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Probes of the vacuum structure of quantum fields in classical backgrounds

L. Sriramkumar

Racah Institute of Physics, Hebrew University,
Givat Ram, Jerusalem 91904, Israel[*].

and

T. Padmanabhan

IUCAA, Post Bag 4, Ganeshkhind, Pune 411 007, India.
E-mail: paddy@iucaa.ernet.in.

We compare the different approaches presently available in literature to probe the vacuum structure of quantum fields in classical electromagnetic and gravitational backgrounds. We compare the results from the Bogolubov transformations and the effective Lagrangian approach with the response of monopole detectors (of the Unruh-DeWitt type) in non-inertial frames in flat spacetime and in inertial frames in different types of classical electromagnetic backgrounds. We also carry out such a comparison in inertial and rotating frames when boundaries are present in flat spacetime. We find that the results from these different approaches do not, in general, agree with each other. We attempt to identify the origin of these differences and then go on to discuss its implications for classical gravitational backgrounds.

[*] Present Address: Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta T6G 2J1, Canada. E-mail: slakshm@phys.ualberta.ca.

1. INTRODUCTION

The vacuum state of a quantum field develops a non-trivial structure in a classical electromagnetic or gravitational background. As a result, essentially, two different types of phenomena occur in a classical background: (i) polarization of the vacuum and (ii) production of particles corresponding to the quantum field. Apart from these two effects, there is another feature that one encounters in a gravitational background: the concept of a particle turns out to be coordinate dependent. (For a detailed discussion on these different aspects of quantum field theory in classical backgrounds, see the following books [1, 2, 3, 4, 5, 6, 7] and the recent reviews [8, 9].) A classic example of vacuum polarization is the Casimir effect [10] while Hawking radiation from collapsing black holes is the most famous example of particle production [11]. The coordinate dependence of the particle concept that arises in a gravitational background is well illustrated by the fact that the Rindler vacuum turns out to be inequivalent to the Minkowski vacuum [12].

Different approaches have been formulated in literature to study the evolution of a quantum field in a classical electromagnetic or gravitational background. On the one hand, the Bogolubov transformations [13] and the effective Lagrangian approach [14, 15, 16] offer us formal methods to probe the vacuum structure of the quantum field. On the other, studying the response of detectors coupled to the quantum field provides us with an operational tool for understanding the concept of a particle [17, 18]. Often in literature, one of these approaches has been used to study the behavior of a quantum field in a classical background and, apart from a few instances (see, for e.g., Refs. [19, 20, 21, 22, 23]), the results from these different approaches have not been compared. Due to this reason, the possibility that these approaches can lead to different results has not been adequately emphasized. (As we shall see later, these different approaches do, in general, lead to different results.) Our motivation in this paper is to compare the results from these different approaches in a variety of situations, identify the origin of the differences that arise and understand its implications for classical gravitational backgrounds.

A detailed outline of the contents of this paper is as follows. In Section 2, we shall briefly review the three different approaches that are available at present to study the evolution of a quantum field in a classical background, viz. (i) the Bogolubov transformations, (ii) the response of detectors and (iii) the effective Lagrangian approach. We shall confine our attention in this paper to monopole detectors of the Unruh-DeWitt type.

In Section 3, we shall compare the results from these different approaches in non-inertial frames in flat spacetime. In Section 3.1, following Refs. [19, 20], we construct different non-inertial trajectories in flat spacetime which are integral curves of timelike Killing vector fields. In Section 3.2, we evaluate the effective Lagrangian in coordinates adapted to these trajectories. We compare these results with the results from the Bogolubov transformations and the response of the Unruh-DeWitt detector that have been obtained in literature before [19, 20, 21]. In Section 3.3, we express the response of the Unruh-DeWitt detector in terms of the Bogolubov coefficients [22] and identify the origin of the differences that arise in the results from these two approaches.

In Section 4, we shall carry out such a comparison when boundaries are present in flat spacetime. In Section 4.1, we compare the response of an inertial Unruh-DeWitt detector in the Casimir vacuum with the result obtained from the effective Lagrangian approach. In Section 4.2, we briefly discuss as to how the response of a rotating detector would compare with the effective Lagrangian when a boundary condition is imposed on the horizon in the rotating frame.

