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# Rate Gap Analysis for Rate-adaptive Antenna Selection and Beamforming Schemes

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***Abstract*—We analyze the asymptotic performance of rate adaptation for Transmit Antenna Selection (TAS) and Maximum Eigenmode Beamforming (MEB) schemes in Multiple-Input Multiple-Output (MIMO) systems under imperfect channel state information (CSI) and feedback delay. The rate is adapted according to a target outage probability. We derive lower and upper bounds to this rate. We also asymptotically characterize the multi-step prediction error when MMSE prediction is used to combat feedback delay. Using the bounds and the prediction error asymptotics, we show that the rate gap from the ideal CSI scenario asymptotically grows logarithmically with SNR. The slope is at most the target outage probability. We find that when the target outage probability is decreased faster than an identified growth rate and prediction error goes to zero, then the rate gap remains bounded.**

## I. Introduction

Several adaptive transmission schemes based on channel state information (CSI) have been proposed for Multiple-Input Multiple-Output (MIMO) wireless systems. Two of those schemes are Maximum Eigenmode Beamforming (MEB) [1] and Transmit antenna selection (TAS) [2]. The MEB scheme involves beamforming along the eigen vector corresponding to the largest singular value of the channel matrix. The implementation of the MEB scheme requires feedback of at least the beamforming vector assuming all other computations are done at the receiver. The TAS scheme involves selecting the best transmit antenna in terms of the maximum channel norm. The TAS scheme has reduced complexity and requires feedback of only the index of the transmit antenna to be chosen. Also, the TAS scheme has been shown to achieve full diversity asymptotically [3].

Rate or power adaptation can be employed along with the above two schemes further to enhance performance [4]. We analyze the performance of rate-adaptive MEB and TAS systems in the presence of imperfections in CSI. When imperfections in CSI at the receiver (due to estimation errors) and feedback delay are introduced, even rate adaptation cannot always result in an outage free transmission due to the mismatch between estimates at the transmitter and the receiver. In other words, the transmitter gets delayed information about changes in the channel while receiver has information about both the current and the past channel conditions. Transmitter adaptation has to take place, under this uncertainty about CSI, at the transmitter. One way to combat this problem is to use prediction. Probability of outage given a fixed rate at the transmitter has been analysed for the MEB scheme in [5] for various imperfect CSI assumptions. The average rate of a rate-adaptive TAS scheme based on a fixed target outage probability has been numerically calculated in [6].

In this paper, we derive analytical results for rate-adaptive MEB and TAS schemes. First, we formally define the rate adaptation scheme applied to the MEB and TAS systems. We unify notation for the MEB and TAS schemes and define the ergodic rate gap to be the expected difference between the rate with perfect CSI and rate with imperfect CSI and feedback delay. We derive lower and upper bounds to the adapted rate with imperfect CSIT and show that the rate gap has a $\log(SNR)$ growth. The slope of the rate gap is upper bounded by the target outage probability. From this, we conclude that when (1) the outage probability is decreased with SNR at a rate faster than $\frac{1}{\log(SNR)}$, and (2) channel prediction drives the mismatch between CSI at the receiver (CSIR) and CSI at the transmitter (CSIT) to zero when SNR becomes high, the rate gap remains bounded.

As part of the above analysis, we also quantify the asymptotics of the mismatch (prediction error from noisy past estimates) between CSIR and CSIT under multi-step MMSE prediction for the Jakes fading model for the channel. The asymptotics for one-step prediction with past values in noise have been characterised in [7], [8] for *Doppler* processes. We extend, by analytical calculations, the result to multi-step prediction. We observe that the exact asymptotic variation of prediction error with SNR does not have any implications for the rate gap asymptotics as long as the prediction error goes to zero.

The organisation of this paper is as follows. We present the system model first in Section II, followed by rate adaptation for both schemes in Section III. Then, we derive bounds on the adapted rate for both the schemes in Section IV. We characterize the asymptotics of prediction error with $SNR$ in Section V. In Section VI, we analyze the ergodic rate gap, characterise it asymptotically, and present some numerical results. Conclusions are drawn in Section VII.

