

Quantum critical behavior of clean itinerant ferromagnets*

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We consider the quantum ferromagnetic transition at zero temperature in clean itinerant electron systems. We find that the Landau-Ginzburg-Wilson order parameter field theory breaks down since the electron-electron interaction leads to singular coupling constants in the Landau-Ginzburg-Wilson functional. These couplings generate an effective long-range interaction between the spin or order parameter fluctuations of the form $1/r^{2d-1}$, with d the spatial dimension. This leads to unusual scaling behavior at the quantum critical point in $1 < d \leq 3$, which we determine exactly. We also discuss the quantum-to-classical crossover at small but finite temperatures, which is characterized by the appearance of multiple temperature scales. A comparison with recent results on disordered itinerant ferromagnets is given.

I. INTRODUCTION

The physics of quantum phase transitions has been a subject of great interest lately. In contrast to the usual classical or thermal phase transitions, quantum phase transitions occur at zero temperature as a function of some nonthermal control parameter, and the fluctuations that drive the transition are of quantum nature rather than thermal in origin. Among the transitions that have been investigated are various metal-insulator transitions, the superconductor-insulator transition in thin metal films, and a variety of magnetic phase transitions. Early work in this field¹ established that if the quantum phase transition has a classical analog at finite temperature, then the former tends to have a simpler critical behavior in the physical dimensions $d = 3$ or $d = 2$ than the latter. The reason for this tendency is that the coupling between statics and dynamics that is inherent to quantum statistics problems effectively increases the dimensionality of the system from d to $d + z$, with z the dynamical critical exponent. This reduces the upper critical dimension d_c^+ , which is the dimension above which mean-field theory yields the exact critical behavior, by z from its value for the classical transition.

One of the most obvious examples of a quantum phase transition, and the first one studied in detail, is the ferromagnetic transition of itinerant electrons that occurs as

a function of the exchange coupling between the electron spins. In a pioneering paper, Hertz¹ derived a Landau-Ginzburg-Wilson (LGW) functional for this transition by considering a simple model of itinerant electrons that interact only via the exchange interaction in the particle-hole spin-triplet channel. Hertz analyzed this LGW functional by means of renormalization group (RG) methods that generalize Wilson's treatment of classical phase transitions. He concluded that the critical behavior in the physical dimensions $d = 3$ and $d = 2$ is mean-field like, since the dynamical critical exponent $z = 3$ decreases the upper critical dimension from $d_c^+ = 4$ for the classical case to $d_c^+ = 1$ in the quantum case. In order to study nontrivial quantum critical behavior, Hertz then considered a model with a magnetization that is defined on a space of arbitrary dimension d , while the correlation functions that determine the coefficients in the LGW functional are taken to be those of a $3-d$ Fermi gas. For this model, he calculated non mean-field like critical behavior in $d < 1$ by means of a $1 - \epsilon$ expansion. Despite the somewhat artificial nature of this model, there is a general belief that the qualitative features of Hertz's analysis, in particular the fact that there is mean-field like critical behavior for all $d > 1$, apply to real itinerant quantum ferromagnets as well.

In this paper we reexamine the ferromagnetic quantum phase transition of itinerant electrons and show that the above belief is qualitatively mistaken. We first consider a model that is more realistic than Hertz's, viz. with an electron-electron interaction that is not restricted to the particle-hole spin-triplet channel. We find that the LGW approach breaks down due to the presence of soft modes

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in addition to the critical modes, namely particle-hole excitations that couple to the conserved order parameter. These soft modes are integrated out in the derivation of the LGW functional, and this leads to singular vertices in the order parameter field theory. This is a rather general observation; analogous effects are expected for a large class of quantum phase transitions, and in general they invalidate the application to quantum phase transitions of the classical LGW philosophy of deriving an effective local field theory entirely in terms of the order parameter.² However, for the present problem we find that the mathematical difficulties generated by the singular couplings can be handled. The resulting nonlocal field theory contains an effective long-range interaction between the order parameter fluctuations, and we are able to determine the critical behavior exactly for all $d > 1$. For $1 < d \leq 3$ the result is different from either mean-field critical behavior or classical Heisenberg critical behavior. We then show that the same is true for Hertz's original model. Our results invalidate the $1-\epsilon$ expansion in that paper, and it corroborates and explains an observation made recently by Sachdev,³ who noted that Hertz's results in $d < 1$ cannot be correct since they violate an exact exponent equality.

The outline of this paper is as follows. In Sec. II we first define our model of itinerant electrons, and then discuss how to derive an order parameter description for the ferromagnetic phase transition starting from a fermionic field theory. We discuss the behavior of the coefficients in the resulting LGW functional and show that they have long-range properties. In Sec. III the quantum critical behavior of the resulting nonlocal field theory at zero temperature is determined exactly for dimensions $d > 1$. For $d > 3$, mean-field exponents are obtained, while for $1 < d < 3$, d -dependent exponents are found. In Sec. IV we investigate the behavior at small but finite temperatures which is characterized by multiple independent temperature scales. In Sec. V we reexamine Hertz's original model of the ferromagnetic quantum phase transition. We show that upon renormalization it acquires the same features as the more realistic model studied in Secs. II - IV. In Sec. VI we discuss our results, and in particular the relation of the present paper to recent work on *disordered* itinerant quantum magnets.⁴ A technical detail concerning perturbation theory is relegated to Appendix A, while in Appendix B we discuss logarithmic corrections to scaling that exist in $d = 3$ as well as for all dimensions $1 < d < 3$. A short account of some of our results has previously been published.⁵

II. ORDER PARAMETER FIELD THEORY FOR THE QUANTUM FERROMAGNETIC TRANSITION

In the first part of this section we define our model of itinerant interacting electrons in terms of a fermionic

field theory. Since we are interested in a phase transition to a magnetically ordered phase, we choose the spin density fluctuation as our order parameter, and proceed to derive an LGW or order parameter description of this transition. We then derive and discuss the coefficients in this LGW functional. As we will see, the interactions in the effective LGW theory are long-ranged due to soft modes that have been integrated out in deriving the LGW functional.

A. The Model

The partition function of any fermionic system can be written in the form,⁶

$$Z = \int D\bar{\psi} D\psi \exp(S[\bar{\psi}, \psi]) \quad , \quad (2.1a)$$

where $\bar{\psi}$ and ψ are Grassmannian (i.e., anticommuting) fields. $D\bar{\psi} D\psi$ denotes the Grassmannian functional integration measure, and S is the action,

$$S = \int_0^\beta d\tau \int d\mathbf{x} \bar{\psi}^i(\mathbf{x}, \tau) \frac{\partial}{\partial \tau} \psi_i(\mathbf{x}, \tau) - \int_0^\beta d\tau H(\tau) \quad . \quad (2.1b)$$

Here \mathbf{x} denotes position and τ imaginary time, $H(\tau)$ is the Hamiltonian in imaginary time representation, $\beta = 1/T$ is the inverse temperature, $i = 1, 2$ denotes spin labels, and summation over repeated covariant and contravariant spin indices is implied. Throughout this paper we use units such that $k_B = \hbar = e^2 = 1$. We start with a microscopic model of itinerant, interacting electrons,

$$H(\tau) = \int d\mathbf{x} \bar{\psi}^i(\mathbf{x}, \tau) \left[-\frac{1}{2m} \nabla^2 - \mu \right] \psi_i(\mathbf{x}, \tau) + \frac{1}{2} \int d\mathbf{x} d\mathbf{y} u(\mathbf{x} - \mathbf{y}) \bar{\psi}^i(\mathbf{x}, \tau) \bar{\psi}^j(\mathbf{y}, \tau) \psi_j(\mathbf{y}, \tau) \times \psi_i(\mathbf{x}, \tau) \quad . \quad (2.2)$$

Here m is the electron mass, μ is the chemical potential, and $u(\mathbf{x} - \mathbf{y})$ is the electron-electron interaction potential. More realistic models to describe itinerant electron magnetism including, e.g., band structure, can be considered along the same lines. The salient points of the present paper, however, are due to long-wavelength effects and therefore are independent of microscopic details like the band structure. For our purposes it therefore is sufficient to study the model defined in Eq. (2.2).