In Section 5, we shall compare the results in inertial frames in different types of classical electromagnetic backgrounds. In Section 5.1, following Ref. [24], we discuss the response of a monopole detector that is coupled to the quantum field through a gauge-invariant and non-linear interaction. In Sections 5.2, 5.3 and 5.4, we study the response of this detector (when it is in inertial motion) in a time-dependent electric field, a time-independent electric field and a time-independent magnetic field, backgrounds, respectively. We also discuss as to how the response of the detector compares with the results expected from the Bogolubov transformations and the effective Lagrangian approach.

In the concluding Section 6, we shall first briefly summarize the results of our analysis in Section 6.1. Then, in Section 6.2, we shall go on to discuss the implications of these results for classical gravitational backgrounds.

Our conventions and notations are as follows. Throughout this paper, we shall set $\hbar = c = 1$. We shall always work in $(3+1)$ dimensions and the metric signature we shall adopt is $(+,-,-,-)$. Also, for the sake of convenience and clarity in notation, we shall denote the set of coordinates x^μ as $\tilde{x}$ and we shall write the derivative $(\partial/\partial x)$ simply as ∂_x. Finally, we shall denote complex conjugation and Hermitian conjugation by an asterisk and a dagger, respectively.

2. PROBES OF THE VACUUM STRUCTURE

In this section, we shall briefly review the three different probes of the vacuum structure of quantum fields in classical backgrounds, viz. (i) the structure of the Bogolubov transformations, (ii) the response of the Unruh-DeWitt detector and (iii) the effective Lagrangian approach. We shall gather here the results that will prove to be essential for our discussion later on.

2.1. Bogolubov transformations

Consider a quantum scalar field $\hat{\Phi}$ of mass m evolving in a given classical background. Let the quantum field $\hat{\Phi}$ satisfy the following equation of motion:

$$\left(\hat{H} + m^2\right)\hat{\Phi} = 0, \tag{1}$$

where $\hat{H}$ is a differential operator whose form depends on the classical background. A conserved current corresponding to this equation of motion can then be used to define a scalar product for the modes of the quantum field. Let $\{u_i(\tilde{x})\}$ and $\{\bar{u}_k(\tilde{x})\}$ be two complete sets of positive *norm*, orthonormal modes corresponding to such a scalar product[1]. When two such complete sets of modes exist, one set of modes can be expressed in terms of the other using the Bogolubov transformations as follows (see, for e.g., Ref. [1], Sec. 3.2):

$$\bar{u}_k(\tilde{x}) = \sum_i \left[\alpha_{ki}\, u_i(\tilde{x}) + \beta_{ki}\, u_i^*(\tilde{x})\right], \tag{2}$$

or conversely

$$u_i(\tilde{x}) = \sum_k \left[\alpha_{ki}^*\, \bar{u}_k(\tilde{x}) - \beta_{ki}\, \bar{u}_k^*(\tilde{x})\right]. \tag{3}$$

The quantities α_{ki} and β_{ki} are called the Bogolubov coefficients [13]. Using the orthonormality of the modes and the relation (2), the Bogolubov coefficients can be expressed as

$$\alpha_{ki} = \left(\bar{u}_k(\tilde{x}), u_i(\tilde{x})\right) \qquad \text{and} \qquad \beta_{ki} = -\left(\bar{u}_k(\tilde{x}), u_i^*(\tilde{x})\right), \tag{4}$$

where the brackets denote scalar products.

A *real* quantum scalar field $\hat{\Phi}$, for instance, can be decomposed in terms of the two sets of modes $\{u_i(\tilde{x})\}$ and $\{\bar{u}_k(\tilde{x})\}$ as follows:

$$\hat{\Phi}(\tilde{x}) = \sum_i \left[\hat{a}_i\, u_i(\tilde{x}) + \hat{a}_i^{\dagger}\, u_i^*(\tilde{x})\right] \tag{5}$$

[1] The definition of positive norm modes will *not*, in general, coincide with the definition of positive frequency modes. It is the former property rather than the latter that has to be taken into account in constructing the Fock space of a quantum field.