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## II. System Model

A MIMO system with $N_t$ transmit antennas and $N_r$ receive antennas is considered. We assume a block rayleigh fading channel. The correlation between different blocks follows the Jakes fading model. The channel matrix for a particular block is denoted by $\mathbf{H}$ and has i.i.d entries distributed as $\mathcal{CN}(\mathbf{0}, 1)$. The received vector $\mathbf{y}$ ($N_r \times 1$) is given by:

$$\mathbf{y} = \sqrt{P}\mathbf{H}\mathbf{x} + \mathbf{z} \tag{1}$$

where $\mathbf{x}$ is the transmitted signal vector and $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2\mathbf{I})$. The power used per training symbol is $P_t$ and the power used per data symbol is $P_d$. The estimated channel at receiver is denoted $\mathbf{H_r}$. $H_r^{ij} \sim \mathcal{CN}(0, \sigma_r^2)$. The CSIT is denoted $\mathbf{H_t}$ and $H_t^{ij} \sim \mathcal{CN}(0, \sigma_t^2)$. We distinguish between two cases:

*1) Perfect CSIR and no feedback delay:* Here $\mathbf{H_r} = \mathbf{H_t}$ and $\sigma_r^2 = \sigma_t^2 = 1$.

*2) Imperfect CSIR and feedback delay:* The feedback delay is $\Delta$ blocks. $\mathbf{H_t}$ is obtained using a $\Delta$-step channel prediction from past values of $\mathbf{H_r}$. Let $\rho$ be the entry wise correlation between $\mathbf{H_r}$ and $\mathbf{H_t}$. $\sigma_r^2 = \frac{P_t}{P_t+\sigma_n^2}$ and $\sigma_t^2 = \mathbf{p}^H\mathbf{w}$, where $\mathbf{w}$ is the $L$-tap Wiener filter used. $\mathbf{p}$ is the cross-correlation vector between the current $H_r^{ij}$ and past values with delay $\Delta$. The following relation holds ($\mathbf{H_t}$ and $\mathbf{H_r}$ are jointly Gaussian) [5]:

$$\mathbf{H_r} = \sigma_r \left[\frac{\rho}{\sigma_t}\mathbf{H_t} + \sqrt{1-\rho^2}\mathbf{E}\right] \tag{2}$$

where $E_{ij} \sim \mathcal{CN}(0, 1)$.

In the MEB scheme, the beamforming vector which corresponds to the largest singular value of the channel matrix (CSIT) $\mathbf{H_t}$ is selected. Let this beamforming vector be $\mathbf{u}$. Then, the transmit vector $\mathbf{x} = \mathbf{u}x$, where $x$ is the transmitted data symbol. In the TAS scheme, only one antenna is selected for transmission. Therefore, $\mathbf{x}$ has only one non-zero entry corresponding to the selected antenna.

## III. Rate Adaptation

Rate adaptation for the TAS scheme was considered in [6]. Similarly, we consider rate adaptation for MEB scheme and unify the rate gap analysis for both schemes. The transmission rate is chosen based on a lower bound on the mutual information and a target outage probability. For the MEB scheme, the mutual information achievable at the receiver can be lower bounded by [9],[10]:

$$I(x, \mathbf{y}/\mathbf{H_t}, \mathbf{H_r}) \geq \log(1 + \Gamma\mathbf{u}^H\mathbf{H_r}^H\mathbf{H_r}\mathbf{u}) \tag{3}$$

where $\Gamma = \frac{P_d}{P_d\sigma_e^2+\sigma_n^2}$. For a given $\mathbf{H_t}$, let the rate to be chosen by the transmitter be $R(\mathbf{H_t})$. The probability of outage for this rate can be upper bounded as in [6]:

$$P(outage/\mathbf{H_t}) \leq P\left(A < 2\beta\left(\frac{1+\mu}{\sigma_r^2}\right)\right), \tag{4}$$

where $A = \|\sqrt{\frac{2\mu}{\sigma_t^2}}\mathbf{H_t}\mathbf{u} + \sqrt{2}\mathbf{E}\mathbf{u}\|^2$, $\mu = \frac{\rho^2}{1-\rho^2}$, and $\beta = \frac{e^{R(\mathbf{H_t})}}{\Gamma}$. In order to ensure an upper bound on the outage probability, the rate $R(\mathbf{H_t})$ is decided by equating the upper bound to a fixed outage probability $P_{out}$ and is given by:

$$R_0(\mathbf{H_t}) = log\left(1 + \Gamma\frac{\sigma_r^2}{2(1+\mu)}F^{-1}_{nc-\chi^2,2N_r,\delta}(P_{out})\right), \tag{5}$$

where $F^{-1}_{nc-\chi^2,2N_r,\delta}$ is the inverse CDF of the non-central $\chi^2$ distribution with $2N_r$ degrees of freedom and centrality parameter $\delta = \frac{2\mu}{\sigma_t^2}\|\mathbf{H_t}\mathbf{u}\|^2$. The $\beta$ corresponding to the above $R_0$ is denoted $\beta_0$.

The outage probability bound for the TAS scheme is also very similar to the bound in equation (4). The only difference is in the expression for $A$. $\mathbf{H_t}\mathbf{u}$ gets replaced by $\mathbf{H_{tsel}}$ corresponding to the maximum norm column of $\mathbf{H_t}$ at the transmitter and $\mathbf{E}\mathbf{u}$ gets replaced by $\mathbf{E_{sel}}$. The statistics of $\mathbf{E}\mathbf{u}$ and $\mathbf{E_{sel}}$ are identical. Both are $N_r \times 1$ circularly symmetric complex gaussian with variance $\frac{1}{2}$ per dimension. Therefore, conditioned on the CSIT ($\mathbf{H_t}$), the distribution of random variable $A$ is identical for both MEB and TAS schemes. Let $\hat{\mathbf{H}}$ denote $\mathbf{H_{tsel}}$ and $\mathbf{H_t}\mathbf{u}$ in their respective cases. Let $\hat{\mathbf{E}}$ denote $\mathbf{E_{sel}}$ and $\mathbf{E}\mathbf{u}$ in their respective schemes. The distribution of $A$ conditioned on $\mathbf{H_t}$ is a non-central chi-squared distribution with $2N_r$ degrees of freedom and centrality parameter $\delta = \frac{2\mu}{\sigma_t^2}\|\hat{\mathbf{H}}\|^2$.

Since the mutual information lower bound (outage upper bound) is used for calculating the rate, we have

$$P_{out} > P(outage/\mathbf{H_t}). \tag{6}$$

Let the perfect CSI ($\mathbf{H_t} = \mathbf{H_r} = \mathbf{H}$) rate be denoted by $R_{ideal}$. For the MEB scheme, the perfect CSI rate is:

$$R_{ideal} = \log(1 + SNR\|\breve{\mathbf{H}}\|^2), \tag{7}$$

where $SNR = P_d/\sigma_n^2$, and $\breve{\mathbf{H}} = \mathbf{H}\mathbf{u}$, $\mathbf{u}$ is the singular vector corresponding to the maximum singular-value of $\mathbf{H}$, the actual channel matrix. Since $\frac{\hat{\mathbf{H}}}{\sigma_t}$ in the imperfect CSI case and $\breve{\mathbf{H}}$ in the perfect CSI case have the same PDF, we first define $\Delta R$ for a given $\bar{H}$ as follows:

$$\Delta R(\bar{H}) = R_{ideal}(\bar{H}) - (1 - P(outage/\bar{H}\sigma_t))R_0(\bar{H}\sigma_t), \tag{8}$$

where $R_{ideal}(\bar{H})$ is the perfect CSI rate when $\breve{\mathbf{H}} = \bar{H}$, $R_0(\bar{H}\sigma_t)$ is the imperfect CSI rate when $\frac{\hat{\mathbf{H}}}{\sigma_t} = \bar{H}$, and $(1 - P(outage/\bar{H}\sigma_t)$ accounts for the possibility of outage with imperfect CSIT. The *ergodic rate gap* will be $E_{\bar{\mathbf{H}}}[\Delta R(\bar{H})]$, where $\bar{\mathbf{H}}$ has the same PDF as $\breve{\mathbf{H}}$ and $\frac{\hat{\mathbf{H}}}{\sigma_t}$ above.