In order to describe magnetism, it is convenient and standard practice to break the interaction part of the action S , which we denote by S_{int} , into particle-hole spin-singlet, particle-hole spin-triplet and particle-particle or Cooper channel contributions, which we denote by $S_{int}^{(s)}$, $S_{int}^{(t)}$ and $S_{int}^{(c)}$, respectively.⁷ For simplicity, we assume

that the interactions are short-ranged in all of these channels. In a metallic system this is justified due to screening, and an effective model with a short-ranged interactions can be derived starting from a bare Coulomb interaction.⁷ The spin-triplet interaction warrants special attention, since the interactions between spin density fluctuations that are described by $S_{int}^{(t)}$ are what causes ferromagnetism. We therefore consider this part of the action separately, and write

$$S = S_0 + S_{int}^{(t)} \quad , \quad (2.3)$$

with

$$S_{int}^{(t)} = \frac{\Gamma_t}{2} \int d\mathbf{x} d\tau \mathbf{n}_s(\mathbf{x}, \tau) \cdot \mathbf{n}_s(\mathbf{x}, \tau) \quad , \quad (2.4a)$$

where \mathbf{n}_s is the electron spin density vector with components,

$$n_s^a(\mathbf{x}, \tau) = \frac{1}{2} \bar{\psi}^i(\mathbf{x}, \tau) \sigma_{ij}^a \psi^j(\mathbf{x}, \tau) \quad . \quad (2.4b)$$

Here the σ^a ($a = 1, 2, 3$) are the Pauli matrices, and Γ_t is the spin-triplet interaction amplitude. S_0 in Eq. (2.3) contains the free electron part and all interaction parts other than the particle-hole spin-triplet contribution to the action. It reads explicitly,

$$\begin{aligned} S_0 = & \int_0^\beta d\tau \int d\mathbf{x} \left[\bar{\psi}^i(\mathbf{x}, \tau) \frac{\partial}{\partial \tau} \psi_i(\mathbf{x}, \tau) \right. \\ & \left. - \bar{\psi}^i(\mathbf{x}, \tau) [-\nabla^2/2m - \mu] \psi_i(\mathbf{x}, \tau) \right] \\ & - \frac{\Gamma_s}{2} \int_0^\beta d\tau \int d\mathbf{x} n(\mathbf{x}, \tau) n(\mathbf{x}, \tau) \\ & - \Gamma_c \int_0^\beta d\tau \int d\mathbf{x} \bar{n}_c^{ij}(\mathbf{x}, \tau) n_{c,ij}(\mathbf{x}, \tau) \quad . \quad (2.5a) \end{aligned}$$

Here n is the electron number or charge density,

$$n(\mathbf{x}, \tau) = \bar{\psi}^i(\mathbf{x}, \tau) \psi_i(\mathbf{x}, \tau) \quad , \quad (2.5b)$$

and $n_{c,ij}$ is the anomalous or Cooper channel density,

$$n_{c,ij}(\mathbf{x}, \tau) = (1 - \delta_{ij}) \psi_i(\mathbf{x}, \tau) \psi_j(\mathbf{x}, \tau) \quad , \quad (2.5c)$$

(no summation). To avoid double counting, only the long-wavelength fluctuations of the charge density and the anomalous density must be taken into account in Eq. (2.5a).⁸ Γ_s and Γ_c are the interaction amplitudes in the particle-hole spin-singlet and in the Cooper channel, respectively. For our purposes it is sufficient to treat Γ_t , Γ_s , and Γ_c simply as numbers, although in a more complete model they are complicated short-ranged interaction potentials.^{7,9} For later reference we note that they result from averaging the original interaction over different regions of phase space, so that the three interaction amplitudes are independent parameters.

B. Order parameter field theory

The standard theory of continuous *thermal* phase transitions proceeds from a microscopic model by identifying the order parameter relevant for the transition, and by integrating out all degrees of freedom except for the long wavelength order parameter fluctuations, or critical modes. The result of this procedure is a LGW theory, i.e. an effective field theory for the critical modes.¹⁰ The rationale behind this approach is that these fluctuations, which are slowly varying in space, determine the behavior near the critical point. The same basic idea has been applied to *quantum* phase transitions, with the only principal difference being that the critical modes are now slowly varying in both space and (imaginary) time. While we will use this approach here, motivated in part by previous work on itinerant electronic systems,¹ we mention that in general one should worry about both the critical modes, and all other slow or soft modes, even if these other soft modes are not ‘critical’ in the sense that they change their character at the phase transition. Of course this concern is not restricted to quantum phase transitions; additional soft modes can and do occur also at thermal phase transitions. However, while this is an exceptional occurrence in the case of the latter, we will argue below that it poses a more serious problem for the former, since at $T = 0$ there are more soft modes than at finite temperature. We will see that in the present problem, and in a large class of other quantum phase transitions, such additional modes are indeed present and lead to complications if one insists on a description entirely in terms of the order parameter. For the problem under consideration, however, we will be able to overcome these problems and determine the critical behavior within the framework of an LGW theory.

The techniques for deriving an order parameter field theory, starting with Eqs. (2.1) through (2.5), are standard.¹ We introduce a classical vector field $\mathbf{M}(\mathbf{x}, \tau)$ with components M^a , the average of one of which (M^3 , say) is proportional to the magnetization m , and decouple the four fermion term in $S_{int}^{(t)}$ by performing a Hubbard-Stratonovich transformation. Subsequently, all degrees of freedom other than \mathbf{M} are integrated out. This procedure in particular integrates out soft particle-hole excitations, which are the additional soft modes mentioned above. We obtain the partition function Z in the form,

$$Z = e^{-F_0/T} \int D[\mathbf{M}] \exp(-\Phi[\mathbf{M}]) \quad , \quad (2.6a)$$

with F_0 the noncritical part of the free energy. The LGW functional Φ reads,

$$\begin{aligned} \Phi[\mathbf{M}] = & \frac{\Gamma_t}{2} \int dx \mathbf{M}(x) \cdot \mathbf{M}(x) \\ & - \ln \left\langle \exp \left[-\Gamma_t \int dx \mathbf{M}(x) \cdot \mathbf{n}_s(x) \right] \right\rangle_{S_0} \quad . \quad (2.6b) \end{aligned}$$

Here we use a four-vector notation with $x = (\mathbf{x}, \tau)$, and $\int dx = \int d\mathbf{x} \int_0^\beta d\tau$, and $\langle \dots \rangle_{S_0}$ denotes an average taken with the action S_0 . A formal expansion of Φ in powers of \mathbf{M} takes the form,

$$\begin{aligned} \Phi[\mathbf{M}] = & \frac{1}{2} \int dx dy \left[\frac{1}{\Gamma_t} \delta(x-y) - \chi^{(2)}(x-y) \right] \\ & \times \mathbf{M}(x) \cdot \mathbf{M}(y) \\ & + \sum_{n=3}^{\infty} b_n \int dx_1 \dots dx_n \\ & \times \chi_{a_1 \dots a_n}^{(n)}(x_1 - x_n, \dots, x_{n-1} - x_n) \\ & \times M^{a_1}(x_1) \dots M^{a_n}(x_n) \quad , \quad (2.7a) \end{aligned}$$

where $b_n = (-1)^n/n!$ and we have scaled \mathbf{M} with Γ_t^{-1} . The coefficients $\chi^{(n)}$ in Eq. (2.7a) are connected n-point spin density correlation functions of a reference system whose action is given by S_0 . They are defined as,

$$\chi_{a_1 \dots a_n}^{(n)}(x_1, \dots, x_n) = \langle n_s^{a_1}(x_1) \dots n_s^{a_n}(x_n) \rangle_{S_0}^c \quad , \quad (2.7b)$$

where the superscript ‘c’ denotes a cumulant or connected correlation function. The tensor structure of $\chi^{(n)}$ is restricted by rotational invariance in spin space. For instance, $\chi^{(3)}$ is proportional to the completely antisymmetric third-rank tensor ϵ^{abc} . For our simple model, the reference ensemble S_0 consists of free electrons with short-ranged particle-hole spin-singlet and particle-particle model interactions. As mentioned in Sec. II A above, the model can be made more realistic, if desired, by e.g. including band structure effects. To this end one would simply replace the $\chi^{(n)}$ above with correlation functions for band electrons.

