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Resistance matrices of balanced directed graphs

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Abstract

Let G be a strongly connected and balanced directed graph. The Laplacian matrix of G is then the matrix (not necessarily symmetric) $L := D - A$, where A is the adjacency matrix of G and D is the diagonal matrix such that the row sums and the column sums of L are equal to zero. Let $L^\dagger = [l^\dagger_{ij}]$ be the Moore-Penrose inverse of L. We define the resistance between any two vertices i and j of G by $r_{ij} := l^\dagger_{ii} + l^\dagger_{jj} - 2l^\dagger_{ij}$. In this paper, we derive some interesting properties of the resistance and the corresponding resistance matrix $[r_{ij}]$.

Keywords. Balanced directed graph, Laplacian matrix, Moore-Penrose inverse, cofactor sums

AMS CLASSIFICATION. 05C50

1 Introduction

Let $G = (V, E)$ be a simple connected graph with finite set of vertices $V = \{1, \dots, n\}$ and edge set E, the set of undirected edges. To each edge $(i, j) \in E$ assign a weight w_{ij} which is a positive number. If $i, j \in V$, define $A := [a_{ij}]$ where

$$a_{ij} := \begin{cases} w_{ij} & (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Define $D := \mathrm{Diag}(A\mathbf{1})$, where $\mathbf{1}$ is the column vector of all ones in $\mathbb{R}^n$. The Laplacian matrix of G is then the symmetric matrix $S := D - A$. If $x \in \mathbb{R}^n$, then it can be verified that

$$x^T S x = \sum_{(i,j)\in E} a_{ij}(x_i - x_j)^2,$$

and hence S is positive semidefinite with null-space span$\{\mathbf{1}\}$. The algebraic connectivity of G is the second smallest eigenvalue of the Laplacian matrix S and the associated eigenvector is called the Fiedler vector which is used to bisect the graph into two connected partitions based on the sign of its components, see Fiedler [1]. We shall denote the Moore-Penrose inverse of S by $S^\dagger$ and its entries by $s^\dagger_{ij}$. To define the distance between any two vertices i and j in G, it is natural to consider the length of the shortest path connecting them. This is the classical distance and we shall denote it by d_{ij}. The function $f : V \times V \to \mathbb{R}$ defined by $f(i, j) := d_{ij}$ is a metric on the vertex set V. There are several reasons why the shortest distance d_{ij} is important. In chemistry, the classical distance d_{ij} is used to represent the structure of a molecule as a metric space: see [2] and references therein. Here is another application in a data communication problem [3]. If v is a vertex of G, and N is a natural number, define

$$A(v) := (a_1, \ldots, a_N), \text{where} \quad a_i \in \{0, 1, *\}.$$

Let

$$\rho(A(v), A(v')) := |\{\nu : \{a_\nu, a_\nu^{'}\} = \{0, 1\}| \quad \forall v, v' \in V.$$

It is known that for some large N, there exists a function ρ such that

$$\rho(A(v), A(v')) = d_{vv'} \quad \forall v, v' \in V. \tag{1}$$

Now the question is to determine the minimum N for which equation (1) holds. A known result states that

$$N \geq \max\{n_+, n_-\},$$

where n_+ and n_- are the number of positive and negative eigenvalues of the symmetric matrix $[d_{ij}]$: see [4]. Suppose there are multiple paths connecting i and j in G. In a network, this may indicate that the nodes i and j are better communicated. Thus, it makes more sense to define a distance between i and j which is shorter than the classical distance d_{ij}. There are several other possible metrics that can be defined on the vertex set V of G. In a seminal paper, Klein and Rándic [5] introduced the resistance distance R_{ij} between any two vertices i and j of G. This is defined via $S^\dagger$, the Moore-Penrose inverse of the Laplacian matrix S of G:

$$R_{ij} := s^\dagger_{ii} + s^\dagger_{jj} - 2s^\dagger_{ij}. \tag{2}$$

In resistive electrical networks, R_{ij} is interpreted as the effective electrical resistance between the nodes i and j of a network N corresponding to G, with resistor of magnitude w_{ij} taken over the edge (i, j) of N. It can be proved that the resistance distance is at most the classical distance and if G is acyclic, then $R_{ij} = d_{ij}$ for all i and j. Resistance distance have several interesting properties. These are discussed in chapter 9 of [6]. In this paper, we generalize the concept of resistance distance to directed graphs.

Let $G = (V, E)$ be a simple directed graph with vertex set $V = \{1, \ldots, n\}$ and edge set E containing directed edges. We write $(i, j) \in E$ if there is a directed edge from vertex i to vertex j. If i and j are any two vertices, we define

$$a_{ij} = \begin{cases} 1 & (i, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $A := [a_{ij}]$ will be called the adjacency matrix of G. The indegree and the outdegree of a vertex k is the sum of all the entries in the k^{th} column and the k^{th} row of the adjacency matrix A. A vertex j in V is said to be balanced if its indegree and the outdegree are equal. Now the graph is said to be balanced if all the vertices are balanced. Recall that a directed graph is strongly connected, if each pair of vertices is connected by a directed path. In the sequel, we assume that G is a strongly connected and balanced directed graph. The Laplacian of G is now defined by $L := \mathrm{Diag}(A\mathbf{1}) - A$. The algebraic connectivity concept is generalized to directed graphs via this definition of the Laplacian matrix and have many other applications like in networks of chaotic systems: see [7]. We now propose a *semi-distance* in directed graphs using the Moore-Penrose inverse of the Laplacian matrix L.

Definition 1. The resistance between any two vertices i and j in V is defined by

$$r_{ij} := l^{\dagger}_{ii} + l^{\dagger}_{jj} - 2l^{\dagger}_{ij}, \tag{3}$$

where $l^{\dagger}_{ij}$ is the $(i,j)^{\text{th}}$ entry in the Moore-Penrose inverse of L.

The matrix $R := [r_{ij}]$ will be called the resistance matrix of G. The *reversal* of G is the directed graph obtained by reversing the orientation of all the edges. The adjacency matrix of the reversal is then the transpose of the matrix A, and thus the resistance matrix of the reversal of G is the transpose of R. Because r_{ij} and r_{ji} are not equal in general, r_{ij} is not necessarily a metric on V and therefore, the resistance matrices we consider here are not symmetric in general. The symmetric part of the Laplacian matrix of G defined by $S := \frac{1}{2}(L + L')$ has a combinatorial interpretation. Define a simple undirected graph H from G as follows. Let the vertex set of H be V. If $i, j \in V$, then we shall say that i and j are adjacent in H, if $(i,j) \in E$ or $(j,i) \in E$. Because G is strongly connected, H is connected. Let F be the set of all edges of H. Now to each edge $(i,j) \in F$, define w_{ij} as follows:

$$w_{ij} = \begin{cases} 1 & (i,j) \in E \ \text{ and } \ (j,i) \in E \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

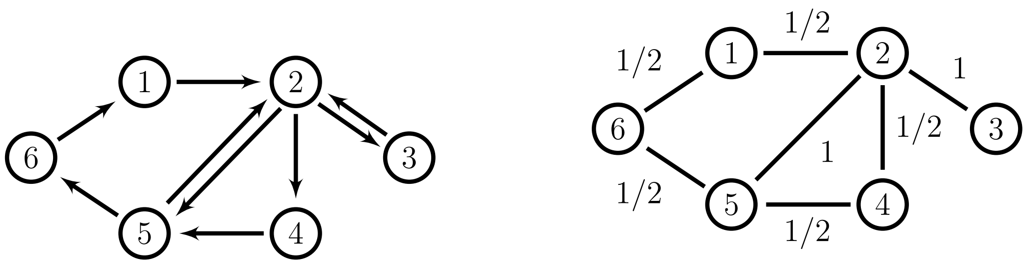
Now, S is the Laplacian of the weighted graph H. Hence for any $x \in \mathbb{R}^n$,

$$x^T L x = x^T S x = \sum_{(i,j) \in F} w_{ij}(x_i - x_j)^2.$$

Thus, the null-space of L and null-space of L' are equal to $\mathrm{span}\{\mathbf{1}\}$ and $L + L'$ is positive semidefinite. To illustrate, we give an example.