Similarly, for the TAS scheme, the perfect CSI rate is again given by (7). However, here $\|\breve{\mathbf{H}}\|^2$ is the maximum channel norm over all transmit antennas. Again, considering the imperfect CSI case in this scheme, the variable $\|\frac{\hat{\mathbf{H}}}{\sigma_t}\|^2$ is statistically same as $\|\breve{\mathbf{H}}\|^2$. Hence, $\Delta R(\bar{H})$ is given by the same expression (8).

Although the rate gap expression is identical for both schemes, the variables involved $\{\breve{\mathbf{H}}, \hat{\mathbf{H}}/\sigma_t\}$ are different statistically. The TAS scheme is characterized by the statistics of the maximum channel norm. The MEB scheme is characterized by the statistics of the gaussian channel matrix multiplied by the singular vector corresponding to the largest singular value.

Hence, when ergodic rate gap is computed, the rate gap will be averaged by different probability distributions.

The main result of the paper is to show the following asymptotics for both TAS and MEB schemes:

$$E_{\bar{\mathbf{H}}}[\Delta R(\bar{H})] < P_{out} log(SNR) + O(1). \quad (9)$$

## IV. Bounds on the rate

To compute $\Delta R(\bar{H})$, we need to characterise $R_0(\bar{H}\sigma_t)$ or equivalently $R_0(\hat{H})$. The inverse CDF function in (5) is numerically computable but difficult to characterise analytically. Therefore, to analytically characterize the asymptotics, we bound the right-hand side of (4) in the following lemma.

**Lemma 1.** *The conditional rate $R_0(\hat{\mathbf{H}})$ has the following bounds:*

$$R_0 < \log\left(1 + \Gamma\rho^2 \frac{\sigma_r^2}{\sigma_t^2}\|\hat{\mathbf{H}}\|^2 + \Gamma\frac{(1-\rho^2)\sigma_r^2}{2} F_{\chi^2}^{-1}(2P_{out})\right) \quad (10)$$

*and*

$$R_0 > \log\left(1 + \Gamma\sigma_r^2\left(\sqrt{\frac{1-\rho^2}{2}} F_{\chi^{-1}}(P_{out}) - \frac{\rho}{\sigma_t}\|\hat{\mathbf{H}}\|\right)^2\right) \quad (11)$$

*where $F_{\chi^2}^{-1}$ is the inverse CDF of the central $\chi^2$ distribution with $2N_r$ degrees of freedom, and $F_\chi^{-1}$ is the inverse CDF of the chi distribution with $2N_r$ degrees of freedom.*

*Proof:* Consider the event $\mathcal{E} = \{2\|\hat{\mathbf{E}}\|^2 < 2\beta_0\frac{1+\mu}{\sigma_r^2} - \frac{2\mu}{\sigma_t^2}\|\hat{\mathbf{H}}\|^2\}$ . Since $\hat{\mathbf{E}}$ is angularly symmetric (direction wise),

$$P(\mathcal{E}_+ = \{Re(\hat{\mathbf{H}}^H\hat{\mathbf{E}}) > 0\}) = P(\mathcal{E}_- = \{Re(\hat{\mathbf{H}}^H\hat{\mathbf{E}}) < 0)\}.$$

Also, we have

$$P(\mathcal{E}_- \cap \mathcal{E}) < P\left(A < 2\beta_0\frac{1+\mu}{\sigma_r^2}\right),$$

and

$$P(\mathcal{E}_- \cap \mathcal{E}) = P(\mathcal{E}_+ \cap \mathcal{E}).$$

Therefore, using the above three equations, we get

$$\frac{1}{2}P(\mathcal{E}) < P\left(A < 2\beta_0\frac{1+\mu}{\sigma_r^2}\right)$$

If the lower bound on left hand side of the previous equation is equated to the target outage probability $P_{out}$, then a rate greater than $R_0$ will be obtained. Therefore, we get the upper bound as given in the lemma. The inverse CDF $F_{\chi^2}^{-1}$ is due to the statistics of $2\|\hat{\mathbf{E}}\|^2$.

