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Reduced Order Models in Analysis of Stochastically Parametered Linear Dynamical Systems

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Abstract

This study focusses on the development of reduced order models, which minimize the computational costs without compromising on the accuracy in the numerical analysis of stochastically parametered linear dynamical systems. A scheme based on polynomial chaos expansion (PCE) and system equivalent reduction expansion process (SEREP) has been developed that enable formulation of reduced order models. Further measures for enhancing the computational efficiency include using sparse grids in conjunction with code parallelization. Interfacing algorithms have been developed that enable finite element (FE) modeling of complex systems using commercial FE softwares and the developed codes.

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Keywords: Reduced order modelling;SEREP; PCE, PETSc.

1. Introducion

Complex dynamical systems can be modeled fairly accurately using high fidelity finite element (FE) models, characterized by large number of degrees-of-freedom (dof). As a result, dimension of the global system matrices developed from the FE discretized equations of motion become very large. Consequently, the analysis of these dynamical systems, even if they are linear, is computationally intensive. In addition, if one is interested in estimating the failure probability of the structure or estimating the accuracy associated with the response analysis, uncertainty quantification (UQ) analysis is essential. Unfortunately, this further increases the computational complexities associated with the numerical analysis.

The focus of the present study is to develop reduced order models for linear dynamical systems having parameter uncertainties. The uncertainties in specification of the structure parameters are incorporated into the numerical model by modeling them as random variables. The structural system is modeled using FE. The corresponding FE structure matrices are functions of the stochastic parameters and hence are random. Reduction in the computational costs is achieved by reducing the size of the system matrices using SEREP [1]. The time histories of the response are obtained

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by adopting a modal based analysis carried out in the reduced subspace. Obviously, this implies the need for the solution of the random eigenvalue problem. A solution for the random eigenvalue problem is obtained within the PCE framework [2–4]. Reduction in the stochastic dimension is achieved by retaining only the first few dominant stochastic modes. A sparse grid based stochastic collocation is used to estimate the projections along the stochastic basis functions. Further computational reduction is achieved by parallelizing the developed codes using PETSc [5,6] in C platform. Interfacing algorithms are developed that enable FE modeling using commercial FE softwares and importing the structure FE matrices and integrating them with the developed algorithms.

2. Methodology

Consider a stochastic linear dynamical system whose FE discretized equations of motion are given as

$$\mathbf{M}(\theta)\ddot{\mathbf{X}}(t,\theta) + \mathbf{C}(\theta)\dot{\mathbf{X}}(t,\theta) + \mathbf{K}(\theta)\mathbf{X}(t,\theta) = \mathbf{F}(t,\theta). \tag{1}$$

Here, $\mathbf{M}(\theta)$, $\mathbf{C}(\theta)$, $\mathbf{K}(\theta) \in \Re^{N\times N}$ are system matrices, $\mathbf{F}(t,\theta)$, $\mathbf{X}(t,\theta)$, $\dot{\mathbf{X}}(t,\theta)$, $\ddot{\mathbf{X}}(t,\theta) \in \Re^{N\times 1}$ are vectors and θ denotes the random dimension. The problem can be converted into the modal domain by transforming the problem into the space of the natural coordinates, through the transformation

$$\mathbf{X}(t,\theta) = \mathbf{\Phi}(\theta)\mathbf{Z}(t,\theta), \tag{2}$$

where, $\mathbf{\Phi}(\theta)$ is obtained by solving the random eigenvalue problem

$$\mathbf{K}(\theta)\mathbf{\Phi}(\theta) = \mathbf{\Lambda}(\theta)\mathbf{M}(\theta)\mathbf{\Phi}(\theta). \tag{3}$$