Example 1. Consider the directed graph G with six vertices given in Figure 1(a). G is strongly connected and balanced. The adjacency and the Laplacian matrices of G are:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$



(a) (b)

Figure 1: (a) Graph G and (b) Graph H

The Moore-Penrose inverse of L is

$$L^{\dagger} = \begin{bmatrix} \frac{5}{9} & \frac{1}{18} & -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} & -\frac{5}{18} \\ -\frac{5}{18} & \frac{2}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & -\frac{1}{9} \\ -\frac{4}{9} & \frac{1}{18} & \frac{8}{9} & -\frac{1}{9} & -\frac{1}{9} & -\frac{5}{18} \\ -\frac{7}{36} & -\frac{7}{36} & -\frac{13}{36} & \frac{23}{36} & \frac{5}{36} & -\frac{1}{36} \\ -\frac{1}{36} & -\frac{1}{36} & -\frac{7}{36} & -\frac{7}{36} & \frac{11}{36} & \frac{5}{36} \\ \frac{7}{18} & -\frac{1}{9} & -\frac{5}{18} & -\frac{5}{18} & -\frac{5}{18} & \frac{5}{9} \end{bmatrix}.$$

The resistance matrix $R = [r_{ij}] = [l^{\dagger}_{ii} + l^{\dagger}_{jj} - 2l^{\dagger}_{ij}]$ is given by

$$R = \begin{bmatrix} 0 & \frac{2}{3} & \frac{5}{3} & \frac{17}{12} & \frac{13}{12} & \frac{5}{3} \\ \frac{4}{3} & 0 & 1 & \frac{3}{4} & \frac{5}{12} & 1 \\ \frac{7}{3} & 1 & 0 & \frac{7}{4} & \frac{17}{12} & 2 \\ \frac{19}{12} & \frac{5}{4} & \frac{9}{4} & 0 & \frac{2}{3} & \frac{5}{4} \\ \frac{11}{12} & \frac{7}{12} & \frac{19}{12} & \frac{4}{3} & 0 & \frac{7}{12} \\ \frac{1}{3} & 1 & 2 & \frac{7}{4} & \frac{17}{12} & 0 \end{bmatrix}.$$

The undirected graph H obtained from G is given in Figure 1(b). The Laplacian matrix S of H is given by

$$S = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 3 & -1 & -\frac{1}{2} & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -1 & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

It can be verified that $S = \frac{1}{2}(L + L')$.

Suppose $G' = (V, F)$ is a simple undirected and connected graph. Let $[R_{ij}]$ be the resistance matrix of G', where R_{ij} is defined in (2). Now $[R_{ij}]$ is the resistance matrix of a strongly connected and balanced directed graph. To see this, we proceed as follows. Let L be the Laplacian matrix of G'. From the edge set F, we shall define a set of directed edges. For each edge $(i, j) \in F$, define two directed edges, viz, (i,j) and (j,i) and let E' be the set of all such directed edges. Then the directed graph $G' := (V, E')$ is strongly connected and balanced. It can be easily seen that the adjacency matrices of G and G' are equal and hence their Laplacian matrices are equal. This means that between any two vertices i and j, the resistance distance in G and the resistance in G' defined by (2) and (3), respectively are same. To illustrate, we give an example.

Example 2. Let G be the graph with five vertices given in Figure 2(a). The di-

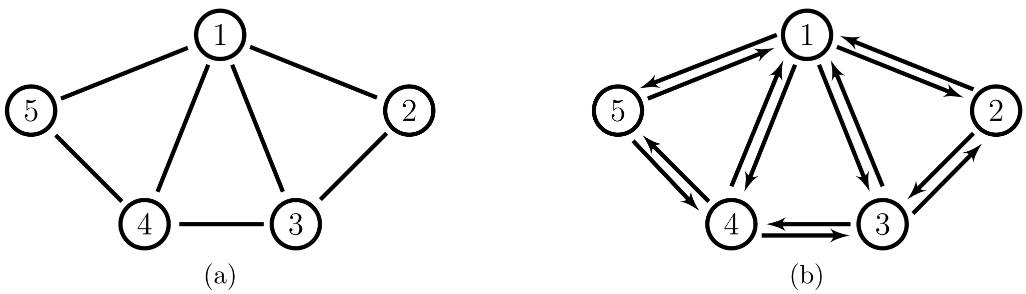


Figure 2: (a) Graph G and (b) Graph G'

rected graph G' constructed from G is shown in Figure 2(b). The adjacency and the Laplacian matrices of G and G' are given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

1.1 Results obtained in the paper

- In our first result, we show that the resistance r_{ij} defined in (3) has the following properties.

  (i) If i and j are any two distinct vertices of G, then $r_{ij} > 0$.

  (ii) If i, j, k are any three vertices, then

$$r_{ik} \le r_{ij} + r_{jk} \quad \forall i, j, k.$$

- In our next result, we compute an identity for the inverse of the resistance matrix $[r_{ij}]$. The motivation for obtaining this identity starts from a classical result of Graham and Lovász [4]. This states the following.

  Theorem 1. *Let T be a tree with $V(T) = \{1, \ldots, n\}$. Let d_{ij} be the length of the shortest path between vertices i and j, and L be the Laplacian of T. Set $D := [d_{ij}]$. Then,*

$$D^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\tau\tau',$$

where $\tau = (2-\delta_1, \dots, 2-\delta_n)'$ *and* δ_i *is the degree of the vertex* i.

Theorem 1 is extended to connected graphs in [8] for resistance matrices.

Theorem 2. *Let* G *be a simple connected graph with vertex set* $V = \{1, \dots, n\}$ *and edge set* E. *Let* S *be the Laplacian of* G *and* R_{ij} *be the resistance distance defined in* (2). *Define* $\widetilde{R} := [R_{ij}]$. *Then,*

$$\widetilde{R}^{-1} = -\frac{1}{2}S + \frac{1}{\tau'\widetilde{R}\tau}\tau\tau',$$

where $\tau_i = 2 - \sum_{(i,j)\in E} R_{ij}$.

Motivated by the above two results, we find the following inverse formula for the resistance matrix $[r_{ij}]$.

Theorem 3. *Let* $G = (V, E)$ *be a strongly connected and balanced directed graph. Let* r_{ij} *be the resistance between the vertices* i *and* j *defined in* (3) *and* $R := [r_{ij}]$. *Then,*

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R\tau}(\tau(\tau' + \mathbf{1}'\operatorname{diag}(L^{\dagger})M)),$$

where $M = L - L'$, *and* $\tau_i := 2 - \sum_{(i,j)\in E} r_{ji}$.

Since the resistance matrix of a connected graph can be written as a resistance matrix of a strongly connected and balanced directed graph, Theorem 1 and 2 are special cases of Theorem 3. Using Theorem 3, we find a formula for computing $\det(R)$.

- In our final result, we investigate the sum of all the cofactors in an $s \times s$ submatrix of $R = [r_{ij}]$. The motivation for this consideration comes from an alternate method to compute the resistance distance defined in (2). This method gives an elegant formula to compute R_{ij}:

$$R_{ij} = \frac{1}{\delta}\det(S(\{i,j\},\{i,j\})), \tag{4}$$

where $S(\{i,j\},\{i,j\})$ is the principal submatrix of S obtained by deleting rows and columns indexed by $\{i,j\}$ and δ is the number of spanning trees in G. A far reaching generalization of (4) is obtained in [9]. This is stated below.