In order to derived the rate lower bound, consider the following triangle inequality:

$$\|\sqrt{2}\hat{\mathbf{E}} + \sqrt{\frac{2\mu}{\sigma_t^2}}\hat{\mathbf{H}}\|^2 \geq \left(\sqrt{2}\|\hat{\mathbf{E}}\| - \sqrt{\frac{2\mu}{\sigma_r^2}}\|\hat{\mathbf{H}}\|\right)^2. \quad (12)$$

Let $P\left(A < \frac{2\beta_0(1+\mu)}{\sigma_r^2}\right)$ be denoted by $P_A$. The following upper bound holds:

$$\begin{aligned} P_A < \; & P\left(\left(\sqrt{2}\|\hat{\mathbf{E}}\| - \sqrt{\frac{2\mu}{\sigma_r^2}}\|\hat{\mathbf{H}}\|\right)^2 < 2\beta_0\frac{1+\mu}{\sigma_r^2}\right) \\ = \; & P\left(\left|\sqrt{2}\|\hat{\mathbf{E}}\| - \sqrt{\frac{2\mu}{\sigma_r^2}}\|\hat{\mathbf{H}}\|\right| \leq \sqrt{2\beta_0\frac{1+\mu}{\sigma_r^2}}\right) \\ \leq \; & P\left(\sqrt{2}\|\hat{\mathbf{E}}\| \leq \sqrt{2\beta_0\frac{1+\mu}{\sigma_r^2}} + \sqrt{\frac{2\mu}{\sigma_t^2}}\|\hat{\mathbf{H}}\|\right). \end{aligned}$$

Equating the right hand side of the above inequality to $P_{out}$, we get the lower bound on the rate. $F_\chi^{-1}$ is due to $\sqrt{2}\|\hat{\mathbf{E}}\|$. ■

We denote the upper bound by $R_{upp}(\hat{\mathbf{H}})$ and lower bound by $R_{low}(\hat{\mathbf{H}})$ from now on.

## V. Asymptotic Mismatch between CSIR and CSIT

In order to charaterize the asymptotic behavior of $R_{upp}$ and $R_{low}$, the asymptotics of $1-\rho^2$ needs to be characterised in the imperfect CSI case. In this section, we show the following assuming prediction using the entire past:

$$1 - \rho^2 = O(\ln(SNR)^{2(\Delta-1)} SNR^{-(1-\frac{\omega_m}{\pi})}). \quad (13)$$

Each $H_t^{ij}[n]$ is predicted based on past CSIR $\{H_r^{ij}[n-\Delta], H_r^{ij}[n-\Delta-1]...., H_r^{ij}[n-\Delta-L+1]\}$. Note that $H_r^{ij}$ is the MMSE estimate of $H^{ij}$. We assume that $H^{ij}$ is a Doppler process with spectrum $F(e^{j\omega})$. $H_r^{ij}$ is nothing but a Doppler process $H^{ij}$ in noise. For the Doppler process in noise, the power spectral density is given by:

$$S(e^{j\omega}) = \begin{cases} \dfrac{P_t^2 F(e^{jw})}{(P_t+\sigma_n^2)^2} + \dfrac{P_t\sigma_n^2}{(P_t+\sigma_n^2)^2} & |\omega| < \omega_m \\ \dfrac{P_t\sigma_n^2}{(P_t+\sigma_n^2)^2} & \omega_m < |\omega| < \pi \end{cases}, \quad (14)$$

Specifically, for the Jakes correlation model $F(e^{j\omega})$ has the following form:

$$F(e^{j\omega}) = \frac{2}{\omega_m\sqrt{1-\left(\dfrac{\omega}{\omega_m}\right)^2}} \quad (15)$$

for $|\omega| < \omega_m$. Since MMSE prediction is used, we have $\sigma_t^2 + \sigma_p^2 = \sigma_r^2$, where $\sigma_p^2$ is the prediction error and $\rho = \frac{\sigma_t}{\sigma_r}$. Therefore, $1-\rho^2 = \frac{\sigma_p^2}{\sigma_r^2}$. Since, $\frac{1}{\sigma_r^2} = O(1)$ at high SNR, $1-\rho^2$ is dependent on the prediction error. This characterizes the mismatch between CSIR and CSIT. The noise power in the case where $H_r^{ij}$ is normalised with $\sigma_r$ is $\frac{1}{1+SNR}$.