Here, $\mathbf{\Lambda}(\theta) \in \Re^{N\times N}$ is the random eigenvalue matrix, $\boldsymbol{\phi}(\theta) \in \Re^{N\times 1}$ are the corresponding eigenvectors and $\mathbf{\Phi}(\theta) \in \Re^{N\times N}$ is the eigenmatrix. A solution to this problem is achieved by projecting the uncertain parameters into the Hilbert space and using polynomial chaos expansions (PCE) to obtain approximations for the random eigenvalues and eigenvectors. The $j-th$ eigenvalue and eigenvector can be represented using PCE as [4]

$$\lambda_j = \sum_{i=0}^{\infty} {\lambda_j}^i \psi_i(\xi), \quad \phi_j = \sum_{i=0}^{\infty} {\phi_j}^i \psi_i(\xi), \tag{4}$$

where, ψ_i are the Hermite polynomials of standard normal random variables $\boldsymbol{\xi}$, ${\lambda_j}^i$ and ${\phi_j}^i$ are the unknown deterministic chaos coefficients. The series in Eq.(4) are truncated to N terms such that only the stochastic dominant modes are retained. SEREP is next used to find a low dimension subspace $\mathbf{T}(x_j) \in \Re^{N\times m}$ with $m<<N$ in order to approximate the state vector $\mathbf{X}(t,x_j)$ in a reduced subspace *i.e.*, $\mathbf{X}(t,x_j) = \mathbf{T}(x_j)\mathbf{X_a}(t,x_j)$. Here x_js are the collocation points. Projecting Eq.(1) on this subspace leads to a set of reduced order differential equations of the form,

$$\mathbf{M}_a(x_j)\ddot{\mathbf{X}}_a(t,x_j) + \mathbf{C}_a(x_j)\dot{\mathbf{X}}_a(t,x_j) + \mathbf{K}_a(x_j)\mathbf{X}_a(t,x_j) = \mathbf{F}_a(t,x_j), \tag{5}$$

and

$$\mathbf{X}(t,x_j) = \begin{Bmatrix} \mathbf{X}_a(t,x_j) \\ \mathbf{X}_d(t,x_j) \end{Bmatrix} = \begin{Bmatrix} \mathbf{\Phi}_a(x_j) \\ \mathbf{\Phi}_d(x_j) \end{Bmatrix} {\mathbf{\Phi}_a}^g(t,x_j)\mathbf{X}_a(t,x_j) = \mathbf{T}(x_j)\mathbf{X_a}(t,x_j). \tag{6}$$

The nodal response is also expressed using PCE in terms of the bases in Eq.(4), with the corresponding projections being the unknowns. Rather than using the stochastic Galerkin approach which necessitates cumbersome algebraic manipulations, especially on account of the nonlinearity in the stochastic parameters, the stochastic collocation technique is used to estimate these projections [7,8]. This involves solving the forward problem a large number of times corresponding to the tensorial grid points used in collocation. The number of such evaluations are reduced by adopting a sparse grid based approach [9]. The computational efficiency is further enhanced by parallelizing the numerical codes using PETSc formulation in C. The next section briefly discusses about PETSc and different eigenvalue solvers used for the parallel computing.

2.1. Introduction to PETSc

PETSc is an acronym for Portable Extensible toolkit for Scientific Computation and is a package of data structures and routines to address mathematical problems arising out of partial differential equations, solve them in parallel over millions of nodes and produce graphical outputs for users. The library supports C, C++, FORTRAN, PYTHON and MATLAB programming languages and can be interfaced with various external packages such as BLAS, LAPACK - used for linear algebra routines, MUMPS - used for massively parallel sparse direct solvers, CUSP - a C++ template used to store matrix library for GPUs, MATLAB, superLU - a parallel or sequential matrix decomposition routine and many more. The basic parallel communication routines for PETSc are derived from MPI. The linear algebra routines are derived from the parent libraries such as BLAS and LAPACK. PETScs tailor made routines are very useful in formulating problems of partial differential equations, time stepping, linear equations, matrix and vector operations, eigenvalue problem solvers etc [10]. The problems under consideration in this study include linear algebra operations, eigenvalue computations, numerical integration and interfacing with commercial software like MATLAB.