Theorem 4. *Let* G *be a connected graph with vertex set* $\{1, 2, \dots, n\}$. *Let* S *be the Laplacian matrix of* G *and* $\widetilde{R} := [R_{ij}]$ *its resistance matrix. Let* $\Omega_1, \Omega_2 \subset \{1, 2, \dots, n\}$ *be non-empty, and let* $|\Omega_1| = |\Omega_2|$. *Put* $\eta := |\Omega_1|$. *Suppose* $\alpha(\Omega_1)$ *and* $\alpha(\Omega_2)$ *are the sum of all the elements in* Ω_1 *and* Ω_2, *respectively. Let*

$S[\Omega_1, \Omega_2]$ denote the $\eta \times \eta$ submatrix of S with rows and columns indexed by Ω_1 and Ω_2, respectively, and $\widetilde{R}[\Omega_2^c, \Omega_1^c]$ be the $(n-\eta) \times (n-\eta)$ submatrix of R with rows and columns indexed by Ω_2^c and Ω_1^c, respectively. Then,

$$\operatorname{cofsum}(\widetilde{R}[\Omega_1, \Omega_2]) = (-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)+\eta-1} \frac{2^{\eta-1}}{\delta} \det(S[\Omega_2^c, \Omega_1^c]), \tag{5}$$

where δ is the number of spanning trees in G.

Equation (4) is a special case of (5). This follows by setting $\Omega_1 = \Omega_2 = \{i, j\}$ and observing that $(-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)} = 1$, $\operatorname{cofsum}(\widetilde{R}[\Omega_1, \Omega_2]) = -2R_{ij}$, $\eta = 2$ and $S[\Omega_2^c, \Omega_1^c] = S(\{i, j\}, \{i, j\})$. In this paper, we generalize Theorem 4 to resistance matrices of directed graphs.

1.2 Outline of the paper

In section 2, we mention the preliminaries that are needed for further discussion. In section 3, we discuss the properties of the resistance. In section 4, we present the inverse formula stated in Theorem 3 and illustrate it by an example. In the final section, we deduce a formula for finding the cofactor sums of the resistance matrix.

2 Preliminaries

We now list a few notation used in this paper and gather some tools to prove our results.

(P1) Let Ω_1 and Ω_2 be non-empty subsets of $\{1, \ldots, n\}$. If W is an $n \times n$ matrix, then $W[\Omega_1, \Omega_2]$ will be the submatrix of W with rows and columns indexed by Ω_1 and Ω_2, respectively. If $\Omega \subseteq \{1, \ldots, n\}$ is non-empty, then $\alpha(\Omega)$ will denote the sum of all elements in Ω.

(P2) The complement of a set Ω is written Ω^c. The transpose and the Moore-Penrose inverse of a matrix A are denoted by A' and $A^\dagger$, respectively. All vectors are regarded as column vectors.

(P3) If $A = [a_{ij}]$ is a square matrix, then $\operatorname{diag}(A)$ is the diagonal matrix with diagonal entries equal to a_{ii}. If $s := (s_1, s_2, \ldots, s_n)' \in \mathbb{R}^n$, then $\operatorname{Diag}(s)$ will be the diagonal matrix with diagonal entries equal to s_i.

(P4) The sum of all the cofactors of an $m \times m$ matrix A is represented by $\operatorname{cofsum}(A)$. The determinant and the classical adjoint of A are written $\det(A)$ and $\operatorname{adj}(A)$, respectively.

(P5) The notation $\mathbf{1}$ will stand for the vector $(1,1,\ldots,1)'$ in $\mathbb{R}^n$ and $J := \mathbf{1}\mathbf{1}'$. The orthogonal projection onto the hyperplane $\{\mathbf{1}\}^\perp$ is denoted by P. It is easy to observe that $P = I - \frac{1}{n}J$, where I is the $n \times n$ identity matrix. If $1 \leq m < n$, then the vector of all ones in $\mathbb{R}^m$ and the $m \times m$ identity matrix will be denoted by $\mathbf{1}_m$ and I_m, respectively.

(P6) The Jacobi's identity on non-singular matrices is the following:

Theorem 5. *Let A be an $n \times n$ non-singular matrix. Let Ω_1, $\Omega_2 \subset \{1,\ldots,n\}$ be non-empty such that $|\Omega_1| = |\Omega_2|$. Then,*

$$\det(A^{-1}[\Omega_2^c, \Omega_1^c]) = (-1)^{\alpha(\Omega_2)+\alpha(\Omega_1)} \frac{1}{\det(A)} \det(A[\Omega_1, \Omega_2]).$$

See Brualdi and Schneider [10].

(P7) An $n \times n$ matrix B is called a $\mathbf{Z}$-matrix, if every off-diagonal entry of B is non-positive. If L is the Laplacian matrix of a strongly connected and balanced directed graph, then L is a $\mathbf{Z}$-matrix. As already noted, $L + L'$ is positive semidefinite, $L\mathbf{1} = L'\mathbf{1} = 0$ and $\operatorname{rank}(L) = n-1$.

(P8) Suppose S is an $n \times n$ matrix such that $S\mathbf{1} = S'\mathbf{1} = 0$ and $\operatorname{rank}(S) = n-1$. Then $S^\dagger\mathbf{1} = {S^\dagger}'\mathbf{1} = 0$, $SS^\dagger = S^\dagger S = P = I - \frac{1}{n}J$ and all the cofactors of S are equal. If L is a $\mathbf{Z}$-matrix such that $L\mathbf{1} = L'\mathbf{1} = 0$ and $\operatorname{rank}(L) = n-1$, then we shall write $L \in \mathbf{Z}(\mathcal{L})$. If $L \in \mathbf{Z}(\mathcal{L})$, then it can be verified that $L^\dagger + {L^\dagger}'$ is positive semidefinite and $\operatorname{trace}(L^\dagger) > 0$.

(P9) Let A be an $n \times n$ matrix. If u and v belong to $\mathbb{R}^n$, then $\det(A + uv') = \det(A) + v' \operatorname{adj}(A) u$. For a proof, see Lemma 1.1 in [11].

(P10) Let $B = [b_{ij}]$ be an $n \times n$ matrix. Then,

(a) B is row diagonally dominant if for each $i = 1,\ldots,n$

$$|b_{ii}| \geq \sum_{\{j : i \neq j\}} |b_{ij}| \;\; \forall j = 1, \ldots n.$$

(b) B is diagonally dominant of its row entries if

$$|b_{ii}| \geq |b_{ij}|$$

for each $i = 1,\ldots,n$ and $j \neq i$.

(c) B is diagonally dominant of its column entries if B' is diagonally dominant of its row entries.

By Theorem 2.5.12 in [12], if B is non-singular and row diagonally dominant, then B^{-1} is diagonally dominant of its column entries.

(P11) Let $G = (V, E)$ be a directed graph with vertex set $V = \{1, 2, ..., n\}$. An *oriented spanning tree* of G rooted at vertex i is a spanning subgraph T such that

(i) Every vertex j of T such that $j \neq i$ has outdegree 1.

(ii) The vertex i has outdegree 0.

(iii) T has no oriented cycles.

The matrix-tree theorem for directed graphs (Theorem 1 in [13]) is the following.

Theorem 6. *Let $G = (V, E)$ be a directed graph with vertex set $V = \{1, 2, ..., n\}$. Let $\kappa(G, i)$ denote the number of oriented spanning trees of G rooted at i. If L is the Laplacian matrix of G, then*

$$\kappa(G, i) = \det(L[\{i\}^c, \{i\}^c]).$$

Suppose G is also strongly connected and balanced. Then all the cofactors of L are equal and therefore $\kappa(G, i)$ is independent of i. We denote $\kappa(G, i)$ by $\kappa(G)$ in the rest of the paper.

3 Properties of the resistance

To establish the desired properties of the resistance defined in (3), we need the following identity. The proof is omitted as it is a direct verification.

Lemma 1. *Let $L \in \mathbf{Z}(\mathcal{L})$. Then L can be partitioned as*

$$L = \begin{bmatrix} B & -Be \\ -e'B & e'Be \end{bmatrix},$$

where B is a square matrix of order $n-1$ and $e = \mathbf{1}_{n-1}$ and

$$L^\dagger = \begin{bmatrix} B^{-1} - \frac{1}{n}ee'B^{-1} - \frac{1}{n}B^{-1}ee' & -\frac{1}{n}B^{-1}e \\ -\frac{1}{n}e'B^{-1} & 0 \end{bmatrix} + \frac{e'B^{-1}e}{n^2}\mathbf{1}\mathbf{1}'.$$

The following theorem is an application of Lemma 1.