C. The coefficients of the LGW functional

To proceed, we have to calculate the coefficients in the LGW functional, Eq. (2.7a), i.e. the connected spin density correlation functions of the reference system S_0 . Normally, one would localize the individual terms in Eq. (2.7a) about a single point in space and time, and expand the correlation functions in powers of gradients and frequencies. However, for the system under consideration such a gradient expansion does not exist. To illustrate this point, let us first consider the spin susceptibility $\chi_{ab}^{(2)} \equiv \delta_{ab} \chi_s$, whose Fourier transform is defined by

$$\begin{aligned} \chi_s(\mathbf{q}, \Omega_n) = & \int d(\mathbf{x}_1 - \mathbf{x}_2) d(\tau_1 - \tau_2) e^{-i\mathbf{q} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} \\ & \times e^{i\Omega_n(\tau_1 - \tau_2)} \chi_s(x_1 - x_2) \quad . \quad (2.8) \end{aligned}$$

where $\Omega_n = 2\pi Tn$ is a bosonic Matsubara frequency. In order to study the critical behavior we have to determine $\chi^{(2)}$ in the long-wavelength and low-frequency limit. In a system with a conserved order parameter, the frequency must be taken to zero *before* the wavenumber, otherwise

one never reaches criticality.¹¹ In the critical region we therefore have $|\Omega_n| \ll q$. In this limit the spin susceptibility has the structure

$$\chi_s(\mathbf{q}, \Omega_n) = \chi_0(q) [1 - |\Omega_n|/q + \dots] \quad , \quad (2.9)$$

where q and Ω_n are being measured in suitable units, and $\chi_0(q)$ is the static spin susceptibility of the reference system.

The static spin susceptibility of non-interacting electrons, i.e. the Lindhard function, as a function of the wavenumber q is analytic at $q = 0$. However, for any electron system with a nonvanishing particle-hole spin-triplet interaction, $\Gamma_t \neq 0$, there is a nonanalytic correction to the static spin susceptibility.¹² Although our bare reference system S_0 does not contain such an interaction amplitude, in a generic reference system a nonvanishing Γ_t is generated in perturbation theory. For being generic in this sense, it suffices that the bare system has a nonvanishing Cooper channel interaction amplitude $\Gamma_c \neq 0$. This is shown in Appendix A. Effectively, we therefore have to calculate the static spin susceptibility of a paramagnetic Fermi liquid with a nonvanishing Γ_t . It has been shown in Ref. 12 that the result is,

$$\chi_0(q \rightarrow 0) = c_0 - c_{d-1} q^{d-1} - c_2 q^2 + \dots \quad , \quad (2.10a)$$

where $c_2 > 0$, and c_{d-1} is proportional to Γ_t^2 for small Γ_t .¹³ The nontrivial, and for our purposes most interesting, contribution in Eq. (2.10a) is the nonanalytic term $\sim q^{d-1}$. The above form holds for $1 < d < 3$. In $d = 3$ the nonanalyticity is of the form $q^2 \ln q$, and for $d > 3$ the leading q -dependence is given by the quadratic term.

This feature of a Fermi liquid is of crucial importance for what follows. It therefore warrants some discussion, even though it has recently been discussed in great detail.¹² It is well known that in a Fermi liquid, both in $d = 3$ and in $d = 2$, the specific heat and the quasiparticle lifetime are nonanalytic functions of the temperature.¹⁴ The physical reason for these effects is, in the language of the present paper, the soft particle-hole excitations that are always present in a electron system at zero temperature. Although they become massive at nonzero temperature (see Eq. (2.10b) below), this is sufficient to make various observables nonanalytic functions of temperature at $T = 0$. The nonanalytic wavenumber dependence of the spin susceptibility at $T = 0$ is just another manifestation of this effect. It is analogous to a feature of *disordered* electron systems, for which it is known that $\chi_0(q \rightarrow 0) \sim \text{const.} + q^{d-2}$.^{4,15} Since the physical reason for this effect is again the presence of soft modes, it is not qualitatively tied to disordered systems. The only difference between disordered and clean systems in this respect is the nature of the soft modes: In the former they are diffusive, leading to an exponent $d - 2$, while in the latter they are ballistic and lead to an exponent $d - 1$.

In the present context, the nonanalytic behavior of χ_0 implies that the standard gradient expansion mentioned at the beginning of this subsection does not exist. Rather, the interaction between the order parameter

fluctuations is effectively of long range, and in real space takes the form $r^{-(2d-1)}$. This is true only at $T = 0$. At finite temperature, where one has to perform a frequency sum rather than a frequency integral to calculate the correlation function, the nonanalytic term is replaced by a term of the schematic structure,¹²

$$q^{d-1} \rightarrow (q + T)^{(d-1)} \quad , \quad (2.10b)$$

so for fixed $T > 0$ an analytic expansion about $q = 0$ exists, and the standard local LGW functional is obtained.

We now turn to the higher spin density correlation functions $\chi^{(n)}$. The same physics that causes the non-analyticity in the spin susceptibility, Eq. (2.10a), leads to an even stronger effect in the higher correlation functions, and results in $\chi^{(n)}$ for $d < n - 1$ not being finite in the limit of zero frequencies and wave numbers. One finds that in this limit $\chi^{(n)}$ is schematically given by

$$\chi^{(n)} \sim u_n + v_n p^{d+1-n} \quad , \quad (2.11a)$$

where u_n and v_n are finite numbers and we have cut off the infrared divergence by means of a cut-off momentum \mathbf{p} . Rotational symmetry in spin space requires $u_{2n+1} = 0$. Again, the nonanalytic behavior is confined to zero temperature, at finite temperature $\chi^{(n)}$ is given by

$$\chi^{(n)} \sim u_n + v_n (T + p)^{d+1-n} \quad , \quad (2.11b)$$

which is finite for $p \rightarrow 0$. Equations (2.10a) and (2.11a) imply that our LGW functional, Eq. (2.7a), contains a nonanalyticity which has the form of a power series in M/p . In order to specify the LGW functional, we still need a physical interpretation of the infrared cutoff momentum \mathbf{p} . This will be given in the next section.

To conclude this section we show that the static correlation functions discussed above provide more important nonanalyticities than higher order terms in a frequency expansion. To see this, we anticipate a result from the next section. We will discuss a fixed point of the renormalization group transformation where the dynamical exponent is $z = d$, i.e. Ω_n scales like q^d . Now let us look at an expansion of $\chi^{(n)}$ in powers of frequency. In our clean system with ballistic modes the term of m^{th} order in Ω_n carries an additional factor of q^m in the denominator, as can be seen in Eq. (2.9) for the spin susceptibility. Thus, if the term of zeroth order in the frequency scales like q^{d+1-n} near criticality, then the m^{th} order term scales like $q^{d+1-n+m(d-1)}$. For $d > 1$ the static susceptibility therefore has a stronger divergence than the frequency corrections. However, in $d = 1$ the leading divergence is provided by a frequency dependent term, as will be discussed in Sec. V below.

III. THE CRITICAL BEHAVIOR AT ZERO TEMPERATURE

Here we discuss the quantum critical behavior of the Gaussian part of the LGW theory defined in the last section. We do so both by explicitly solving the Gaussian

theory, and by studying the renormalization group properties of the Gaussian fixed point. We then analyze the non-Gaussian terms in the field theory, and show that they are irrelevant, in the renormalization group sense, with respect to the Gaussian fixed point for all dimensions $d > 1$. This implies that the Gaussian theory yields the exact critical behavior for all of these dimensions, except for logarithmic corrections to scaling in $d = 3$ that are discussed in Appendix B. We then construct the equation of state near the critical point. This requires a more detailed discussion of the irrelevant non-Gaussian terms in the field theory, since the equation of state is determined in part by dangerous irrelevant variables.

A. The Gaussian fixed point

According to Eqs. (2.7), (2.9), and (2.10a), the Gaussian part of the LGW functional $\Phi[\mathbf{M}]$ is,

$$\begin{aligned} \Phi_2[\mathbf{M}] = & \frac{1}{2} \sum_{\mathbf{q}} \sum_{\Omega_n} \mathbf{M}(\mathbf{q}, \Omega_n) [t_0 + a_{d-1} q^{d-1} \\ & + a_2 q^2 + a_\Omega |\Omega_n|/q] \cdot \mathbf{M}(-\mathbf{q}, -\Omega_n) \quad , \quad (3.1a) \end{aligned}$$

where

$$t_0 = 1 - \Gamma_t \chi_s(\mathbf{q} \rightarrow 0, \Omega_n = 0) \quad , \quad (3.1b)$$

is the bare distance from the critical point, and a_{d-1} , a_2 , and a_Ω are positive constants.