2.2. Eigenvalue solvers

PETSc provides an easy interfacing with many packages which provide efficient eigenvalue solver e.g, SLEPc, LAPACK, Elemental. The eigensolvers provided by LAPACK and SLEPc [11] have been used in this study. SLEPc is built on top of PETSc and hence the syntax are the same as PETSc. PETSc has also incorporated generalized eigenvalue solver provided by LAPACK into it. Eigenvalue solvers used in this study are EPS and LAPACKE sygvx. EPS solvers are provided by SLEPc and can run in parallel. LAPACKE sygvx are not designed to run in parallel. However, this study consider both of them and use them according to the demands of the problem as in many cases the parallel eigensolvers take more time than sequential solvers because of the hidden costs associated with parallelization. Commercial software MATLAB [MATLAB (2010)] also uses LAPACK sygvx routine at the back end of eig (A,B) command. The study performed by [12] shows that MATLAB has an impressive performance for applications where system dimensions are of order 10^4 to 10^5 and where few eigenvalues are required. SLEPc on the other hand, proves to be superior when the system dimensions exceed 10^6 as seen from Fig. 1. Among the eigenvalue solvers tested by the authors, SLEPc has been found to consume least memory in eigenvalue computations; see Fig. 2. The present study demonstrates the PETSc formulation of the random eigenvalue analysis using LAPACK as well as SLEPc.

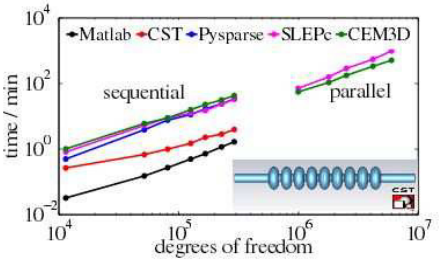

Fig. 1. comparion of computational time taken by different solvers.

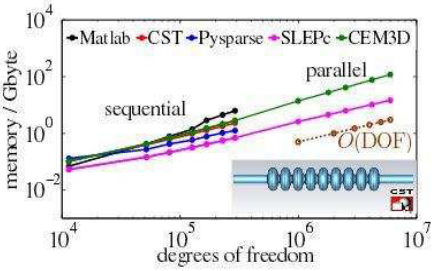

Fig. 2. comparison of memory used by different solvers.

3. Results and discussions

To demonstrate the proposed method, a simple numerical example involving a cantilever Euler-Bernoulli beam is considered. A random eigenvalue analysis has been carried out first to identify the dominant modes and then those are used to evaluate the nodal responses by modal analysis. For modeling the randomness, two cases are considered: Case 1, where the Youngs modulus denoted by E_e is assumed to be random and Case 2, where Youngs modulus E_e, and the mass density denoted by ρ_e are assumed to be random. The standard deviations of the random parameters are assumed to be 10% of their mean values. Solutions for the governing equations are evaluated using Newmark-Beta

method. Stability parameters taken for the Newmark-Beta method for case 1 and case 2 are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{3}{2}$, $\frac{4}{5}$ respectively. The FE models for the systems have been developed in a commercial FE software and have been integrated with the developed codes using interfacing algorithms. The programs have been executed on a 32 GB RAM, 3.40 GHz 12 Intel i7 processor computer. The results have been validated using full scale Monte Carlo simulations (MCS).

The beam is taken to be of length 310mm and has a rectangular cross section of 25.65mm × 3.25mm. The concentrated tip load is taken to be of the form $P\sin(\omega t)$ where the amplitude P is 10 units and the excitation frequency ω is 600rad/s respectively. In developing the FE equations of motion of the form Eq. (1) for case (1), the beam is discretized into 5 elements with three dofs (u, v, θ) at each node. Thus, the size of the global matrices $N = 18$. The model is reduced using SEREP by considering only first four dominant modes and last four displacement dofs. For case (2), two dofs (v, θ) at each node are considered since the axial displacements (u) are negligible for the beam element and the total dofs for the beam becomes 12. Here, the reduced model consists of first three dominant modes and last three transverse displacement dofs. In both cases, displacement of the nodal response at the tip of the beam is chosen for comparing the results.