Theorem 7. *Let $L \in \mathbf{Z}(\mathcal{L})$, $L := [l_{ij}]$ and $L^\dagger := [l^\dagger_{ij}]$. Define $r_{ij} := l^\dagger_{ii} + l^\dagger_{jj} - 2l^\dagger_{ij}$. Then,*

(i) $r_{ij} > 0 \ \ \forall i \neq j$.

(ii) $r_{ik} \leq r_{ij} + r_{jk} \quad \forall i, j, k.$

Proof. Define $\Omega := \{1, \ldots, n-1\}$, $B := L[\Omega, \Omega]$ and $C := B^{-1} = [c_{ij}]$. To prove (i), we shall assume without loss of generality that $j = n$ and show that $r_{in} > 0$ for any $i \in \Omega$. Put $e := \mathbf{1}_{n-1}$. By Lemma 1,

$$L^{\dagger} = \begin{bmatrix} B^{-1} - \frac{1}{n}ee'B^{-1} - \frac{1}{n}B^{-1}ee' & -\frac{1}{n}B^{-1}e \\ \\ -\frac{1}{n}e'B^{-1} & 0 \end{bmatrix} + \frac{e'B^{-1}e}{n^2}\mathbf{1}\mathbf{1}'. \tag{6}$$

By a well-known result on $\mathbf{Z}$-matrices, B^{-1} is a non-negative matrix. Therefore, $B^{-1}e$ is a positive vector. Let $x := B^{-1}e$ and $y' := e'B^{-1}$. For any $i \in \Omega$, by (6) we have

$$\begin{aligned} r_{in} &= l^{\dagger}_{ii} + l^{\dagger}_{nn} - 2l^{\dagger}_{in} \\ &= c_{ii} - \frac{1}{n}y_i - \frac{1}{n}x_i + \frac{2}{n}x_i \\ &= c_{ii} - \frac{1}{n}y_i + \frac{1}{n}x_i. \end{aligned} \tag{7}$$

It can be seen that B is row diagonally dominant. In view of (P10), C is diagonally dominant of its column entries and therefore,

$$c_{ii} \geq c_{ji} \;\; \forall j = 1, \ldots, n-1.$$

Thus,

$$nc_{ii} \geq (n-1)c_{ii} \geq \sum_{j=1}^{n-1} c_{ji} = y_i.$$

Hence,

$$c_{ii} \geq \frac{y_i}{n}.$$

Since $x_i > 0$, it follows from (7) that $r_{in} > 0$. This completes the proof of (i).
We now prove (ii). We shall show that if $j, k \in \Omega$, then

$$r_{nk} \leq r_{nj} + r_{jk},$$

and the proof can be completed by using a similar argument applied to any other r_{ik}.
Since

$$\begin{aligned} r_{nk} - r_{nj} - r_{jk} &= l^{\dagger}_{nn} + l^{\dagger}_{kk} - 2l^{\dagger}_{nk} - l^{\dagger}_{nn} - l^{\dagger}_{jj} + 2l^{\dagger}_{nj} - l^{\dagger}_{jj} - l^{\dagger}_{kk} + 2l^{\dagger}_{jk} \\ &= -2(l^{\dagger}_{nk} + l^{\dagger}_{jj} - l^{\dagger}_{nj} - l^{\dagger}_{jk}), \end{aligned}$$

it suffices to show that $l^{\dagger}_{nk} + l^{\dagger}_{jj} - l^{\dagger}_{nj} - l^{\dagger}_{jk} \geq 0$. In view of (6), it follows that

$$\begin{aligned} l^{\dagger}_{nk} + l^{\dagger}_{jj} - l^{\dagger}_{nj} - l^{\dagger}_{jk} &= -\frac{1}{n}y_k + c_{jj} - \frac{1}{n}y_j - \frac{1}{n}x_j + \frac{1}{n}y_j - c_{jk} + \frac{1}{n}y_k + \frac{1}{n}x_j \\ &= c_{jj} - c_{jk}. \end{aligned} \tag{8}$$

Since B' is row diagonally dominant, by (P10), C is diagonally dominant of its row entries, and hence $c_{jj} \geq c_{jk}$. The proof is complete. $\square$

The main result of this section is now immediate from the above result.

Theorem 8. *Let G be a strongly connected and balanced directed graph and $R := [r_{ij}]$ be the resistance matrix of G. Then, every off-diagonal entry of R is positive and thus R is a non-negative matrix. Furthermore, the resistance r_{ij} satisfies the triangle inequality.*

4 Inverse of the resistance matrix

For a resistance matrix R, we now obtain the inverse formula stated in Theorem 3. Since $r_{ij} = l^{\dagger}_{ii} + l^{\dagger}_{jj} - 2l^{\dagger}_{ij}$ and $R = [r_{ij}]$, we have

$$R = \operatorname{diag}(L^{\dagger})J + J\operatorname{diag}(L^{\dagger}) - 2L^{\dagger}. \tag{9}$$

Define $X := (L + \frac{1}{n}J)^{-1}$ and $\tilde{X} := \operatorname{diag}(X)$. By an easy computation, we find that $L^{\dagger} = X - \frac{1}{n}J$ and hence

$$R = \tilde{X}J + J\tilde{X} - 2X.$$

For $i = 1, 2, .., n$, let

$$\tau_i := 2 - \sum_{\{j:(i,j)\in E\}} r_{ji} \quad\text{and}\quad \tau := (\tau_1, \dots, \tau_n)'.$$

Set $M := L - L'$. The inverse formula will be proved by using the following lemma.

Lemma 2. *The following are true.*

(i) $\tau = L\tilde{X}\mathbf{1} + \frac{2}{n}\mathbf{1}$.

(ii) $\tau' + \mathbf{1}'\tilde{X}M = \mathbf{1}'\tilde{X}L + \frac{2}{n}\mathbf{1}'$.

(iii) $LR + 2I = \tau\mathbf{1}'$.

(iv) $RL + 2I = \mathbf{1}\tau' + J\tilde{X}M$.

(v) $\mathbf{1}'\tau = 2$.

(vi) $\tau' R\tau = 2\tilde{x}'L\tilde{x} + \frac{8}{n}\operatorname{trace}(L^{\dagger})$.

(vii) $\tau' R\tau > 0$.

Proof. Fix $i \in \{1, \dots, n\}$. Define $\delta_i := (A\mathbf{1})_i$. From $\big(L + \frac{1}{n}J\big)X = I$, we have

$$\delta_i x_{ii} - \sum_{\{j:(i,j)\in E\}} x_{ji} + \frac{1}{n}\sum_{j=1}^{n} x_{ji} = 1. \tag{10}$$

As $X\mathbf{1} = X'\mathbf{1} = \mathbf{1}$,

$$\sum_{j=1}^{n} x_{ji} = (X'\mathbf{1})_i = 1.$$

Hence from (10),

$$\delta_i x_{ii} - \sum_{\{j:(i,j)\in E\}} x_{ji} = 1 - \frac{1}{n}.$$

and so,

$$\sum_{\{j:(i,j)\in E\}} x_{ji} = \delta_i x_{ii} - 1 + \frac{1}{n}. \tag{11}$$

Also, we see that

$$\begin{aligned} \tau_i &= 2 - \sum_{\{j:(i,j)\in E\}} r_{ji} \\ &= 2 - \sum_{\{j:(i,j)\in E\}} (x_{ii} + x_{jj} - 2x_{ji}) \\ &= 2 - \sum_{\{j:(i,j)\in E\}} x_{ii} - \sum_{\{j:(i,j)\in E\}} x_{jj} + 2 \sum_{\{j:(i,j)\in E\}} x_{ji}. \end{aligned} \tag{12}$$

Since

$$\sum_{\{j:(i,j)\in E\}} x_{ii} = x_{ii} \sum_{j=1}^{n} a_{ij}, \tag{13}$$

and

$$\sum_{j=1}^{n} a_{ij} = (A\mathbf{1})_i = \delta_i, \tag{14}$$

from (12), (13) and (14), we now have

$$\tau_i = 2 - \delta_i x_{ii} - \sum_{\{j:(i,j)\in E\}} x_{jj} + 2 \sum_{\{j:(i,j)\in E\}} x_{ji}.$$