We first analyze the critical behavior implied by Eqs. (3.1). Later we will show that for $d > 1$ fluctuations are irrelevant, and the critical behavior found this way is exact for these dimensions. Four critical exponents can be directly read off Eq. (3.1a). These are the correlation length exponent ν , defined by $\xi \sim t^{-\nu}$, with ξ the correlation length and t the (renormalized) dimensionless distance from the critical point; the exponent η that determines the wavenumber dependence of the magnetic (i.e., order parameter) susceptibility at criticality, $\langle M_a(\mathbf{q}, 0) M_a(-\mathbf{q}, 0) \rangle \sim q^{-2+\eta}$; the dynamical scaling exponent z that characterizes critical slowing down by relating the divergence of the relaxation time, τ_r , to that of the correlation length, $\tau_r \sim \xi^z$; and the exponent γ that describes the t -dependence of the static magnetic susceptibility $\chi_m = \langle M_a(0, 0) M_a(0, 0) \rangle \sim t^{-\gamma}$. An inspection of Eq. (3.1a) yields,

$$\nu = \begin{cases} 1/(d-1) & \text{for } 1 < d < 3 \\ 1/2 & \text{for } d > 3 \end{cases} \quad , \quad (3.2a)$$

$$\eta = \begin{cases} 3-d & \text{for } 1 < d < 3 \\ 0 & \text{for } d > 3 \end{cases} \quad , \quad (3.2b)$$

$$z = \begin{cases} d & \text{for } 1 < d < 3 \\ 3 & \text{for } d > 3 \end{cases} \quad , \quad (3.2c)$$

$$\gamma = 1 \quad \text{for all } d > 1 \quad . \quad (3.2d)$$

In $d = 3$ there are logarithmic corrections to scaling, see Appendix B.

For later reference, we also discuss the critical behavior given by Eqs. (3.1) and (3.2) from a renormalization group point of view. Let b be the renormalization group length rescaling factor. Under renormalization, all quantities change according to $A \rightarrow A(b) = b^{[A]} A$, with $[A]$ the scale dimension of A . The scale dimension of the order parameter is,

$$[\mathbf{M}(\mathbf{q}, \Omega_n)] = -1 + \eta/2 \quad , \quad (3.3a)$$

or, equivalently,

$$[\mathbf{M}(\mathbf{x}, \tau)] = (d + 1)/2 \quad . \quad (3.3b)$$

At the critical fixed point, a_Ω and either a_{d-1} (for $1 < d < 3$), or a_2 (for $d > 3$) are not renormalized, i.e. their scale dimensions are zero. Using this, and $[q] = 1$, $[\Omega_n] = z$ immediately yields η and z as given by Eqs. (3.2b, 3.2c). Equation (3.2a) follows from the relevance of t_0 , or its renormalized counterpart, t , at the critical fixed point. That is, the scale dimension of t is positive and given by $1/\nu \equiv [t] = 2 - \eta$.

B. The non-Gaussian terms

We now show that all of the non-Gaussian terms in the field theory are renormalization group irrelevant with respect to the Gaussian fixed point discussed in the last subsection. To do this we determine the scale dimensions of the coefficients u_n and v_n of the higher-order terms in the LGW functional which had been defined in Eq. (2.11a). Since the latter have been defined in Fourier space, we first take the Fourier transform of the n^{th} summand in Eq. (2.7a). This yields

$$\begin{aligned} [\chi^{(n)}] &= -\frac{1}{2}(n-2)[T/V] - n[M(\mathbf{q}, \Omega_n)] \\ &= -\frac{1}{2}(n-2)(d+z) + n(1-\eta/2) \quad , \end{aligned} \quad (3.4)$$

where we have used Eq. (3.3a). Next we need to assign a scale dimension to the cutoff momentum p in Eq. (2.11a). The most obvious guess from a scaling point of view is to identify p with the inverse correlation length ξ^{-1} , which makes $[p] = 1$. We will ascertain in Sec. III C below that this is indeed the correct choice. It then follows from Eq. (2.11a) that of the two parameters u_n and v_n , v_n has the larger scale dimension, and hence is more relevant, for $d < n - 1$, while u_n is more relevant for $d > n - 1$. For even n (remember that $u_{2n+1} = 0$) and $d > n - 1$ the most relevant parameter thus has a scale dimension

$$[u_{2n}] = -(d+1)(n-1) + 2 \quad , \quad (3.5a)$$

which is always negative. For odd n , and for $d < n - 1$ for even n , we need to consider,

$$[v_n] = \begin{cases} -(n-2)(d-1)/2 & \text{for } 1 < d < 3 \\ 2 - n(d-1)/2 & \text{for } d < 3 \end{cases} \quad . \quad (3.5b)$$

We see that all of the v_n are irrelevant for $d > 1$, while in $d = 1$ they all become marginal. We conclude that for $d > 1$ the critical Gaussian fixed point is stable, and so the exponents given in Eqs. (3.2) are exact. In $d = 1$ an infinite number of operators seems to become marginal, so naively one would conclude that the upper critical dimension is given by $d_c^+ = 1$. However, one has to keep in mind that the functional form of correlation functions in $d = 1$ can be qualitatively different from that in $d > 1$, so our power counting may be valid only for $d > 1$, and the scale dimension of some operators may change discontinuously from irrelevant in $d > 1$ to relevant in $d = 1$. This actually happens, as will be discussed in Sec. V below.

C. The equation of state

We now determine the equation of state, and calculate the critical behavior of the magnetization and the magnetic susceptibility. Since we have shown in the last subsection that fluctuations are irrelevant for the critical behavior for all $d > 1$, we can determine the equation of state by simply calculating the saddle point contribution to the free energy. However, in order to do so, we have to include the higher order terms in the LGW functional. Although they are irrelevant operators for $d > 1$, they are potentially dangerously irrelevant¹⁶ with respect to the magnetization m . We will see below that all of the v_n are indeed dangerously irrelevant, since m is a singular function of the v_n for $v_n \rightarrow 0$.

We determine the mean field equation of state by calculating the saddle point contribution to the free energy. To do so, we replace the order parameter field $\mathbf{M}(x)$ in the LGW functional, Eq. (2.7a), by the average magnetization m and determine the stationary point of Φ with respect to m . This yields the equation of state in the form

$$t m + u_4 m^3 + m^d \sum_{n=3}^{\infty} v_n (m/p)^{n-1-d} = H \quad , \quad (3.6)$$

where an external magnetic field H was added. We have only kept the leading terms in each order of m , and we have suppressed all numerical prefactors since they are unimportant for our purposes. The equation of states now has the form of a power series in m/p . This implies that the cut-off momentum p effectively scales like m . Thus, all higher order terms effectively have the same power of m , viz. m^d , and the equation of states now reads

$$t m + u_4 m^3 + v m^d = H \quad , \quad (3.7)$$

with u_4 from Eq. (2.11a) and v another finite coefficient. From Eq. (3.7) we immediately obtain the exponents β and δ , defined by $m(t, H = 0) \sim t^\beta$, $m(t = 0, H) \sim H^{1/\delta}$, as

$$\beta = \begin{cases} 1/(d-1) & \text{for } 1 < d < 3 \\ 1/2 & \text{for } d > 3 \end{cases}, \quad (3.8a)$$

$$\delta = \begin{cases} d & \text{for } 1 < d < 3 \\ 3 & \text{for } d > 3 \end{cases}. \quad (3.8b)$$

In $d = 3$ logarithmic corrections to scaling occur, see Appendix B.

We are now in a position to determine the exact scale dimension of the cut-off momentum p and to verify the identification of p with ξ^{-1} made in the last subsection. As we have seen after Eq. (3.6), the cutoff p scales like m , so that $[p] = [m]_{\text{eff}}$, where $[m]_{\text{eff}} = \beta/\nu$ is the *effective* scale dimension of m , i.e. the scale dimension after the effects of the dangerous irrelevant variables have been taken into account. From Eqs. (3.8a), (3.2a) we see that $\beta = \nu$, hence $[m]_{\text{eff}} = 1$. Consequently, $[p] = 1$ which justifies the identification p with ξ^{-1} for scaling purposes made in Sec. IIIB. These results complete the proof of the statement that the system is above its upper critical dimension for $d > 1$.

IV. BEHAVIOR AT FINITE TEMPERATURES, AND THE QUANTUM TO CLASSICAL CROSSOVER

In the first part of this section we discuss various sources of temperature dependence in our field theory and the corresponding scaling behavior of the temperature. In the following two subsections we apply the results to a calculation of the equation of state at finite temperatures and of the specific heat, and we discuss the crossover from the quantum Gaussian fixed point to the classical Gaussian fixed point. In the last subsection we discuss the analogous temperature dependence in systems with quenched disorder.