As noted before, for Doppler processes, asymptotics of one step ($\Delta = 1$) prediction error in noise (with power $\frac{1}{SNR}$) has been characterised in [7], [8]. This analysis is the best case scenario when the entire past is used in prediction (holds almost for large $L$). We quote the result here:

$$1 - \rho^2 = \sigma_p^2 = O(SNR^{-(1-\frac{\omega_m}{\pi})}). \quad (16)$$

We now derive a similar result for multi-step prediction using results from [11]. The correlation model assumed in (14) obeys *Paley-Wiener* condition (this can occur when $F$ is absolutely log integrable in its support) which is:

$$\int_{-\pi}^{\pi} \log(S(e^{j\omega}))d\omega > -\infty$$

It can be noted that the Jakes spectrum in noise also satisfies this criterion. The following hold for $S(e^{j\omega})$

$$\ln(S(e^{j\omega})) = \sum_{n=-\infty}^{n=\infty} c_n e^{-j\omega n}$$

and, therefore, we have

$$\begin{aligned} S(e^{j\omega}) &= e^{\left(\frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-j\omega n}\right)} e^{\left(\frac{c_0}{2} + \sum_{n=-1}^{-\infty} c_n e^{-j\omega n}\right)} \\ &= S^+(e^{j\omega})S^-(e^{j\omega}). \end{aligned}$$

Let $f_n$ be defined such that:

$$S^+(e^{j\omega}) = \sum_{n=0}^{\infty} f_n e^{-j\omega n}.$$

Expanding $S^+$ to get $f_n$, we have:

$$\begin{aligned} S^+(e^{j\omega}) &= e^{\frac{c_0}{2}} \left[ \sum_{k=0}^{\infty} \frac{\left(\sum_{n=1}^{\infty} c_n e^{-j\omega n}\right)^k}{k!} \right] \quad (17) \\ &= e^{\frac{c_0}{2}} \left[ 1 + c_1 e^{-j\omega} + \left(c_2 + \frac{c_1^2}{2!}\right) e^{-2j\omega} + .. \right. \\ & \left. \left( \sum_{k=1}^{n} \sum_{i_1+..i_k=n} \frac{(c_{i_1}..c_{i_k})}{k!} \right) e^{-j\omega n} + .. \right]. \end{aligned}$$

Now, $c_n$ is evaluated as follows:

$$\begin{aligned} c_n &= \frac{1}{2\pi} \left( \int_{-\omega_m}^{\omega_m} \ln(\epsilon^2 + F(e^{j\omega n}))e^{j\omega n} d\omega + \right. \\ & \left. \int_{-\pi}^{-\omega_m} \ln(\epsilon^2)e^{j\omega n} d\omega + \int_{\omega_m}^{\pi} \ln(\epsilon^2)e^{j\omega n} d\omega \right) \\ &= O(1) + O(\ln(\epsilon^2)) \\ &= O(1) + O(\ln(SNR)), \end{aligned}$$

where $\epsilon^2 = \frac{P_t\sigma_n^2}{(P_t+\sigma_n^2)^2} = O(SNR^{-1})$ is the noise variance. Let $E_N$ denote the $N$- step prediction error. Then, we have

$$E_N = \sum_{n=0}^{N-1} f_n^2. \quad (18)$$

Also, $e^{c_0} = O(SNR^{-(1-\frac{\omega_m}{\pi})})$. When $f_n$ is expressed in terms of $c_n$'s using equation (17), it is a summation of terms of the form $c_{i_1}...c_{i_k}$ (product of some $c_n$'s). The dominant term in $f_n^2$, from terms like $c_{i_1}...c_{i_k}$, is $O(\ln(SNR)^{2n} SNR^{-(1-\frac{\omega_m}{\pi})})$. This is due to the term $\frac{c_1^n}{n!}$. Therefore, we have

$$1 - \rho^2 = \sigma_p^2 = E_\Delta = O(\ln(SNR)^{2(\Delta-1)} SNR^{-(1-\frac{\omega_m}{\pi})}) \quad (19)$$

Here $\Delta$ is the delay as mentioned before. When $\Delta = 1$, we get back the existing result in (16).