In view of (11),

$$\begin{aligned} \tau_i &= 2 - \delta_i x_{ii} - \sum_{\{j:(i,j)\in E\}} x_{jj} + 2\delta_i x_{ii} - 2 + \frac{2}{n} \\ &= \delta_i x_{ii} - \sum_{\{j:(i,j)\in E\}} x_{jj} + \frac{2}{n}. \end{aligned} \tag{15}$$

Since

$$\begin{aligned} \sum_{\{j:(i,j)\in E\}} x_{jj} &= \sum_{j=1}^{n} a_{ij} x_{jj} \\ &= (A\tilde{X}\mathbf{1})_i \end{aligned} \tag{16}$$

and

$$(\mathrm{Diag}(A\mathbf{1})\tilde{X}\mathbf{1})_i = \delta_i x_{ii}, \tag{17}$$

equations (15), (16) and (17) imply

$$\begin{aligned}\tau_i &= ((\mathrm{Diag}(A\mathbf{1}) - A)\tilde{X}\mathbf{1})_i + \frac{2}{n} \\ &= (L\tilde{X}\mathbf{1} + \frac{2}{n}\mathbf{1})_i.\end{aligned}$$

Thus,

$$\tau = L\tilde{X}\mathbf{1} + \frac{2}{n}\mathbf{1}.$$

The proof of (i) is complete.
We have

$$\begin{aligned}\mathbf{1'}\tilde{X}M + \tau' &= \mathbf{1'}\tilde{X}L - \mathbf{1'}\tilde{X}L' + \mathbf{1'}\tilde{X}L' + \frac{2}{n}\mathbf{1'} \\ &= \mathbf{1'}\tilde{X}L + \frac{2}{n}\mathbf{1'}.\end{aligned}$$

The proof of (ii) is complete.
To prove (iii), recall that

$$R = \tilde{X}J + J\tilde{X} - 2X.$$

As $X = L^\dagger + \frac{1}{n}J$, we have

$$R = \tilde{X}J + J\tilde{X} - 2L^\dagger - \frac{2}{n}J \tag{18}$$

In view of (P8), $LL^\dagger = I - \frac{1}{n}J$. Since $LJ = 0$, by (18),

$$\begin{aligned}LR &= L\tilde{X}J - 2I + \frac{2}{n}J \\ &= L\tilde{X}\mathbf{1}\mathbf{1'} - 2I + \frac{2}{n}\mathbf{1}\mathbf{1'} \\ &= (L\tilde{X}\mathbf{1} + \frac{2}{n}\mathbf{1})\mathbf{1'} - 2I.\end{aligned} \tag{19}$$

By (i),

$$\tau = L\tilde{X}\mathbf{1} + \frac{2}{n}\mathbf{1}.$$

Hence by (19), $LR = \tau\mathbf{1'} - 2I$. This completes the proof of (iii).
To prove (iv), first we observe that

$$\begin{aligned}RL + 2I &= (\tilde{X}J + J\tilde{X} - 2X)L + 2I \\ &= J\tilde{X}L - 2XL + 2I.\end{aligned} \tag{20}$$

Using $X(L+\frac{1}{n}J)=I$ and $XJ=J$, we have $XL=I-\frac{1}{n}J$. Hence by (20),

$$\begin{aligned} RL+2I &= J\tilde{X}L-2(I-\frac{1}{n}J)+2I \\ &= J\tilde{X}L+\frac{2}{n}J. \end{aligned} \tag{21}$$

By (i),

$$\begin{aligned} \mathbf{1}\tau' &= \mathbf{1}(\mathbf{1}'\tilde{X}L'+\frac{2}{n}\mathbf{1}') \\ &= J\tilde{X}L'+\frac{2}{n}J. \end{aligned} \tag{22}$$

From (21) and (22), we get

$$RL+2I-J\tilde{X}L=\mathbf{1}\tau'-J\tilde{X}L',$$

and hence

$$RL+2I=\mathbf{1}\tau'+J\tilde{X}M.$$

The proof of (iv) is done. Using part (i),

$$\mathbf{1}'\tau=\mathbf{1}'L\tilde{X}\mathbf{1}+\frac{2}{n}\mathbf{1}'\mathbf{1}=2.$$

This proves (v).
By using (i), (ii) and $M=L-L'$, we have

$$\begin{aligned} \tau'R\tau &= \Big(\mathbf{1}'\tilde{X}L+\frac{2}{n}\mathbf{1}'-\mathbf{1}'\tilde{X}M\Big)R\Big(L\tilde{X}\mathbf{1}+\frac{2}{n}\mathbf{1}\Big) \\ &= \mathbf{1}'\tilde{X}L'RL\tilde{X}\mathbf{1}+\frac{2}{n}\mathbf{1}'\tilde{X}L'R\mathbf{1}+\frac{2}{n}\mathbf{1}'RL\tilde{X}\mathbf{1}+\frac{4}{n^2}\mathbf{1}'R\mathbf{1}. \end{aligned} \tag{23}$$

As $L\mathbf{1}=0$, $L'\mathbf{1}=0$ and $R=\tilde{X}J+J\tilde{X}-2X$,

$$\begin{aligned} \mathbf{1}'\tilde{X}L'RL\tilde{X}\mathbf{1} &= \mathbf{1}'\tilde{X}L'(\tilde{X}J+J\tilde{X}-2X)L\tilde{X}\mathbf{1} \\ &= -2\mathbf{1}'\tilde{X}L'XL\tilde{X}\mathbf{1}. \end{aligned} \tag{24}$$

As $XL=I-\frac{1}{n}J$, by (24),

$$\begin{aligned} \mathbf{1}'\tilde{X}L'RL\tilde{X}\mathbf{1} = -2\mathbf{1}'\tilde{X}L'\Big(I-\frac{1}{n}J\Big)\tilde{X}\mathbf{1} &= -2\mathbf{1}'\tilde{X}L'\tilde{X}\mathbf{1} \\ &= -2\tilde{x}'L\tilde{x}. \end{aligned} \tag{25}$$

By (21),

$$\begin{aligned} \mathbf{1}'RL\tilde{X}\mathbf{1} = \mathbf{1}'\Big(J\tilde{X}L+\frac{2}{n}J-2I\Big)\tilde{X}\mathbf{1} \quad &= (n\mathbf{1}'\tilde{X}L)\tilde{X}\mathbf{1} \\ = n\tilde{x}'L\tilde{x}. \end{aligned} \tag{26}$$

Since $L'\mathbf{1} = 0$ and $X\mathbf{1} = \mathbf{1}$,

$$\begin{aligned}\mathbf{1}'\tilde{X}L'R\mathbf{1} &= \mathbf{1}'\tilde{X}L'(\tilde{X}J + J\tilde{X} - 2X)\mathbf{1} \\ &= n\mathbf{1}'\tilde{X}L'\tilde{X}\mathbf{1} + 2\mathbf{1}'\tilde{X}L'\mathbf{1} \\ &= n\tilde{x}'L\tilde{x}.\end{aligned} \tag{27}$$

From $R = \tilde{X}J + J\tilde{X} - 2X$ and $X\mathbf{1} = 1$, we have

$$\mathbf{1}'R\mathbf{1} = 2n \text{ trace}(X) - 2n = 2n \text{ trace}(L^{\dagger}). \tag{28}$$

Substituting (25), (26), (27) and (28) in (23), we get

$$\tau' R\tau = 2\tilde{x}'L\tilde{x} + \frac{8}{n}\text{trace}(L^{\dagger}).$$

Since $L + L'$ is positive semidefinite, $\tilde{x}'L\tilde{x} \geq 0$. As trace of $L^{\dagger}$ is also positive, we get (vii). The proof is complete. □

Theorem 9.

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R\tau}(\tau(\tau' + 1' \operatorname{diag}(L^{\dagger})M)),$$

where $M = L - L^T$.