and

$$\hat{\Phi}(\tilde{x}) = \sum_{k} \left[\hat{\bar{a}}_k\, \bar{u}_k(\tilde{x}) + \hat{\bar{a}}_k^{\dagger}\, \bar{u}_k^{*}(\tilde{x})\right]. \tag{6}$$

Using these expansions and the Bogolubov transformations (3), it can be easily shown that

$$\hat{\bar{a}}_k = \sum_{i} \left(\alpha_{ki}^{*}\, \hat{a}_i - \beta_{ki}^{*}\, \hat{a}_i^{\dagger}\right). \tag{7}$$

It is clear from this expression that the Fock spaces based on the two sets of modes $\{u_i(\tilde{x})\}$ and $\{\bar{u}_k(\tilde{x})\}$ will prove to be different whenever the Bogolubov coefficient β turns out to be non-zero. When β is non-zero, the expectation value of the number operator $\left(\hat{\bar{a}}_k^{\dagger}\hat{\bar{a}}_k\right)$ in the vacuum state annihilated by the operator $\hat{a}_i$ is given by

$$\left\langle\left(\hat{\bar{a}}_k^{\dagger}\hat{\bar{a}}_k\right)\right\rangle = \sum_{i} |\beta_{ki}|^2. \tag{8}$$

In a gravitational background, the Bogolubov transformations can either relate the modes of a quantum field at two different times in the same coordinate system or the modes in two different coordinate systems covering the same region of spacetime. When the Bogolubov coefficient β is non-zero, in the latter context, such a result is normally interpreted as implying that the quantization in the two coordinate systems are inequivalent [12]. Whereas, in the former context, a non-zero β is attributed to the production of particles by the background gravitational field [25]. Similarly, in an electromagnetic background, a non-zero β relating the modes of a quantum field at different times (in a particular gauge) implies that the background produces particles (see, for instance, Ref. [4], Sec. 2.1). Though it has been suggested in literature that inequivalent (i.e. gauge-dependent) vacua may arise in electromagnetic backgrounds as well, it has not been explicitly shown as yet (see Ref. [26]; also see Ref. [27], Sec. 4.6).

2.2. Response of the Unruh-DeWitt detector

A detector is an idealized point like object whose motion is described by a classical worldline, but which nevertheless possesses internal energy levels. Such detectors are essentially described by the interaction Lagrangian for the coupling between the degrees of freedom of the detector and the quantum field. The simplest of the different possible detectors is the detector due to Unruh and DeWitt [17, 18]. Consider a Unruh-DeWitt detector that is moving along a trajectory $\tilde{x}(\tau)$, where τ is the proper time in the frame of the detector. The interaction of the Unruh-DeWitt detector with a *real* scalar field Φ is described by the interaction Lagrangian

$$\mathcal{L}_{\rm int} = \bar{c}\,\mu(\tau)\,\Phi\left[\tilde{x}(\tau)\right], \tag{9}$$

where $\bar{c}$ is a small coupling constant and μ is the detector's monopole moment. Let us assume that the quantum field $\hat{\Phi}$ is initially in the vacuum state $|0\rangle$ and the detector is in its ground state $|\bar{E}_0\rangle$ corresponding to an energy eigen value $\bar{E}_0$. Then, up to the first order in perturbation theory, the amplitude of transition of the Unruh-DeWitt detector to an excited state $|\bar{E}\rangle$, corresponding to an energy eigen value $\bar{E}\,(>\bar{E}_0)$, is described by the integral (see, for instance, Ref. [1], Sec. 3.3)

$$\mathcal{A}(\mathcal{E}) = \mathcal{M}\int\limits_{-\infty}^{\infty} d\tau\, e^{i\mathcal{E}\tau}\,\langle\Psi|\hat{\Phi}[\tilde{x}(\tau)]|0\rangle, \tag{10}$$

where $\mathcal{M} \equiv i\bar{c}\,\langle\bar{E}|\hat{\mu}(0)|\bar{E}_0\rangle$, $\mathcal{E} = (\bar{E}-\bar{E}_0) > 0$ and $|\Psi\rangle$ is the state of the quantum scalar field after its interaction with the detector. (Since the quantity $\mathcal{M}$ depends only on the internal structure of the detector and does not depend on its motion, we shall drop this quantity hereafter.) The transition probability of the detector to all possible final states $|\Psi\rangle$ of the quantum field is given by