## VI. Ergodic Rate Gap Analysis

In this section, we characterise asymptotics of $\Delta R(\bar{H})$ and show the result stated in (9). The following lemma holds.

**Lemma 2.** *If $R_{ideal}(\bar{H})$ denotes the rate under perfect CSI, and $R_{low}(\bar{H}\sigma_t)$ and $R_{upp}(\bar{H}\sigma_t)$ are the upper and lower bounds on the rate for imperfect CSI $R_0(\bar{H}\sigma_t)$ with delay $\Delta$, then*

$$\lim_{SNR\to\infty} R_{ideal} - R_{low} = \lim_{SNR\to\infty} R_{ideal} - R_{upp} = \log(1+\frac{1}{\eta})$$

*where $P_d = \eta P_t$.*

*Proof:* Let $x = SNR\|\bar{H}\|^2$, and $y = \Gamma\rho^2\sigma_r^2\|\bar{H}\|^2 + \Gamma\sigma_r^2 \frac{1-\rho^2}{2} F_{\chi^2}^{-1}(2P_{out})$. We observe that

$$\lim_{SNR\to\infty} R_{ideal} - R_{upp} = \log\left(\frac{1+x}{1+y}\right) = \log\left(1 + \frac{\frac{x}{y} - 1}{1 + \frac{1}{y}}\right). \quad (20)$$

Hence, it is enough to characterise the ratio $\frac{y}{x}$. Also, note that

$$\Gamma\rho^2\sigma_r^2 = \Gamma\sigma_r^2 - \Gamma(1-\rho^2)\sigma_r^2,$$

$$\lim_{SNR\to\infty} \frac{\Gamma\sigma_r^2}{SNR} = \frac{1}{1+\frac{1}{\eta}}, \quad \text{and}$$

$$\lim_{SNR\to\infty} 1 - \rho^2 = 0.$$

The last equality follows from (19). Therefore, the following holds:

$$\lim_{SNR\to\infty} \frac{y}{x} = \frac{1}{1+\frac{1}{\eta}} \quad (21)$$

By eqns (21) and (20), we have

$$\lim_{SNR\to\infty} R_{ideal} - R_{upp} = \log(1 + \frac{1}{\eta})$$

Similarly, the lower bound result can also be proved by using $y = \Gamma\sigma_r^2 \left(\sqrt{\frac{1-\rho^2}{2}} F_{\chi^{-1}}(P_{out}) - \rho\|\bar{H}\|\right)^2$. ■

In the above result, note that while the prediction error goes to zero, the rate at which the prediction error tends to zero does not matter. This is different from the diversity-multiplexing gain tradeoff in [5] where the rate at which $\rho$ tends to 1 is important. The rate gap as defined in (8) can be bounded on both sides using (6) and $R_{low}, R_{upp}$ as:

$$\begin{aligned} & R_{ideal}(\bar{H}) - (1 - P(outage/\bar{H}\sigma_t))R_{upp}(\bar{H}\sigma_t) < \Delta R(\bar{H}) \\ & < R_{ideal}(\bar{H}) - (1 - P_{out})R_{low}(\bar{H}\sigma_t) \end{aligned}$$

$$\begin{aligned} \Rightarrow & P(outage/\bar{H}\sigma_t)R_{ideal} + (1 - P(outage/\bar{H}\sigma_t))(R_{ideal} - \\ & R_{upp}) < \Delta R < P_{out}R_{ideal} + (1 - P_{out})(R_{ideal} - R_{low}). \end{aligned}$$