Proof. By item (iii) of Lemma 2,

$$LR + 2I = \tau\mathbf{1}'.$$

In view of item (v) of the previous Lemma, $\mathbf{1}'\tau = 2$. So,

$$LR\tau + 2\tau = (\mathbf{1}'\tau)\tau = 2\tau.$$

This implies $LR\tau = 0$ and since $L \in \mathbf{Z}(\mathcal{L})$, there exists $0 \neq \alpha \in \mathbb{R}$ such that $R\tau = \alpha\mathbf{1}$. As $\tau'\mathbf{1} = 2$, we get $\alpha = \frac{1}{2}\tau' R\tau$. Therefore,

$$R\tau = \frac{\tau' R\tau}{2}\mathbf{1}. \tag{29}$$

Since $M\mathbf{1} = 0$, from item (iv) of Lemma 2, we deduce that

$$\begin{aligned}(\tau' + \mathbf{1}'\tilde{X}M)(RL + 2I) &= (\tau' + \mathbf{1}'\tilde{X}M)(\mathbf{1}\tau' + J\tilde{X}M) \\ &= 2(\tau' + \mathbf{1}'\tilde{X}M).\end{aligned}$$

After simplification the above equation leads to

$$(\tau' + \mathbf{1}'\tilde{X}M)RL = 0.$$

We now claim that $(\tau' + \mathbf{1}'\tilde{X}M)R \neq 0$. If not, then $\tau' R\tau + \mathbf{1}'\tilde{X}MR\tau = 0$. By (29), $R\tau$ is a multiple of $\mathbf{1}$. So, $MR\tau = 0$ and hence $\tau' R\tau = 0$. This contradicts the previous Lemma. Hence, $(\tau' + \mathbf{1}'\tilde{X}M)R \neq 0$. As $L \in \mathbf{Z}(\mathcal{L})$, it follows that

$$(\tau' + \mathbf{1}'\tilde{X}M)R = \beta\mathbf{1}',$$

for some $\beta \neq 0$. Since $\mathbf{1}'\tau = 2$, $\beta = \frac{1}{2}\tau' R\tau$. Thus,

$$(\tau' + \mathbf{1}'\tilde{X}M)R = \frac{\tau' R\tau}{2}\mathbf{1}'. \tag{30}$$

Now, item (iii) of Lemma 2 and (30) imply

$$\begin{aligned}\Big(-\frac{1}{2}L + \frac{\tau(\tau' + 1'\tilde{X}M)}{\tau' R\tau}\Big)R &= -\frac{1}{2}LR + \frac{1}{\tau' R\tau}\tau(\tau' + 1'\tilde{X}M)R \\ &= I - \frac{1}{2}\tau\mathbf{1}' + \frac{1}{\tau' R\tau}\Big(\frac{\tau' R\tau}{2}\Big)\tau\mathbf{1}' \\ &= I.\end{aligned}$$

Since $L^{\dagger} = X - \frac{1}{n}J$, $\mathbf{1}'\tilde{X}M = \mathbf{1}'(\operatorname{diag}(L^{\dagger}) + \frac{1}{n}I)M = \mathbf{1}'\operatorname{diag}(L^{\dagger})M$. The proof is complete. $\square$

To illustrate the inverse formula in Theorem 9, we consider the resistance matrix of Example 1.

Example 3. Consider the resistance matrix in Example 1.

$$R = \begin{bmatrix} 0 & \frac{2}{3} & \frac{5}{3} & \frac{17}{12} & \frac{13}{12} & \frac{5}{3} \\ \frac{4}{3} & 0 & 1 & \frac{3}{4} & \frac{5}{12} & 1 \\ \frac{7}{3} & 1 & 0 & \frac{7}{4} & \frac{17}{12} & 2 \\ \frac{19}{12} & \frac{5}{4} & \frac{9}{4} & 0 & \frac{2}{3} & \frac{5}{4} \\ \frac{11}{12} & \frac{7}{12} & \frac{19}{12} & \frac{4}{3} & 0 & \frac{7}{12} \\ \frac{1}{3} & 1 & 2 & \frac{7}{4} & \frac{17}{12} & 0 \end{bmatrix}. \tag{31}$$

Then we have the following:

$$\begin{aligned}\tau &= \begin{bmatrix} \frac{2}{3} & -\frac{5}{6} & 1 & \frac{2}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}', \\ \tau' + 1'\operatorname{diag}(L^{\dagger})M &= \begin{bmatrix} \frac{1}{3} & -\frac{3}{4} & 1 & \frac{3}{4} & \frac{1}{12} & \frac{7}{12} \end{bmatrix},\end{aligned}$$

and

$$\tau' R\tau = \frac{67}{12}. \tag{32}$$

We now have

$$\begin{aligned}R^{-1} &= -\frac{1}{2}L + \frac{1}{\tau' R\tau}(\tau(\tau' + 1'\operatorname{diag}(L^{\dagger})M)) \\ &= \begin{bmatrix} -\frac{185}{402} & \frac{55}{134} & \frac{8}{67} & \frac{6}{67} & \frac{2}{201} & \frac{14}{201} \\ -\frac{10}{201} & -\frac{93}{67} & \frac{47}{134} & \frac{26}{67} & \frac{98}{201} & -\frac{35}{402} \\ \frac{4}{67} & \frac{49}{134} & -\frac{43}{134} & \frac{9}{67} & \frac{1}{67} & \frac{7}{67} \\ \frac{8}{201} & -\frac{6}{67} & \frac{8}{67} & -\frac{55}{134} & \frac{205}{402} & \frac{14}{201} \\ \frac{2}{201} & \frac{32}{67} & \frac{2}{67} & \frac{3}{134} & -\frac{401}{402} & \frac{104}{201} \\ \frac{209}{402} & -\frac{3}{67} & \frac{4}{67} & \frac{3}{67} & \frac{1}{201} & -\frac{187}{402} \end{bmatrix}.\end{aligned}$$

4.1 Determinant of the resistance matrix

By using Theorem 9, we compute an expression for the determinant of the resistance matrix.

Corollary 1.

$$\det(R) = (-1)^{n-1}2^{n-3}\frac{\tau' R\tau}{\kappa(G)}.$$

Proof. By using Theorem 9 and (P9), we have

$$\begin{aligned}\det(R^{-1}) &= \frac{1}{\tau' R\tau}(\tau' + \mathbf{1}' \operatorname{diag}(L^{\dagger})M) \operatorname{adj}(-\frac{1}{2}L)\tau \\ &= \Big(-\frac{1}{2}\Big)^{n-1}\frac{\kappa(G)}{\tau' R\tau}(\tau' + \mathbf{1}' \operatorname{diag}(L^{\dagger})M)J\tau \\ &= \Big(-\frac{1}{2}\Big)^{n-1}\frac{\kappa(G)}{\tau' R\tau}\tau' J\tau.\end{aligned}$$

Since $\mathbf{1}'\tau = 2$, it follows that

$$\det(R) = (-1)^{n-1}2^{n-3}\frac{\tau' R\tau}{\kappa(G)}.$$

$\square$

Example 4. Consider the directed graph G on six vertices given in Figure 1(a). G has two oriented spanning trees T_1 and T_2 (see Figure 3(a) and 3(b), respectively) rooted at vertex 1. Thus, $\kappa(G) = 2$. From Example 3, $\tau' R\tau = \frac{67}{12}$. By Corollary 1,

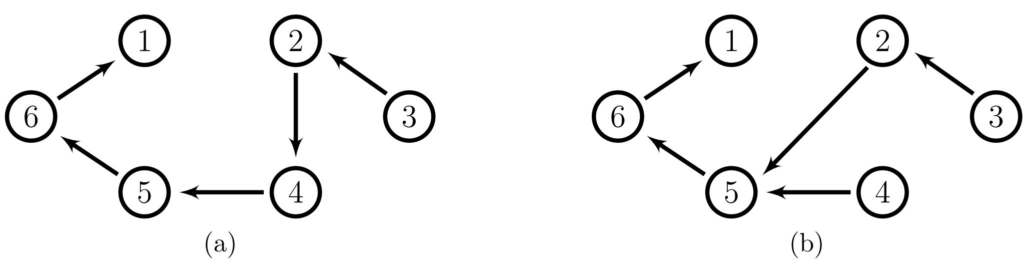


Figure 3: (a) spanning tree T_1 (b) spanning tree T_2

we have

$$\begin{aligned}\det(R) &= (-1)^{n-1}2^{n-3}\frac{\tau' R\tau}{\kappa(G)} \\ &= -\frac{67}{3}.\end{aligned} \tag{33}$$

5 Cofactor sums of the resistance matrix

Let $\Omega_1, \Omega_2 \subset \{1, 2, \ldots, n\}$ be non-empty and $|\Omega_1| = |\Omega_2|$. Define $\eta := |\Omega_1| = |\Omega_2|$. We now derive an identity for computing the sum of all the entries in the cofactor matrix of $R[\Omega_1, \Omega_2]$. We shall use the following elementary lemma repeatedly. The proof is immediate.