3.1. Case 1 - one random variable, E_e

E_e is assumed to be Gaussian distributed with a mean value of 50GPa in the 5^{th} element. A 5^{th} order Hermite polynomial is used to obtain the PCE (full) and the PCE (SEREP) models. $10,000$ samples are used for full scale MCS. The mean natural frequencies obtained from the reduced model (PCE - SEREP) are compared with the full model (PCE and MCS) and are listed in Table. 1. Probability density function (pdf) of the first natural frequency has been evaluated and compared in Fig. 3. Nodal responses are calculated using modal analysis. Fig. 4 compares the mean time history of the displacement and a good match is observed between the reduced model and the full model. The eigenvalue solvers used by PETSc takes more time to solve the given system matrices compared to that of MATLAB but consumes 70% less memory on the disk as compared to MATLAB. The parallelization of the solver, like EPS, pays off only when the system matrices exceed the order of 10^6 [10], and hence EPS solvers do not show time performance improvement for the cantilever beam problem either. PETSc formulations are validated for this simple case and will be used for large ordered system in future to show its computational advantages.

Table 1. Comparison of mean of first 4 natural frequencies obtained from different software platforms for full and reduced models.

MATLAB (MCS-full)	PETSc (MCS-full)	COMSOL (MCS- full)	MATLAB (PCE-full)	MATLAB (PCE-SEREP)
23.6770	23.6770	23.6770	23.6770	23.6770
148.4521	148.4519	148.4528	148.4520	148.4520
416.9388	416.9384	416.9753	416.9386	416.9386
823.6157	823.6147	824.1198	823.6154	823.6154

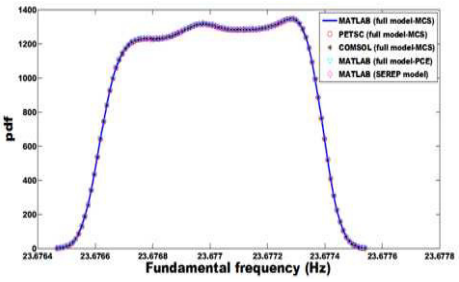

Fig. 3. comparison of the pdf of fundamental frequency.

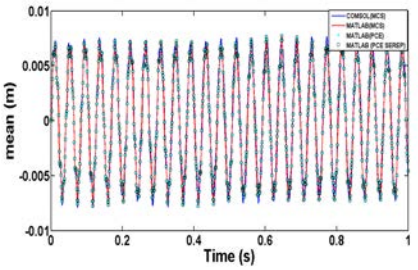

Fig. 4. comparison of the time history of tip mean response.

Gaussian random variable is easy to model and numerically simulate, but is not an appropriate choice for modeling material properties as there is a non-zero probability of the material properties attaining physically impossible negative values. Instead, we next assume the random variables to have Lognormal distribution; this ensures strict positivity

of the material properties. Here, E_e is distributed throughout the length of the beam. From the convergence study it is observed that 3 PCE terms are sufficient to represent the PCE (full) and PCE (SEREP) nodal responses. A normalized error of the nodal responses from the reduced system is found to be of the order 10^{-9} which is excellent. The results are shown in Figs. 5-10. Fig. 5 compares the mean response obtained from the reduced model and the full model. Fig. 6 represents a magnified part of this plot for improved clarity. From these two plots it can be observed that, the mean estimates from the full and the reduced models calculated by PCE show an excellent match with the predictions obtained from Monte Carlo simulations (MCS) on the full model, which serves as the benchmark. The standard deviation of the displacement is shown in Fig. 7 with a magnified part in Fig. 8. Here also a good agreement is observed between the estimates obtained from the reduced model (PCE) and the full model (PCE and MCS). Figs. 9 and 10 show the pdf and the failure probability of the extreme value displacement. A good agreement is observed between these estimates obtained from the reduced model and the full model. On a desktop computer, the total

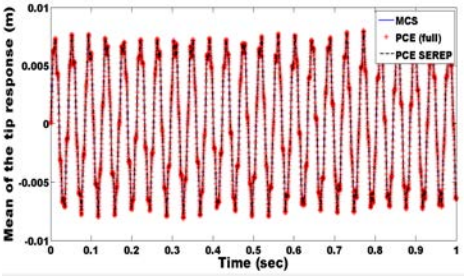

Fig. 5. comparison of the mean displacement.

