > 0$ or $A < 0$) and ξ_o (which fixes the slot location) on velocity gradient $[f_{\eta\eta}(\xi, 0)]$ at the wall for different values of α are presented in Figs. 2–7. In the case of non-uniform slot suction (See Figs. 2, 4 and 6), the velocity gradient ($f_{\eta\eta}(\xi, 0)$) increases as slot starts and attain their maximum values before the trailing edge of the slot. Finally, $f_{\eta\eta}(\xi, 0)$ decreases from its maximum value as the effect of the adverse pressure gradient dominates and $f_{\eta\eta}(\xi, 0)$ decreases to zero. As mentioned earlier, this implies that separation occurs at this point. The enhancement in the velocity gradient is due to the increment of the suction parameter A, but in the case of slot injection

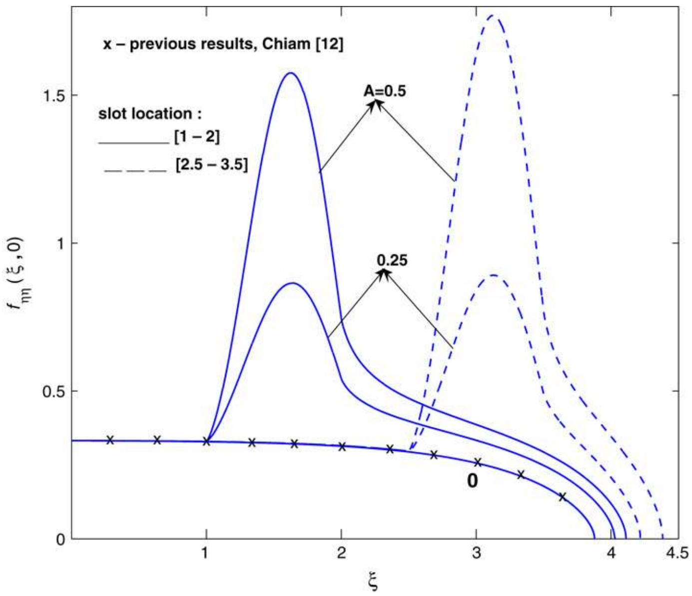

Fig. 4. Effect of suction ($A > 0$) on $f_{\eta\eta}(\xi, 0)$ for $\omega^* = \pi$ and $\alpha = 0.001$.

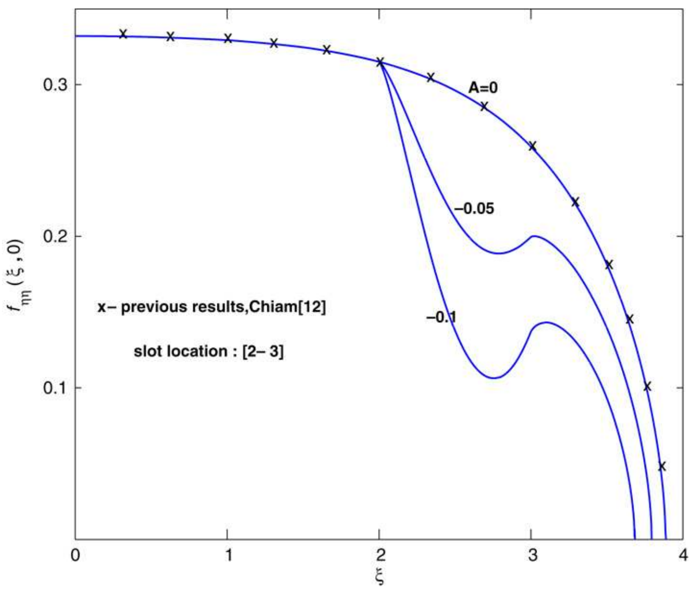

Fig. 5. Effect of injection ($A < 0$) on $f_{\eta\eta}(\xi, 0)$ for $\omega^* = \pi$ and $\alpha = 0.001$.

(See Figs. 3, 5 and 7) the skin friction decreases with the increase of the injection parameter A. The results indicate that the effect of non-uniform slot suction is to move the point of separation downstream, i.e., it delays the separation as can be seen in Figs. 2, 4 and 6. In contrast, the effect of non-uniform slot injection is to move the point of separation upstream as shown in Figs. 3, 5 and 7. In particular, for $\alpha = 0.001$ and $A = 0.5$, it is noticed that the point of separation moves downstream approximately by 14% as the suction parameter A increases from 0 to 0.5. Moreover, the results presented in Figs. 2, 4 and 6 indicate that if we move the location of the slot downstream, the point of separation also moves downstream (i.e.,it delays the separation). Further, comparative studies on Figs. 2–7 show that the point of separation moves downstream approximately from 0.63 to 10.2 corresponding to the decrease of α from 10^{-1} to 10^{-6}. This large change is due to the fact that as α decreases rapidly, the free-stream flow becomes closer to flow over flat plate for extremely small α.