A. Scaling behavior of the temperature

The temperature dependence of the LGW functional, Eq. (2.7a), is due to a number of entirely different effects. First, the spin-density correlation functions $\chi^{(n)}$ that determine the coefficients of the LGW functional are temperature dependent as given in Eqs. (2.10b) and (2.11b). These are correlation functions for the reference ensemble that is far from any critical point. Therefore, the frequency or temperature in $\chi^{(n)}$ scales like the momentum, and consequently we have a temperature scale whose scale dimension is,

$$[T]_{\text{ball}} = 1 \quad . \quad (4.1a)$$

This we will refer to as the ballistic temperature scale.

A second source of temperature dependence is the usual dynamical scaling. This originates from the fact that the time integration in the LGW functional, Eq. (2.7a), extends over the finite interval $[0, 1/T]$. Dynamical scaling is hence equivalent to finite-size scaling in the direction of imaginary time. The scaling behavior of the temperature due to dynamical scaling is described by the dynamical exponent z as given in Eq. (3.2c). We therefore have a second temperature scale whose scale dimension is

$$[T]_{\text{crit}} = z \quad , \quad (4.1b)$$

which below we will refer to as the critical temperature scale.

Finally, the distance from the critical point, t , is temperature dependent. For the bare t_0 , this is simply the usual T^2 dependence that is familiar from Fermi liquid theory. The scale dimension of this Fermi liquid temperature scale is thus

$$[T]_{\text{FL}} = 1/2\nu \quad . \quad (4.1c)$$

However, upon renormalization t acquires a more intricate temperature dependence. Since the loop or fluctuation corrections involve integrals over critical propagators, the latter depends on the critical temperature scale. However, it also depends on the scale dimensions of the vertices u_n and v_n , Eq. (2.11a), that are dangerous irrelevant variables with respect to the T -dependence of the magnetization and the magnetic susceptibility. Millis¹⁷ and Sachdev¹⁸ have shown that the resulting temperature scale has a scale dimension

$$[T]_{\text{fluct}} = z/(1 - \nu\theta) \quad , \quad (4.1d)$$

where θ is the scale dimension of the appropriate dangerous irrelevant operator, u_4 in our case. This we call the fluctuation temperature scale.

Of all the temperature scales that are present in a given quantity, the one with the largest scale dimension will be the dominant one. This means, for instance, that the Fermi liquid scale will be subdominant compared to the ballistic one for all $1 < d < 3$; for $d > 3$ the two scales are indistinguishable. Since $z > 1$ for all $d > 1$ it also means that the critical temperature scale will in general be dominant over both the ballistic scale and the fluctuation scale. There are, however, two possible mechanisms that can invalidate this conclusion. The first possibility is that some quantities do not depend on the critical temperature scale. For those the leading temperature dependence will be given by either the ballistic temperature scale, or by the fluctuation scale, depending on the values of $[T]_{\text{ball}}$ and $[T]_{\text{fluct}}$. As we will see, this possibility is realized for the magnetization and the magnetic susceptibility. The second possibility is that a subdominant temperature scale is dangerously irrelevant. This possibility cannot be ruled out by general scaling considerations.

B. Magnetization, and magnetic susceptibility

Since we work above an upper critical dimensionality, the magnetization m and the magnetic susceptibility χ_m are determined by the $q = \Omega_n = 0$ Fourier component of the order parameter field, and do not depend on the finite-frequency behavior of the critical modes. Therefore, dynamical scaling does *not* enter the temperature dependence of m and χ_m directly. In other words, their behavior is completely determined by the equation of state, Eq. (3.6), whose coefficients acquire finite temperature corrections according to Eqs. (2.10b) and (2.11b). The tm term in Eq. (3.6) has a correction of the form $m(p+T)^{d-1} \sim m(m+T)^{d-1}$. Similarly, the higher order terms $v_n p^{d+1-n} m^{n-1}$ have corrections proportional to $(p+T)^{d+1-n} m^{n-1} \sim (m+T)^{d+1-n} m^{n-1}$. This is the ballistic temperature scale discussed in Sec. IV A above. In addition, there is the fluctuation temperature scale that results from the temperature dependence of t . By using Eq. (3.5a) in Eq. (4.1d) we see that $[T]_{\text{ball}} > [T]_{\text{fluct}}$, and thus is the dominant temperature scale, for $1 < d < 2$, and for $d > 5$, but that for $2 < d < 5$, $[T]_{\text{fluct}}$ is the relevant scale. All of these corrections can be summarized in the following scaling law for the magnetization m ,

$$m(t, T, H) = b^{-\beta/\nu} m(tb^{1/\nu}, Tb^{\phi/\nu}, Hb^{\delta\beta/\nu}) \quad , \quad (4.2a)$$

where b is an arbitrary scale factor. The crossover exponent

$$\phi = \begin{cases} \nu = 1/(d-1) & , \quad \text{for } 1 < d < 2 & , \\ d/2(d-1) & , \quad \text{for } 2 < d < 3 & , \\ 3/(d+1) & , \quad \text{for } 3 < d < 5 & , \\ \nu = 1/2 & , \quad \text{for } d > 5 & , \end{cases} \quad (4.2b)$$

describes the crossover from the quantum critical region to a regime whose behavior is dominated by the classical Gaussian fixed point.^{17,19} Note that, for the reasons explained above, the crossover exponent is *not* given by $z\nu$ which one would expect from dynamical scaling. Also note the complicated behavior of the crossover exponent ϕ as a function of the dimensionality, which is brought about by the competition between the ballistic and the fluctuation temperature scales. The result $\phi = \nu$ that was reported in Refs. 5 and 20 was correct only for $1 < d < 2$ and $d > 5$. By differentiating Eq. (4.2a) with respect to H , we obtain the analogous homogeneity law for the magnetic susceptibility χ_m ,

$$\chi_m(t, T, H) = b^{\gamma/\nu} \chi_m(tb^{1/\nu}, Tb^{\phi/\nu}, Hb^{\delta\beta/\nu}) \quad . \quad (4.3)$$

C. The specific heat

The scaling behavior of the specific heat is determined by the sum of the mean-field and the Gaussian fluctuation contribution to the free energy density f . The mean-field contribution follows immediately from Eq. (3.7).

The Gaussian fluctuation contribution, f_G , which gives the leading nonanalyticity for the specific heat at the critical point, can be calculated in complete analogy to the case of classical ϕ^4 -theory in $d > 4$.²¹ Neglecting an uninteresting constant contribution to f_G , we obtain,

$$f_G = \frac{T}{2V} \sum_{\mathbf{q}, \omega_n} \left[2 \ln(H/m + a_{d-1} q^{d-1} + a_2 q^2 + a_\Omega |\Omega_n|/q) + \ln(x_d H/m - (x_d - 1)t + a_{d-1} q^{d-1} + a_2 q^2 + a_\Omega |\Omega_n|/q) \right] \quad . \quad (4.4)$$

Here $x_d = d$ for $1 < d < 3$, and $x_d = 3$ for $d > 3$. The specific heat coefficient γ_V is conventionally defined by,

$$\gamma_V = c_V/T = -\partial^2 f / \partial T^2 \quad . \quad (4.5)$$

Again we are interested only in scaling properties and not in exact coefficients. Keeping this in mind, an adequate representation of Eqs. (4.4) and (4.5) is given by the integral,

$$\gamma_V = \int_0^\Lambda dq \frac{q^{d-1}}{T + q^d + q^3 + Hq/m} \quad , \quad (4.6)$$

with Λ an ultraviolet cutoff.