Lemma 3. *Let B be an $m \times m$ matrix, $\beta \in \mathbb{R}$ and*

$$A = \begin{bmatrix} B & \frac{1}{\beta}\mathbf{1}_m \\ \frac{1}{\beta}\mathbf{1}'_m & 0 \end{bmatrix}.$$

Then,

$$\mathrm{cofsum}(B) = -\beta^2 \det(A).$$

We now obtain the following identity.

Lemma 4. *Let S be a $n \times n$ matrix. Suppose rank$(S) = n - 1$, $S\mathbf{1} = 0$ and $S'\mathbf{1} = 0$. Then,*

$$\mathrm{cofsum}(S[\Omega_1, \Omega_2]) = (-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)} n^2 \gamma \det(S^\dagger[\Omega_2^c, \Omega_1^c]).$$

where γ is the common cofactor value of S.

Proof. Let

$$A := \begin{bmatrix} S & \frac{1}{\sqrt{n}}\mathbf{1} \\ \frac{1}{\sqrt{n}}\mathbf{1}' & 0 \end{bmatrix}.$$

Then A is non-singular and in fact,

$$A^{-1} = \begin{bmatrix} S^\dagger & \frac{1}{\sqrt{n}}\mathbf{1} \\ \frac{1}{\sqrt{n}}\mathbf{1}' & 0 \end{bmatrix}. \tag{34}$$

Define $\widetilde{S} := S[\Omega_1, \Omega_2]$. By Lemma 3,

$$\mathrm{cofsum}(\widetilde{S}) = -n \det\left(\begin{bmatrix} \widetilde{S} & \frac{1}{\sqrt{n}}\mathbf{1}_\eta \\ \frac{1}{\sqrt{n}}\mathbf{1}'_\eta & 0 \end{bmatrix} \right). \tag{35}$$

Define

$$\Delta_1 := \Omega_1 \cup \{n+1\} \quad \text{and} \quad \Delta_2 := \Omega_2 \cup \{n+1\}.$$

Then,

$$A[\Delta_1, \Delta_2] = \begin{bmatrix} \widetilde{S} & \frac{1}{\sqrt{n}}\mathbf{1}_\eta \\ \frac{1}{\sqrt{n}}\mathbf{1}'_\eta & 0 \end{bmatrix}.$$

By rewriting equation (35), we have

$$\mathrm{cofsum}(\widetilde{S}) = -n \det(A[\Delta_1, \Delta_2]). \tag{36}$$

By Jacobi's formula (P6)

$$\det(A[\Delta_1, \Delta_2]) = (-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)} \frac{\det(A^{-1}[\Delta_2^c, \Delta_1^c])}{\det(A^{-1})}. \tag{37}$$

From (36) and (37), we get

$$\text{cofsum}(\widetilde{S}) = (-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)+1} n \frac{\det(A^{-1}[\Delta_2^c, \Delta_1^c])}{\det(A^{-1})}. \tag{38}$$

Using equation (34),

$$A^{-1}[\Delta_2^c, \Delta_1^c] = S^{\dagger}[\Omega_2^c, \Omega_1^c]. \tag{39}$$

Again applying Lemma 3,

$$\det(A) = -\frac{1}{n}\text{cofsum}(S) = -n\gamma.$$

where γ is the common cofactor value of S. So,

$$\det(A^{-1}) = -\frac{1}{n\gamma}. \tag{40}$$

By (38),(39) and (40),

$$\text{cofsum}(\widetilde{S}) = (-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)} n^2 \gamma \det(S^{\dagger}[\Omega_2^c, \Omega_1^c]).$$

The proof is complete. □

Lemma 5. *Let A be a $n \times n$ matrix and let $P = I - \frac{1}{n}\mathbf{1}\mathbf{1}'$. Define $S := PAP$. Then,*

$$\text{cofsum}(A) = \text{cofsum}(S) \quad \textit{and} \quad \text{cofsum}(A[\Omega_1, \Omega_2]) = \text{cofsum}(S[\Omega_1, \Omega_2]).$$

Proof. We begin by noting that

$$\begin{aligned} S &= (I - \frac{1}{n}\mathbf{1}\mathbf{1}')A(I - \frac{1}{n}\mathbf{1}\mathbf{1}') \\ &= A - \frac{1}{n}A\mathbf{1}\mathbf{1}' - \frac{1}{n}\mathbf{1}\mathbf{1}'A + \frac{\mathbf{1}'A\mathbf{1}}{n^2}(\mathbf{1}\mathbf{1}'). \end{aligned} \tag{41}$$

Let $e := \mathbf{1}_\eta$. By (41),

$$S[\Omega_1, \Omega_2] = A[\Omega_1, \Omega_2] + ue' + ev' + \beta ee', \tag{42}$$

for some vectors u, v in $\mathbb{R}^\eta$ and for some real scalar β. We now claim that if $x \in \mathbb{R}^\eta$, and if B is an $\eta \times \eta$ matrix, then

$$\text{cofsum}(B + x\mathbf{1}_\eta') = \text{cofsum}(B).$$

Using (P9), we get

$$\begin{aligned}\operatorname{cofsum}(B+x\mathbf{1}'_\eta) &= \mathbf{1}'\operatorname{adj}(B+x\mathbf{1}'_\eta)\mathbf{1}\\ &= \det(B+x\mathbf{1}'_\eta+\mathbf{1}_\eta\mathbf{1}'_\eta)-\det(B+x\mathbf{1}'_\eta)\\ &= \det(B+(x+\mathbf{1}_\eta)\mathbf{1}'_\eta)-\det(B+x\mathbf{1}'_\eta)\\ &= \det(B)+\mathbf{1}'_\eta\operatorname{adj}(B)(x+\mathbf{1}_\eta)-\det(B)-\mathbf{1}'_\eta\operatorname{adj}(B)x\\ &= \mathbf{1}'_\eta\operatorname{adj}(B)\mathbf{1}_\eta=\operatorname{cofsum}(B).\end{aligned} \tag{43}$$

Similarly, we see that

$$\operatorname{cofsum}(B+\mathbf{1}_\eta x')=\operatorname{cofsum}(B). \tag{44}$$

Repeatedly using (43) and (44) in (41) and (42), we obtain

$$\operatorname{cofsum}(A)=\operatorname{cofsum}(S) \text{ and } \operatorname{cofsum}(A[\Omega_1,\Omega_2])=\operatorname{cofsum}(S[\Omega_1,\Omega_2]).$$

This completes the proof. □

By Lemma 4 and 5, we now obtain the following result.

Theorem 10. *Let S be an $n\times n$ matrix such that $\operatorname{rank}(S)=n-1$, $S\mathbf{1}=0$ and $S'\mathbf{1}=0$. Define $D=[d_{ij}]$ by*

$$D=\operatorname{diag}(S)J+J\operatorname{diag}(S)-2S.$$

Then,

$$\operatorname{cofsum}(D[\Omega_1,\Omega_2])=(-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)+\eta-1}2^{\eta-1}n^2\gamma\det(S^\dagger[\Omega_2^c,\Omega_1^c]),$$

where γ is the common cofactor value of S.