The second term of each bound above goes to a con-

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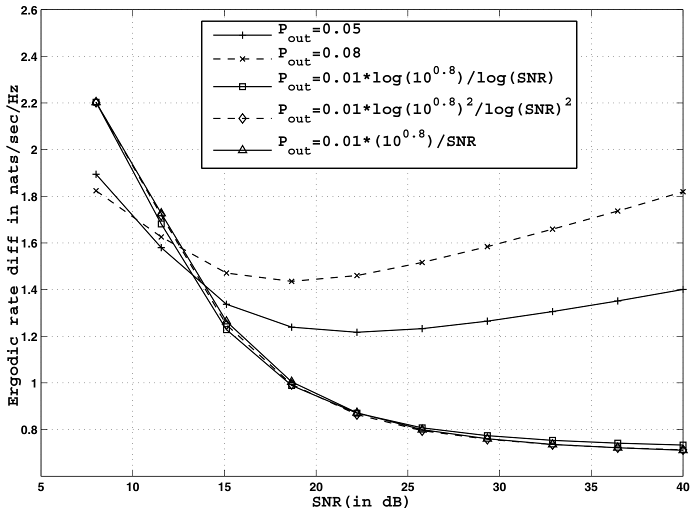


Fig. 1. Ergodic rate gap for a $2 \times 2$ system with $\omega_m = 0.1\pi$,$N_r = N_t = 2$, $\Delta = 2$ and $L = 20$ under TAS scheme

stant by Lemma 2. Therefore, the growth rate of $\Delta R$ is upper bounded by $P_{out} \log(O(SNR))$ and lower bounded by $P(outage/\bar{H}\sigma_t) \log(O(SNR))$, i.e.,

$$\Delta R(x) < \mathcal{K}(x) P_{out} log(SNR)$$

for $SNR > \Theta$ where $x$ is a realisation of the variable $\bar{\mathbf{H}}$ and $\Theta$ is independent of the variable $x$ (from the definition of $O(.)$ notation).

Now, the ergodic rate gap is upper bounded as follows:

$$\int_0^\infty \Delta R(x) f_{\|\bar{\mathbf{H}}\|^2}(x) dx < \int_0^\infty (P_{out} log(SNR) + \mathcal{K}(x)) f_{\|\bar{\mathbf{H}}\|^2}(x) dx \tag{22}$$

for $SNR > \Theta$ where $\Theta$ is independent of the variable $x$. $f_{\|\bar{\mathbf{H}}\|^2}$ is the pdf according to the scheme chosen. On averaging, the second term results in a constant. The first term integrates out to $P_{out} \log(SNR)$. Hence the result in (9) has been shown.

From the above results, we observe that if $P_{out}$ (outage target) is not a function of $SNR$, the ergodic rate gap grows as $\log(SNR)$. However, if $P_{out}$ is decays faster than $(\log(SNR))^{-1}$, then the rate gap is bounded. This means that at high $SNR$, the outage target should be lower. Note that Lemma 2 still holds since all inverse CDF functions also decrease as $P_{out}$ decreases .

We also show the rate gap behaviour with respect to the chosen outage probability through simulations under TAS scheme (ref. Fig.1). The ergodic rate gap (actually with $P_{out}$ substituted in (8) and inverse CDFs of non-central distribution calculated numerically) is computed using monte carlo simulations (to average different realisations $\bar{\mathbf{H}}$) and plotted for various values of $SNR$. $\Delta = 2$, $N_r = N_t = 2$ and $\omega_m = 0.1\pi$ are the parameters used. We observe that the rate gap growth for very high $SNR$ is linear in the constant $P_{out}$ case and the slope increases when $P_{out} = 0.05$ is increased to $P_{out} = 0.08$. Also, we note that for the following growth rates $SNR^{-1}, \log(SNR)^{-1}, \log(SNR)^{-2}$, the rate gap is bounded asymptotically as predicted by the theory.

## VII. Conclusions

We have compared the achievable rates under delay and imperfect CSI of the rate adaptive TAS and MEB schemes with that of the perfect cases respectively. The analysis is common and the ergodic rate gap in both cases are shown to have a $\log(SNR)$ growth. The slope depends on the target outage probability. The rate gap is shown to be bounded if the target outage probability is reduced with $SNR$.

It is instructive to note that the analysis for uplink MAC user selection is similar to the TAS scheme and the analysis for downlink BC user selection is similar to that of the MEB scheme (except that the beamforming vector is the unit channel vector of the chosen user and the maximum norm would be the choosing criteria). The analysis presented here holds good when prediction error goes to zero and MMSE filter with large number of taps almost ensures it. Furthermore, the outage probability has to be decided based on $SNR$ and if suitably chosen can make rate gap bounded.

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