Let us point out two interesting features of this result. First, for all dimensionalities in the range $1 < d \leq 3$, γ_V is logarithmically singular for $T, H \rightarrow 0$. This can be seen most easily from Eq. (4.6), and it is also true for the exact result, Eqs. (4.5), (4.4). Such a d -independent logarithmic singularity is somewhat unusual. Wegner has discussed how logarithmic corrections to scaling arise if a set of scale dimensions fulfills some resonance condition.²² In the present case the relevant relation is that the scale dimension of the free energy, $d + z = 2z$ for $1 < d < 3$, is a multiple of the scale dimension of T , which is $z = d$ in this region. The fact that the logarithm appears in a *range* of dimensions, rather than only for a special value of d , is due to the dynamical exponent being exactly d in that range. Second, as discussed in Sec. IV A, two different temperature scales appear in Eq. (4.6). The first two terms in the denominator indicate that $T \sim \xi^{-d}$, as one would expect from dynamical scaling. However, the last term in Eq. (4.6) contains the magnetization, which in turn depends on $[T]_{\text{ball}}$ and $[T]_{\text{fluct}}$. These two features imply that the scaling equation for γ_V should be written,

$$\gamma_V(t, T, H) = \Theta(3-d) \ln b + F_\gamma(tb^{1/\nu}, Tb^z, Hb^{\delta\beta/\nu}, Tb^{\phi/\nu}) \quad . \quad (4.7)$$

Note that the scale dimension of γ_V , ignoring the logarithm, is zero in all dimensions. Since $z > \phi/\nu$ for all $d > 1$, one can formally ignore the fourth entry in the scaling function since it is subleading compared to the second entry and its effects can be considered as ‘corrections to scaling’. We emphasize that in contrast to the

magnetization and the magnetic susceptibility, the specific heat *does* depend on the critical modes, and hence contains the critical temperature scale. As mentioned in Sec. IV A, the latter is dominant when it is present, and γ_V provides an example for that.

D. The disordered case revisited

Let us finally reconsider the case of systems with quenched disorder.⁴ We do this partly to point out the remarkable analogy between the clean and dirty cases, and partly to correct a mistake in the results of Ref. 4 for dimensions $d > 3$.

In the presence of disorder, the temperature or frequency far from criticality scales like the wavenumber squared. The ballistic temperature scale of Eq. (4.1a) therefore gets replaced by a diffusive one,

$$[T]_{\text{diff}} = 2 \quad . \quad (4.8)$$

The scale dimensions for the other temperature scales, Eqs. (4.1b) - (4.1d) remain valid, but the values of the exponents z , ν , and θ change. In the action, the disorder leads to terms in addition to, and structurally different from, those in Eq. (2.7a).⁴ In particular, at order \mathbf{M}^4 a second term appears whose coupling constant was denoted by v_4 in Ref. 4. This operator, whose scale dimension is $[v_4] = -|d-4|$, is the least irrelevant operator that is dangerous with respect to t , and one therefore has $\theta = -|d-4|$ in the fluctuation temperature scale, Eq. (4.1d). With z and ν as determined in Ref. 4, this leads to a crossover exponent

$$\phi = \begin{cases} 2\nu = 2/(d-2) & , \quad \text{for } 2 < d < \sqrt{5} + 1 & , \\ d/2 & , \quad \text{for } \sqrt{5} + 1 < d < 4 & , \\ 4/(d-2) & , \quad \text{for } 4 < d < 6 & , \\ 2\nu = 1 & , \quad \text{for } d > 6 & , \end{cases} \quad (4.9)$$

The result $\phi = 2\nu$ of Refs. 4, 20 was thus not correct for the (unphysical) dimensionality range $\sqrt{5} + 1 < d < 6$. The scaling behavior of the magnetization and the magnetic susceptibility is given by Eqs. (4.2a) and (4.3), respectively, with ϕ from Eq. (4.9), and all other exponents as given in Ref. 4.

V. HERTZ'S MODEL REVISITED

In this section we reexamine Hertz's original model of the ferromagnetic quantum phase transition.¹ We show that at tree level, the LGW theory for this model breaks down for related, but somewhat different reasons than in the realistic model above. Moreover, starting at one-loop order the renormalization group generates terms that are not in the bare action. As a result, the critical behavior of this model in $d > 1$ is *not* mean-field like, but rather

the same as that of the more realistic model we have studied so far.

Hertz's model differs in two respects from the more realistic one given by Eqs. (2.3) through (2.5). First, the interaction part of the action contains only the particle-hole spin-triplet channel that is decoupled in the derivation of the LGW functional. Consequently, the reference ensemble S_0 consists of noninteracting electrons. Second, the coefficients $\chi^{(n)}$ of the LGW functional are taken to be the correlation functions of a 3- d Fermi gas, irrespective of the dimensionality of the space the magnetization is confined to. For this model the spin susceptibility χ_s of the reference system is simply the Lindhardt function, so Eq. (2.9) gets replaced by

$$\chi_s(\mathbf{q}, \Omega_n) = c_0 - c_2 q^2 - |\Omega_n|/q \quad . \quad (5.1)$$

In comparison to the analogous expression for an interacting Fermi liquid, Eqs. (2.9) and (2.10a), the term proportional to q^{d-1} is missing, and $\chi_s(\mathbf{q}, \Omega_n = 0)$ is now analytic at $\mathbf{q} = 0$. The resulting Gaussian part of the bare LGW functional has the form

$$\Phi_2[\mathbf{M}] = \frac{1}{2} \sum_{\mathbf{q}, \Omega_n} \mathbf{M}(\mathbf{q}, \Omega_n) [t_0 + a_2 q^2 + a_\Omega |\Omega_n|/q] \times \mathbf{M}(-\mathbf{q}, -\Omega_n) \quad . \quad (5.2)$$

This action allows for a Gaussian fixed point with mean-field static exponents and a dynamical exponent $z = 3$.¹

Let us now investigate the stability of this Gaussian fixed point. At tree level, we can do this by calculating the scale dimensions of the coefficients of the higher order terms. Hertz considered the higher order correlation functions $\chi^{(n)}$ only in the limit $\Omega_n = 0$, $q \rightarrow 0$ where they are finite numbers. The usual power counting arguments in analogy to Sec. IIIB show that all of those terms are irrelevant for $d > 1$. The quartic one becomes marginal in $d = 1$ and relevant for $d < 1$. This changes, however, if one considers the first order term in an expansion of the $\chi^{(n)}$ for non-interacting electrons in powers of frequency. The reason is that, at nonzero external frequency, frequency mixing effects occur that are similar to those brought about by an electron-electron interaction (which causes them even at zero external frequency). For power counting purposes, i.e. to determine the scale dimensions of the coefficients, it is not necessary to calculate the $\chi^{(n)}$ completely as functions of $n-1$ frequencies and wavevectors. Guided by more complete calculations for the cases $n = 3, 4, 5$, we have concluded that for power counting purposes it suffices to consider one independent frequency, Ω , and two directionally independent wavevectors of equal length, $|\mathbf{q}_1| = |\mathbf{q}_2| = |\mathbf{q}|$, that form an angle θ . With these simplifications the leading term of the general coefficient is easily calculated. We find that for odd n the linear-in-frequency term in the action can symbolically be written

$$w_n \int dx M^n(x) \Omega/q^{n-1} \quad , \quad (5.3a)$$

with w_n some coupling constant, while the corresponding terms for even n are less relevant. The same power counting arguments that we used in Sec. III B show that the scale dimension of w_n is $[w_n] = -(d-1)(n-2)/2$, which is negative for all $d > 1$. The coupling constants w_n are thus irrelevant for all physical dimensions $d \geq 2$, and seem to become marginal in $d = 1$. However, Eq. (5.3a) holds only if the two independent wavevectors are neither parallel nor antiparallel, i.e. it holds only if $d > 1$. For $\theta = 0, \pi$ one finds a stronger singularity for the terms with even n ,

$$w_n \int dx M^n(x) \Omega/q^{2n-3} \quad , \quad (5.3b)$$

while the terms with odd n are less relevant. In dimensions $d > 1$ the parallel wavevectors form a set of measure zero and this stronger singularity is of no consequence. In $d = 1$, however, Eq. (5.3b) represents the generic behavior of the terms of $O(M^n)$. Power counting yields the scale dimensions to be $[w_n] = n - 2$ in $d = 1$, so all of these coefficients are relevant operators. This is sufficient to conclude that the upper critical dimension is *not* one, but rather that the $1-d$ system is *below* its upper critical dimension, and will show nontrivial critical behavior. This provides a technical explanation for Sachdev's observation³ that Hertz's results in $d < 1$ cannot be correct.

Moreover, the renormalization of the model beyond the tree level qualitatively changes the form of the Gaussian action, Eq. (5.2). Consider, for instance, the one-loop renormalization of Φ_2 by the terms of order \mathbf{M}^3 and \mathbf{M}^4 . The corresponding diagrams are shown in Fig. 1. It is easy to see that these diagrams are equivalent to those that determine the spin susceptibility of interacting electrons, and have been calculated in Ref. 12.²³ Renormalization therefore leads to a term proportional to q^{d-1} in the Gaussian action, and hence to a Φ_2 as given by Eq. (3.1a). We conclude that the critical behavior of Hertz's model for $d > 1$ is *not* mean-field like, but rather the same as that of the more realistic model discussed in Secs. II - IV.