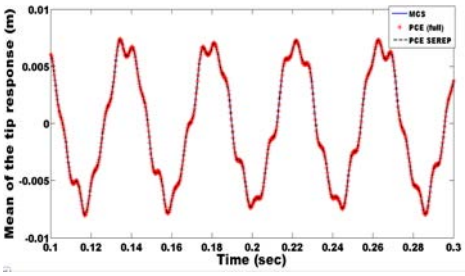

Fig. 6. A magnified part of the mean displacement between 0.1 and 0.3s.

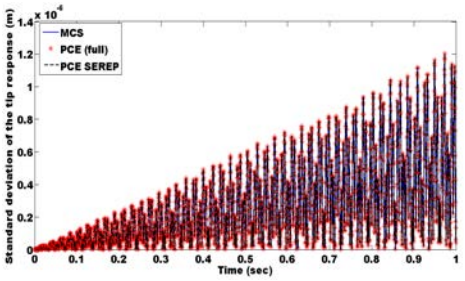

Fig. 7. comparison of the standard deviation.

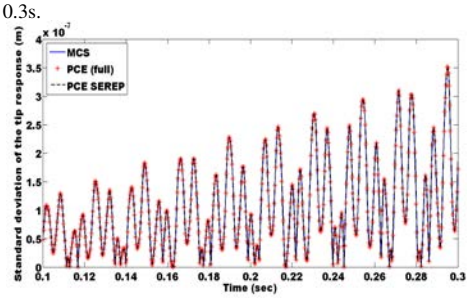

Fig. 8. A magnified part of the standard deviation of the displacement between 0.1 and 0.3s.

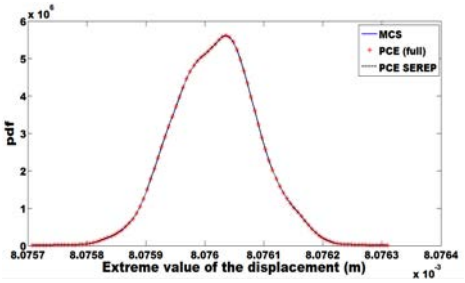

Fig. 9. comparison of the extreme value displacement.

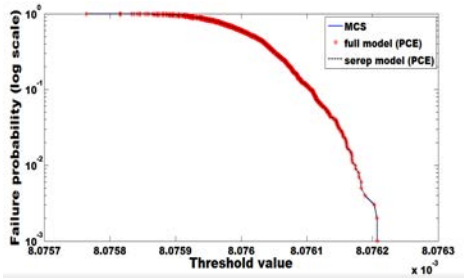

Fig. 10. Comparison of the failure probability of the extreme value displacement.

computational time for the MCS using 1000 realizations was 133.6s, whereas for the proposed PCE based SEREP method, it was only 2.3s, indicating a reduction of 98.28% in the computational cost.