$$\mathcal{P}(\mathcal{E}) = \sum_{|\Psi\rangle}|\mathcal{A}(\mathcal{E})|^2 = \int\limits_{-\infty}^{\infty} d\tau \int\limits_{-\infty}^{\infty} d\tau'\, e^{-i\mathcal{E}(\tau-\tau')}\, G^{+}\left[\tilde{x}(\tau),\tilde{x}(\tau')\right], \tag{11}$$

where $G^{+}\left[\tilde{x}(\tau),\tilde{x}(\tau')\right]$ is the Wightman function defined as

$$G^{+}\left[\tilde{x}(\tau),\tilde{x}(\tau')\right] = \langle 0|\hat{\Phi}\left[\tilde{x}(\tau)\right]\,\hat{\Phi}\left[\tilde{x}(\tau')\right]|0\rangle. \tag{12}$$

For trajectories which are integral curves of timelike Killing vector fields, the Wightman function will be invariant under time translations in frame of the detector. In such a case, a transition probability rate for the detector can be defined as follows:

$$\mathcal{R}(\mathcal{E}) = \int\limits_{-\infty}^{\infty} d\Delta\tau\, e^{-i\mathcal{E}\Delta\tau}\, G^{+}(\Delta\tau), \tag{13}$$

where $\Delta\tau = (\tau-\tau')$.

2.3. The effective Lagrangian approach

The effective Lagrangian approach consists of integrating out the degrees of freedom corresponding to the quantum field thereby obtaining a correction to the Lagrangian describing the classical background [14]. The correction thus obtained, in general, has a real as well as an imaginary part to it [15, 16]. Its real part is interpreted as the 'vacuum-to-vacuum' transition amplitude, i.e. the amplitude for the quantum field to remain in the initial vacuum state at late times and the existence of a non-zero imaginary part is attributed to the instability of the vacuum. In other words, the real part part of the effective Lagrangian reflects the amount of vacuum polarization and the imaginary part is related to the number of particles produced by the classical background.

Consider the case of a *real* quantum scalar field $\hat{\Phi}$ satisfying the equation of motion (1) in a given classical background. For such a case, the correction to the Lagrangian describing the classical background is obtained by integrating the degrees of freedom corresponding to the quantum field $\hat{\Phi}$. In Schwinger's proper time formalism, the correction is given by the integral [15, 16]

$$\mathcal{L}_{\rm corr} = -\left(\frac{i}{2}\right)\int_0^\infty \frac{ds}{s}\; e^{-im^2 s}\; K(\tilde{x},\tilde{x};s), \tag{14}$$

where $K(\tilde{x},\tilde{x};s)$ is the $\tilde{x}' \to \tilde{x}$ limit of the quantity

$$K(\tilde{x},\tilde{x}';s) \equiv \langle \tilde{x}| e^{-i\hat{H}s}|\tilde{x}'\rangle. \tag{15}$$

The quantity $K(\tilde{x},\tilde{x}';s)$ is the path integral kernel of a quantum mechanical system described by the time evolution operator $\hat{H}$ and the integration variable s acts as the time parameter for the quantum mechanical system. The integral (14) yields a divergent expression even in the Minkowski vacuum in flat spacetime. Therefore, the effective Lagrangian for any nontrivial background has to be regularized by subtracting this contribution due to flat spacetime.

Schwinger's proper time formalism can also be used to evaluate the Feynman propagator. The Feynman propagator corresponding to a quantum field $\hat{\Phi}$ satisfying the equation of motion (1) is described by the following integral [15, 16]:

$$G_{\rm F}(\tilde{x},\tilde{x}') = -i\int_0^\infty ds\; e^{-i\left[m^2 s - i(\epsilon/s)\right]}\; K(\tilde{x},\tilde{x}';s), \tag{16}$$

where $\epsilon \to 0^+$ and $K(\tilde{x}, \tilde{x}'; s)$ is the quantum mechanical kernel defined in Eq. (15).