VI. DISCUSSION

In this paper we have shown that clean itinerant quantum ferromagnets at zero temperature do not show, as was previously thought, uninteresting mean-field critical behavior for all dimensionalities $d > 1$. Rather, there are two upper critical dimensions. The first one is $d_c^+ = 1$, above which the critical behavior is described by a Gaussian field theory but is not mean-field like, and the second one is $d_c^{++} = 3$, above which one does find mean-field critical behavior. The reason for this unusual behavior is soft modes that lead to an effective long-range interaction between the order parameter fluctuations. As is the case for classical models with long-range interactions,²⁴ this

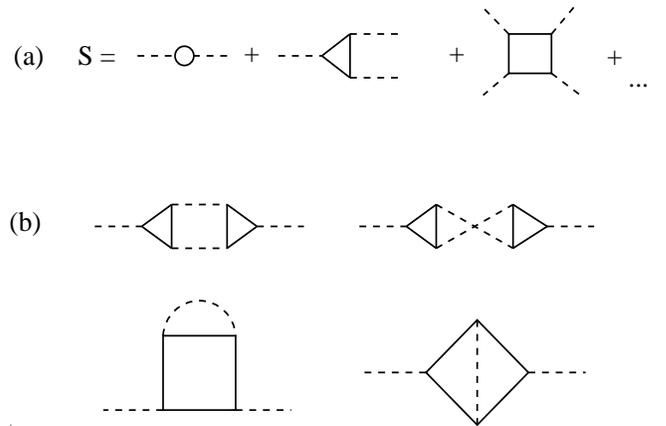


FIG. 1. (a) Symbolic representation of the action. The dashed lines represent the external legs, and the polygons represent the vertices. (b) Renormalization of the two-point vertex to one-loop order.

leads to nontrivial critical behavior that still can be determined exactly. d_c^{++} is the marginal dimension where the soft mode induced long-range interaction coincides with the usual $r^{-(d+2)}$ behavior. In this final section we discuss a few aspects of these results that have not been covered yet.

First of all, both our approach and our results are remarkably similar to a recent treatment of *disordered* itinerant quantum ferromagnets.⁴ In these papers it was found that the disorder induced diffusive excitations in a Fermi system with quenched disorder lead to similar, but stronger effects, with a long-ranged interaction between order parameter fluctuations that falls off like $r^{-(2d-2)}$, and *three* 'upper critical dimensionalities', viz. 2, 6, and 4. The first two are analogous to d_c^+ and d_c^{++} above. In addition, the critical exponents ν , η , and z take on their mean-field values for $d > 4$, while β and δ remain anomalous between $d = 4$ and $d = 6$. In the present case, a different structure of the dangerous irrelevant variables makes the upper two special dimensions coincide. The same difference in the dangerous irrelevant variables leads to a difference in the temperature dependence of the equation of state as a function of dimensionality: In the disordered case, the diffusive temperature scale with $[T]_{\text{diff}} = 2$, which is analogous to the ballistic temperature scale $[T]_{\text{ball}} = 1$ in the clean case, is dominant in the physical dimension $d = 3$. In the clean case, in contrast, the ballistic scale is subdominant compared to the fluctuation temperature scale for $2 < d < 5$.

The present paper shows that the basic concepts of Refs. 4 are not restricted to disordered systems. Indeed, an attempt to construct an effective field theory entirely in terms of the order parameter field for any phase transition will break down (in the sense that it is impossible to construct a *local* effective theory) if there are soft or slow modes other than the order parameter fluctuations that couple to the order parameter. In the present case, the

spin-triplet particle-hole excitations that always exist in an interacting Fermi system, and that are distinct from the order parameter mode in that they are soft even in the paramagnetic phase, are such additional soft modes. They lead to the nonanalytic behavior of spin density correlation functions that is displayed in Eqs. (2.10) and (2.11), and hence to the effective long-ranged interaction between the order parameter modes. In general, the appearance of such additional soft modes would call for the derivation of a different effective theory that does not integrate out as many degrees of freedom, and that keeps all of the soft modes on equal footing. However, as in Ref. 4 we have opted here to work with the order parameter effective theory after all, since it turns out that the difficulties introduced by the nonlocal character of the field theory can be overcome. Nevertheless it would be interesting to treat the same problem by means of a local theory that keeps more degrees of freedom explicitly.

It should also be pointed out that our results depend crucially on the fact that the electronic spin density is a conserved quantity. If this feature was lost, e.g. due to some type of spin flip process, then the spin-triplet particle-hole excitations would acquire a mass or energy gap, and at scales larger than this mass the effective interactions between the order parameter fluctuations would be of short range. The asymptotic critical phenomena would then be described by a local order parameter field theory with mean-field critical behavior in all physical dimensions. At this point one might wonder whether the magnetization in the ordered phase, and magnetic fluctuations in the disordered one, act effectively as magnetic impurities, and why this does not lead to an energy gap that invalidates our conclusions. The answer is that this effect has already been taken into account. In the ordered phase, the magnetization indeed acts as a cutoff, as has been discussed in connection with Eq. (3.6), and this leads to the nonanalyticity in the equation of state. In the disordered phase, the cutoff enters only via fluctuations, which are RG irrelevant with respect to the Gaussian fixed point. The effect therefore manifests itself only in the corrections to scaling, not in the leading behavior.

We also mention that all of the qualitative points discussed in Refs. 4 and 20 that had to do with the fact that one works above an upper critical dimension, apply here as well. In particular, the presence of dangerous irrelevant variables means that some of Sachdev's general results³ are not directly applicable to the transition discussed here. For instance, the Wilson ratio, $W = (m/H)/(C_V/T)$, diverges at criticality, as it does in the disordered case, rather than being a universal number as would be the case in the presence of hyperscaling. However, due to the different structure of the dangerous irrelevant variables that was already mentioned above in connection with the multiple upper critical dimensionalities, some details are different between the disordered and clean cases. For instance, the scaling function F_γ , Eq. (4.7), is a function of T/H (if one neglects the sub-

leading ballistic temperature scale) in agreement with the prediction of Ref. 3, while in the presence of disorder this is not the case.⁴

We add one more remark concerning Hertz's original model that was discussed in Sec. V. Our conclusion that a proper renormalization of that model leads to a critical behavior that is the same as that of the realistic model solves the following paradox that would otherwise arise: Suppose one divided the interaction term in Hertz's model into two structurally identical pieces, one carrying some fraction of the interaction amplitude Γ_t , and the other the rest. Suppose one then applied the Hubbard-Stratonovich decoupling only to one of these pieces, and lumped the other into the reference ensemble. Then the reference ensemble would contain a nonzero Γ_t , and according to Ref. 12 the Gaussian action would contain the nonanalytic q^{d-1} term that leads to non-mean field critical behavior for all $d < 3$. If one decouples all of the interaction term, on the other hand, then the reference ensemble has $\Gamma_t = 0$, and in the absence of any other interaction amplitudes there is no way to generate a nonzero Γ_t by renormalization. If Hertz's model had indeed a critical behavior that is different from that of the realistic model, then the inevitable conclusion would be that the result depends on how exactly one performs the decoupling of the interaction term, which would be physically absurd.