3.2. Case 2 - two random variables, (E_e, ρ_e)

The random parameters are assumed to be lognormaly distributed. The mean value for ρ_e is taken as 2662Kg/m^3.The PCE analysis has been carried out for the beam by full grid as well as sparse grid stochastic collocation methods. In the full grid stochastic method, a 2^{nd} order Gauss Hermite quadrature is used to obtain the PCE (full) and PCE (SEREP) nodal responses. This results in 9 collocation points and function evaluations to estimate the PCE coefficients. A convergence study has been carried out to obtain the number of PCE terms and it can be observed that 6 terms are required to represent the PCE responses (both full and SEREP). Next, the accuracy of the sparse grid collocation method is tested on the same example. A Gauss-Patterson (GP) sparse grid of level 7 is used to evaluate the PCE coefficients. The total number of collocation points and the deterministic runs are 1793 to estimate the 6 PCE coefficients. A normalized error of the nodal responses from the reduced system is of order 10^{-7} which is excellent. Numerical results are plotted in Figs. 11-16. In Fig. 11, time histories of the estimated mean displacement are presented. Fig. 12 represents a magnified part of this plot. From these two plots, it is observed that the mean estimate obtained from the full grid and sparse grid collocation methods for both PCE (full) model and PCE (SEREP) model gave excellent accuracy compared to the Monte Carlo estimates. The standard deviation of the displacement is shown in Fig. 13 with a magnified part in Fig. 14. Here also estimates using all three methods agree for the two models.

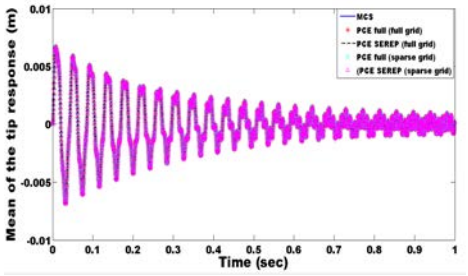

Fig. 11. comparison of the mean displacement.

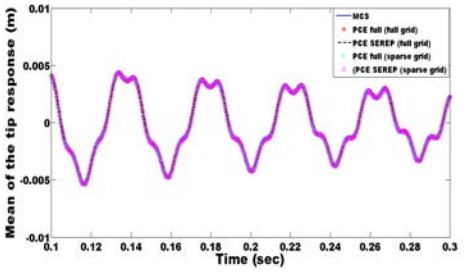

Fig. 12. A magnified part of the mean displacement between 0.1 and 0.3s.

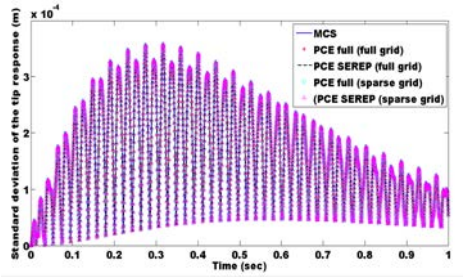

Fig. 13. comparison of the standard deviation.

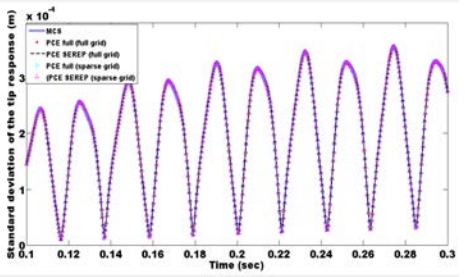

Fig. 14. A magnified part of the standard deviation of the displacement between 0.1 and 0.3s.

Figs. 15 and 16 show the pdf and the failure probability of the extreme value of the displacement and these estimates obtained from the full model and reduced model by PCE are exactly matching with that of MCS full model. From this study, it can be observed that the first four modes are sufficient to capture the dynamics of the system. On a desktop computer, the total computational time for MCS using 10000 realizations was $288.5s$, while for the proposed PCE based SEREP model, it was $37.92s$ in sparse grid and $3.2s$ in full grid. Here, the time required for the full grid is smaller than that for the sparse grid, because different quadrature scheme was used to construct the full (Gauss Hermite-non nested) and sparse grid (GP-nested). It is to be noted that, a full grid with GP of level 7 would need 71289 collocation points compared to 1793 used here for sparse grid and hence would be computationally costly. On

the other hand, a sparse grid constructed with a nested Gauss Hermite quadrature if it were available would be faster than the full grid with the same quadrature.

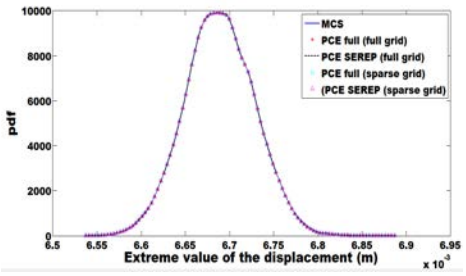

Fig. 15. comparison of the pdf of the extreme value displacement.