Proof. Pre and post multiplying by P in the equation

$$D=\operatorname{diag}(S)J+J\operatorname{diag}(S)-2S,$$

we have

$$PDP=-2S.$$

Thus, by Lemma 5,

$$\begin{aligned}\operatorname{cofsum}(S[\Omega_1,\Omega_2]) &= \operatorname{cofsum}(-\frac{1}{2}D[\Omega_1,\Omega_2])\\ &= \Big(\frac{-1}{2}\Big)^{\eta-1}\operatorname{cofsum}(D[\Omega_1,\Omega_2]).\end{aligned} \tag{45}$$

Using Lemma 4 in (45), we get

$$\operatorname{cofsum}(D[\Omega_1,\Omega_2])=(-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)+\eta-1}2^{\eta-1}n^2\gamma\det(S^\dagger[\Omega_2^c,\Omega_1^c]).$$

The proof is complete. □

It can be noted that Theorem 4 follows from Theorem 10 immediately. Applying Theorem 4 to resistance matrices of strongly connected balanced directed graphs, we get the following.

Theorem 11. *Let G be a strongly connected balanced directed graph with vertex set $\{1,2,\ldots,n\}$, Laplacian matrix L and resistance matrix R. Then the following items hold.*

(i) $\mathrm{cofsum}(R[\Omega_1,\Omega_2]) = (-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)+\eta-1}\frac{2^{\eta-1}}{\kappa(G)}\det(L[\Omega_2^c,\Omega_1^c])$.

(ii) For every distinct $i,j \in \{1,2,..,n\}$,

$$r_{ij}+r_{ji} = \frac{2}{\kappa(G)}\det(L[\{i,j\}^c,\{i,j\}^c]).$$

Proof. (i) Since

$$R = \mathrm{diag}(L^\dagger)J + J\,\mathrm{diag}(L^\dagger) - 2L^\dagger,$$

by Theorem 10 it follows that

$$\mathrm{cofsum}(R[\Omega_1,\Omega_2]) = (-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)+\eta-1}2^{\eta-1}n^2\delta\det(L[\Omega_2^c,\Omega_1^c]), \tag{46}$$

where δ is the common cofactor value of $L^\dagger$. Let

$$A := \begin{bmatrix} L & \frac{1}{\sqrt{n}}\mathbf{1} \\ \frac{1}{\sqrt{n}}\mathbf{1}' & 0 \end{bmatrix}.$$

Then A is non-singular and,

$$A^{-1} = \begin{bmatrix} L^\dagger & \frac{1}{\sqrt{n}}\mathbf{1} \\ \frac{1}{\sqrt{n}}\mathbf{1}' & 0 \end{bmatrix}.$$

By Lemma 3, we have $\mathrm{cofsum}(L^\dagger) = -n\det(A^{-1})$ and $\mathrm{cofsum}(L) = -n\det(A)$. Thus,

$$\mathrm{cofsum}(L) = \frac{n^2}{\mathrm{cofsum}(L^\dagger)} = \frac{1}{\delta}.$$

and hence

$$\kappa(G) = \frac{\mathrm{cofsum}(L)}{n^2} = \frac{1}{n^2\delta}. \tag{47}$$

By (46) and (47), we have

$$\mathrm{cofsum}(R[\Omega_1,\Omega_2]) = (-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)+\eta-1}\frac{2^{\eta-1}}{\kappa(G)}\det(L[\Omega_2^c,\Omega_1^c]).$$

The proof of (i) is complete.

(ii) Let $i, j \in \{1, 2, .., n\}$ be such that $i \neq j$. Substituting $\Omega_1 = \Omega_2 = \{i, j\}$ in (i), we get

$$\operatorname{cofsum}(R[\Omega_1, \Omega_2]) = (-1)^{2i+2j+1} \frac{2}{\kappa(G)} \det(L[\Omega_2^c, \Omega_1^c]). \tag{48}$$

As $\operatorname{cofsum}(R[\Omega_1, \Omega_2]) = -(r_{ij} + r_{ji})$, by (48),

$$r_{ij} + r_{ji} = \frac{2}{\kappa(G)} \det(L[\{i, j\}^c, \{i, j\}^c]).$$

This completes the proof of (ii).

□

To illustrate the above theorem, we present the following example.

Example 5. Consider the directed graph G on four vertices given in Figure 4(a). G

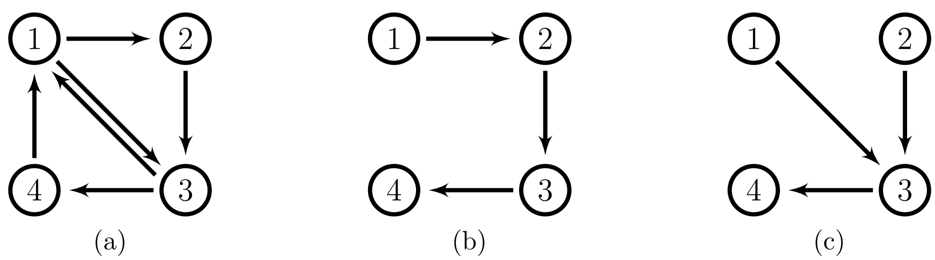


Figure 4: (a) Graph G, (b) spanning tree T_1 and (c) spanning tree T_2

has two oriented spanning trees T_1 and T_2 rooted at vertex 4 (see Figure 4(b) and 4(c)). Thus, $\kappa(G) = 2$. The Laplacian and resistance matrices of G are

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{2} & \frac{5}{4} \\ \frac{5}{4} & 0 & \frac{3}{4} & \frac{3}{2} \\ \frac{1}{2} & \frac{5}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{2} & \frac{5}{4} & 0 \end{bmatrix}.$$

Let $\Omega_1 = \{1, 2\}$ and $\Omega_2 = \{1, 4\}$. Now,

$$R[\Omega_1, \Omega_2] = \begin{bmatrix} 0 & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{2} \end{bmatrix}, \quad \operatorname{cofsum}(R[\Omega_1, \Omega_2]) = -1,$$

$$L[\Omega_2^c, \Omega_1^c] = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}, \quad \det(L[\Omega_2^c, \Omega_1^c]) = 1 \text{ and } \alpha(\Omega_1) + \alpha(\Omega_2) + \eta - 1 = 9.$$

Hence,

$$(-1)^{\alpha(\Omega_1)+\alpha(\Omega_2)+\eta-1} \frac{2^{\eta-1}}{\kappa(G)} \det(L[\Omega_2^c, \Omega_1^c]) = -1 = \operatorname{cofsum}(R[\Omega_1, \Omega_2]).$$

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References

[1] Fiedler M. Algebraic connectivity of graphs. Czech Math J. 1973;23(98):298–305.

[2] Gutman I, Polansky OE. Mathematical concepts in organic chemistry. Berlin: Springer-Verlag; 1986.

[3] Graham RL, Pollak HO. On the addressing problem for loop switching. Bell Syst Tech J. 1971;50:2495–2519.

[4] Graham RL, Lovász L. Distance matrix polynomials of trees. Adv Math. 1978;29:60–88.

[5] Klein DJ, Rándic M. Resistance distance. J Math Chem. 1993;12:81–95.

[6] Bapat RB. Graphs and Matrices. Hindustan Book Agency: TRIM 58; 2012.

[7] Chai Wah Wu. Algebraic connectivity of directed graphs. Linear and Multilinear Algebra. 2005;53:203–223.

[8] Bapat RB. Resistance distance in graphs. Math Student. 1999;68:87–98.

[9] Bapat RB, Sivasubramanian S. Identities for minors of the Laplacian, resistance and distance matrices. Linear Algebra Appl. 2011;435:1479–1489.

[10] Brualdi R, Schneider H. Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley. Linear Algebra Appl. 1983;52,53:769–791.

[11] Ding J, Zhou A. Eigenvalues of rank-one updated matrices with some applications. Appl. Math. Lett. 2007;20:1223–1226.

[12] Horn RA, Johnson CR. Topics in matrix analysis. Cambridge: Cambridge University Press; 1994.

[13] Patrick De Leenheer. An elementary proof of a matrix tree theorem for directed graphs. arXiv:1904.12221v1. 2019.