Finally, it should be mentioned that an experimental corroboration or refutation of our results is probably harder for the clean case discussed in the present paper than for the disordered case treated in Refs. 4. There are several reasons for this. First of all, the zero-temperature behavior can of course not be observed directly. An experimental study would therefore have to concentrate on the crossover from the quantum critical behavior to the classical one that will occur if the classical transition point is at a sufficiently low temperature for the crossover point to be within the critical region. This requires a ferromagnet with as low a Curie temperature as possible. In addition, the parameter t that measures the distance from the critical point is, in the quantum case, *not* the temperature but rather the exchange interaction or some other microscopic parameter that is hard to control. Both of these difficulties can be overcome relatively easily in the disordered case, where a change in the composition of a ferromagnetic alloy changes both the classical transition temperature and t . The quantum critical point is reached in the low-temperature limit in a sample whose composition is such that the Curie temperature just vanishes. As was discussed in Ref. 4, this provides a manageable handle on t that has no obvious analog in the clean case. Furthermore, the differences between the quantum critical exponents in $d = 3$ and the classical Heisenberg exponents are more pronounced in the disordered case than in the clean one. As we have seen, the $3-d$ critical behavior in the latter case is mean-field with logarithmic corrections to scaling. The logarithms would be hard to observe in any case, and the

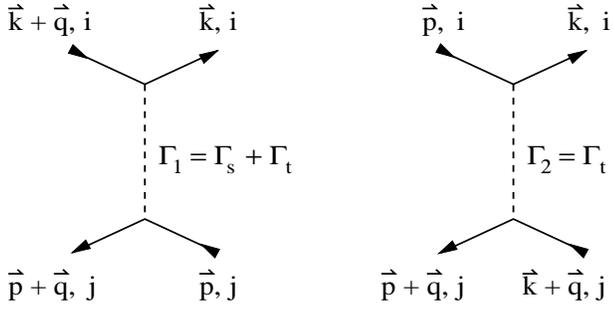


FIG. 2. The three basic interaction vertices. Straight lines with arrows denote particles and holes, wavy lines denote the interaction. The notation is the same as used in the text.

mean-field exponents are much closer to 3-*d* Heisenberg exponents than those obtained in Ref. 4.

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APPENDIX A: RENORMALIZATION OF THE PARTICLE-HOLE SPIN-TRIPLET INTERACTION VERTEX

In this appendix we show that a nonvanishing particle-particle channel interaction, $\Gamma_c \neq 0$, generates a particle-hole spin-triplet interaction Γ_t , even if $\Gamma_t = 0$ in the bare system.

Figure 2 shows the three basic types of interaction vertices. Now suppose that $\Gamma_t = 0$. Then a vertex of this type can be constructed by means of the diagrams shown in Fig. 3, where all of the dashed lines represent Cooper channel interaction amplitudes.

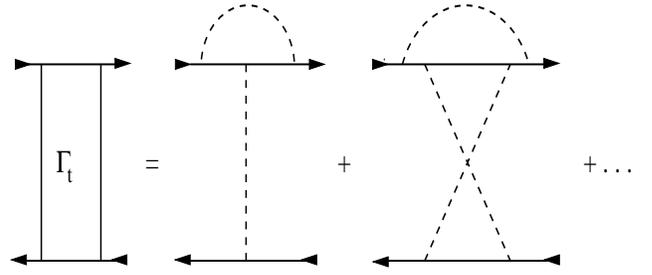


FIG. 3. Construction of a particle-hole spin-triplet vertex out of particle-particle vertices.

APPENDIX B: LOGARITHMIC CORRECTIONS TO SCALING FOR $D = 3$, AND FOR $1 < D < 3$

There are three distinct mechanisms that produce logarithmic corrections to scaling: (1) Marginal operators, (2) Wegner resonance conditions between a set of scale dimensions, and (3) logarithmic corrections to the scale dimension of a dangerous irrelevant operator. The first two mechanisms are well known.²² The third is operative only above an upper critical dimension, and is therefore of particular interest for quantum phase transitions.

In the present case, logarithmic corrections to scaling arise due to mechanisms (2) and (3). Mechanism (2) produces corrections to the scaling of the specific heat in all dimensions $1 < d \leq 3$, as was discussed in Sec. IV C. The third mechanism produces corrections to scaling in $d = 3$. According to Eq. (2.11a), the correlation function $\chi^{(4)}$ contains a term $v_4 \ln p$ in $d = 3$. Via Eq. (3.6) or (3.7) this leads, for instance, to a leading behavior of the spontaneous magnetization,

$$m(t, H = 0) \sim \frac{t^{1/2}}{\sqrt{\ln(1/t)}} \quad , \quad (\text{B1})$$

and at the critical point we have

$$m(t = 0, H) \sim \frac{H^{1/3}}{[\ln(1/H)]^{1/3}} \quad . \quad (\text{B2})$$

¹ J. A. Hertz, Phys. Rev. B **14**, 1165 (1976). For earlier thoughts about quantum phase transitions, see the references therein.

² We use the term ‘LGW theory’ in the narrow sense, in which it is usually used in the literature, of an effective field theory in terms of the order parameter field only. If one defines it as an effective theory for all soft modes, then it is valid in the present case, too.

³ S. Sachdev, Z. Phys. B **94**, 469 (1994).

- ⁴ D. Belitz and T.R. Kirkpatrick, Europhys. Lett. **35**, 201 (1996); T. R. Kirkpatrick and D. Belitz, Phys. Rev. B **53**, 14364 (1996).
- ⁵ T. Vojta, D. Belitz, R. Narayanan, and T. R. Kirkpatrick, Europhys. Lett. **xx**, xxx (1996) (cond-mat/9510146).
- ⁶ See, e.g., J.W. Negele and H. Orland, Quantum Many-Particle Systems, Addison Wesley (New York 1988).
- ⁷ See, e.g., A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York 1975).
- ⁸ This becomes clearer in Fourier space, where the sums over wavenumbers that replace the real space integrals in Eq. (2.5a) extend over small wavenumbers only. See, e.g., Ref. 12 for a more detailed exposition of this point.
- ⁹ Our interaction constants Γ_s , Γ_t , and Γ_c are the zero angular momentum ($l = 0$) components in a multipole expansion of the respective interaction potentials, and hence are related to the $l = 0$ Landau parameters. The physics we are interested in is due to hydrodynamic effects, which are strongest in the $l = 0$ or density channel. This justifies our neglecting the higher Fermi liquid parameters.
- ¹⁰ See, e.g., K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).
- ¹¹ This can be seen as follows. Since the magnetization is conserved, ordering on a length scale L requires some spin density to be transported over that length, which takes a time $t \sim L/v_F$, with v_F the Fermi velocity. Now suppose the coherence length is ξ , and we look at the system at a momentum scale q or a length scale $L \sim 1/q < \xi$. Because of the time it takes the system to order over that scale, the condition for criticality is $L < \text{Min}(v_F t, \xi)$. In particular, one must have $L < v_F t$, or $\Omega \sim 1/t < v_F q$.
- ¹² D. Belitz, T. R. Kirkpatrick, and T. Vojta, cond-mat/9611099.
- ¹³ To lowest order in perturbation theory in Γ_t one finds $c_{d-1} < 0$, see Ref. 12. In that reference, various mechanisms have been discussed that may lead to $c_{d-1} > 0$ for realistic values of the interaction strength. In what follows we discuss only systems whose parameter values are such that $c_{d-1} > 0$, since else the continuous ferromagnetic phase transition we are interested in does not exist.
- ¹⁴ See, e.g., G. Baym and C. Pethick, *Landau Fermi Liquid Theory: Concepts and Applications* (Wiley, New York 1991). Notice that $\chi_0(q = 0)$ as a function of temperature in $d = 3$ does *not* show a corresponding nonanalyticity. This does not contradict the nonanalytic q -dependence at zero temperature, see Ref. 12.
- ¹⁵ T. R. Kirkpatrick and D. Belitz, cond-mat/9602144, to appear in J. Stat. Phys.
- ¹⁶ M. E. Fisher, in *Advanced Course on Critical Phenomena*, edited by F. W. Hahne, Springer (New York 1983).
- ¹⁷ A. J. Millis, Phys. Rev. B **48**, 7183 (1993).
- ¹⁸ S. Sachdev, cond-mat/9606083.
- ¹⁹ For all dimensions $d > 2$, where there is a classical Heisenberg transition, there is still another crossover in the system, namely from the classical Gaussian region to the classical Heisenberg critical region. For $1 < d < 2$, where there is no long-range order at any nonzero temperature, this is not the case. In either case, ϕ describes the leading low-temperature effect due to the relevance of the temperature with respect to the quantum fixed point.
- ²⁰ D. Belitz and T. R. Kirkpatrick, J. Phys. Cond. Matt. **8**, 1 (1996).
- ²¹ See, e.g., S. K. Ma, *Modern Theory of Critical Phenomena*, (Benjamin, Reading, MA 1976).
- ²² F. J. Wegner, in *Phase Transitions and Critical Phenomena*, vol.6, edited by C. Domb and M. S. Green (Academic, New York 1976).
- ²³ Remember that in writing Eq. (5.2) a factor of Γ_t has been absorbed in the field \mathbf{M} , and that the Gaussian vertex is given by $1 - \Gamma_t \chi^{(2)}$. In Ref. 12, the spin susceptibility was calculated to second order in Γ_t , which in the present language also involves some two-loop diagrams.
- ²⁴ M. E. Fisher, S.-K. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).