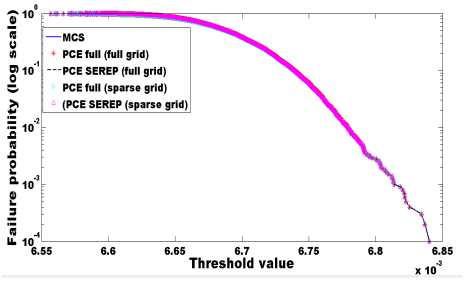

Fig. 16. comparison of the failure probability of the extreme value displacement.

4. Conclusion

A PCE based SEREP model reduction technique has been used to develop a reduced order model for the dynamical system with parametric uncertainties. Usage of the interfacing algorithms enable the system FE modeling using commercially available FE software and importing the system FE matrices to MATLAB. Response prediction using a low order dynamical system have been carried out to demonstrate the efficiency of the proposed method. The proposed method is computationally faster than the Monte Carlo method. The computational complexities associated with MCS have been reduced by parallelising the codes using PETSc. Application to complex larger order systems are currently being studied.

References

[1] P. Avitabile., J. Callahan., E.R. Pan, Effects of various model reduction techniques on computed system response, Proceedings of the Seventh International Modal Analysis Conf, Las Vegas, Nevada. (1989) 1109–1115.

[2] D. Ghosh., R. Ghanem., J. Red-Horse, Analysis of eigenvalues and modal interaction of stochastic systems, AIAA Journal. 43(10),(2005) 2196–2201 .

[3] R. Ghanem., D. Ghosh, Efcient characterization of the random eigenvalue problem in apolynomial chaos decomposition, International Journal for Numerical Methods in Engineering 72(4). (2007) 486–504.

[4] D. Ghosh, Application of random eigenvalue problem in forced response analysis of a linear stochastic structure, Archive of Applied Mechanics. 83, (2013) 1341-1357.

[5] S. Balay., S. Abhyankar., M.F. Adams., J. Brown., P. Brune., K. Buschelman., V. Eijkhout., W.D. Gropp., D. Kaushik., M.G. Knepley., L.C. Mcinnes., K. Rupp., B.F. Smith,., H. Zhang, PETSc users manual, technical report anl-95/11-Revision 3.5. Argonne National Laboratory. URLhttp//www.mcs.anl.gov/petsc (2014a)

[6] S. Balay., S. Abhyankar., M.F. Adams., J. Brown., P. Brune., K. Buschelman., V. Eijkhout., W.D. Gropp., D. Kaushik., M.G. Knepley., L.C. Mcinnes., K. Rupp., B.F. Smith,., H. Zhang, PETSc Web page.http://www.mcs.anl.gov/petsc. URLhttp: //www.mcs.anl.gov/petsc (2014B).

[7] I.M. Babuska., F. Nobile., R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data, SIAM Journal on Numerical Analysis, 45(3),(2007) 1005–1034.

[8] A. Desai., S. Sarkar, Analysis of a nonlinear aeroelastic system with parametric uncertainties using PCE, Mathematical Problems in Engineering Article ID 379472, doi:10.1155/2010/379472 (2010).

[9] T. Gerstner., M. Griebel, Numerical integration using sparse grids, Numerical Algorithms, 18, (1998) 209-232.

[10] S. Balay., W.D. Gropp., L.C. Mcinnes., B.F. Smith, Efficient management of parallelism in object oriented numerical software libraries, Modern Software Toolsin Scientific Computing. Birkhauser Press (1997).

[11] V. Hernandez., J.E. Roman., V. Vidal, SLEPc: Scalable library for eigenvalue problem computations, Lecture Notesin ComputerScience. 2565, (2003) 377-391.

[12] F. Yaman., W. Ackermann., T. Weiland, Comparison of eigenvalue solvers for large sparse matrix pencils (2012).