The effects of slot suction (injection) parameter on velocity profiles ($f(\eta)$) are displayed in Fig. 8. These profiles indicate that the injection decreases the steepness of the profiles but the steepness of the profiles increases with suction. In general the profiles at a distant streamwise location are comparatively less steeper than those at the starting point of the steamwise

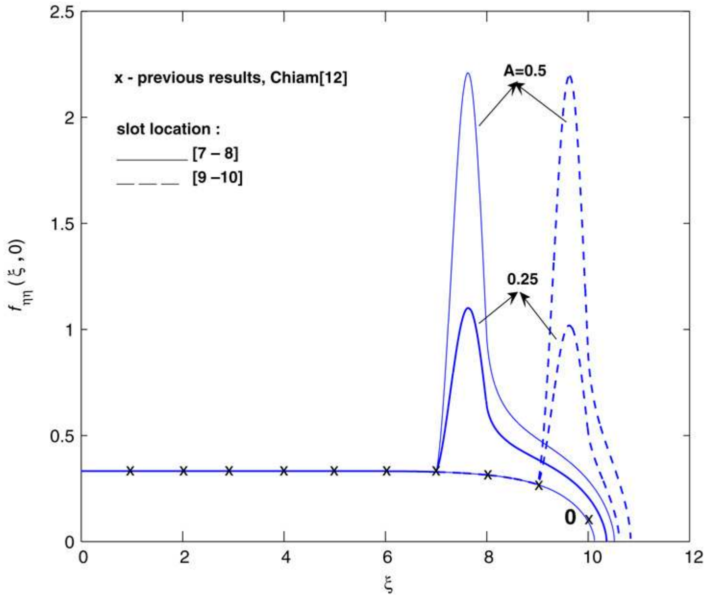

Fig. 6. Effect of suction ($A > 0$) on $f_{\eta\eta}(\xi, 0)$ for $\omega^* = \pi$ and $\alpha = 0.000001$.

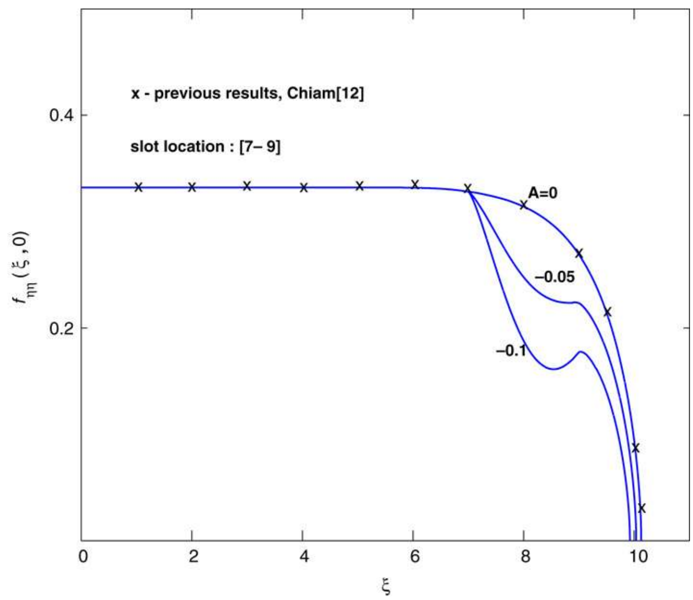

Fig. 7. Effect of injection ($A < 0$) on $f_{\eta\eta}(\xi, 0)$ for $\omega^* = \frac{\pi}{2}$ and $\alpha = 0.000001$.

coordinate. Further Fig. 8 shows that the velocity profile at the separation point have larger η_∞ than those at the previous streamwise locations.

5. Conclusions

Non-similar solution of a steady laminar incompressible boundary layer flow with an exponentially decreasing velocity distribution for non-uniform slot injection/suction has been obtained starting from the origin of streamwise coordinate to the exact point of separation.

- The numerical results are obtained to find the location of the point of separation for various values of α to cover extensive variations in decreasing free-stream velocity distributions.
- The point of separation moves significantly in the downstream direction as α decreases rapidly and finally the free-stream flow becomes closer to flow over a flat plate for extremely small α.

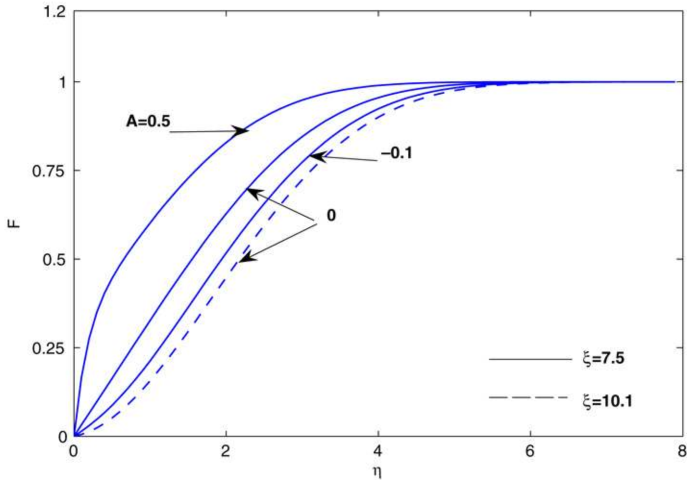

Fig. 8. Velocity profiles at $\xi = 7.5$ and $\xi = 10.1$ for $\epsilon = 0.000001$ with different values of A.

- The point of separation can be delayed by non-uniform slot suction as well as by moving the slot in downstream direction whereas the effect of non-uniform slot injection is just opposite.

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