3. IN NON-INERTIAL FRAMES IN FLAT SPACETIME

Earlier, in Section 2.2, we had mentioned that if the trajectory of the Unruh-DeWitt detector is chosen to be an integral curve of a timelike Killing vector field, then the Wightman function will be invariant under translations in the proper time in the frame of the detector. We had also pointed out that in such a case we can define a transition probability rate for the detector. In Section 3.1, we shall construct integral curves of timelike Killing vector fields in flat spacetime and, as we shall see, these curves correspond to different types of non-inertial trajectories. Then, in Section 3.2, we shall go on to compare the response of Unruh-DeWitt detectors with the results from the Bogolubov transformations and the effective Lagrangian approach in coordinate systems adapted to these non-inertial trajectories.

3.1. Stationary trajectories in flat spacetime

As is well known, there are ten independent timelike Killing vector fields in flat spacetime. They correspond to three types of symmetries—translations, rotations and boosts. Different types of trajectories can be generated by choosing various linear combinations of these Killing vector fields. However, we do not gain anything by treating, say, boosts along the three different axes separately. A sufficiently general Killing vector field in flat spacetime that incorporates effects of translations, rotations and boosts can be written as [19, 20]

$$\xi^\mu(\tilde{x}) = (1 + \kappa x, \kappa t - \lambda y, \lambda x - \rho z, \rho y)\,, \tag{17}$$

where κ, λ and ρ are constants and (t, x, y, z) are the Minkowski coordinates.

Let us now consider some special cases of $\xi^\mu(\tilde{x})$ and the trajectories generated by them. The simplest of the cases is when κ, λ and ρ are all set to zero. For such a case, the Killing vector field $\xi^\mu(\tilde{x})$ reduces to

$$\xi^\mu(\tilde{x}) = (1, 0, 0, 0). \tag{18}$$

The natural coordinate systems corresponding to this Killing vector field are the rectangular Minkowski coordinates and the other curvilinear coordinates. The flat space line element in terms of the Minkowski coordinates is given by

$$ds^2 = dt^2 - d\mathbf{x}^2, \tag{19}$$

where $\mathbf{x} \equiv (x, y, z)$. Other than the inertial trajectory we have just discussed, the Killing vector field $\xi^\mu(\tilde{x})$ also generates five different types of non-inertial trajectories [19, 20]. We shall consider three of them here.

3.1.1. Uniformly accelerated motion

Let us choose $\lambda = \rho = 0$. For such a case, the Killing vector field $\xi^\mu(\tilde{x})$ reduces to

$$\xi^\mu(\tilde{x}) = (1 + \kappa x, \kappa t, 0, 0)\,. \tag{20}$$

The integral curve of such a Killing vector field is given by

$$\tilde{x}(\tau) = \kappa^{-1}\Big(\sinh(\kappa\tau), \cosh(\kappa\tau), 0, 0\Big) \tag{21}$$

which corresponds to the trajectory of a uniformly accelerated observer moving with a proper acceleration κ. A natural coordinate system for such an observer is related to the Minkowski coordinates by the following transformations:

$$t = g^{-1}\,\xi\,\sinh(g\eta)\ \ ;\ \ x = g^{-1}\,\xi\,\cosh(g\eta)\ \ ;\ \ y = y\ \ ;\ \ z = z, \tag{22}$$

where g is a constant. The new coordinates (η, ξ, y, z) are called the Rindler coordinates [28] and the proper acceleration of an observer at the point ξ in this coordinate system is (g/ξ). In terms of the Rindler coordinates, the flat spacetime line element (19) is given by

$$ds^2 = \xi^2\, d\eta^2 - g^{-2}\, d\xi^2 - dy^2 - dz^2. \tag{23}$$

3.1.2. Rotational motion

On setting $\rho = 0$ in Eq. (17), we obtain that

$$\xi^\mu(\tilde{x}) = (1 + \kappa x, \kappa t - \lambda y, \lambda x, 0)\,. \tag{24}$$

The trajectory generated by such a Killing vector field is given by

$$\tilde{x}(\tau) = \sigma^{-2}\Big(\lambda\sigma\tau, \kappa\cos(\sigma\tau), \kappa\sin(\sigma\tau), 0\Big), \tag{25}$$

where $\sigma^2 = (\lambda^2 - \kappa^2)$ and $|\kappa| < |\lambda|$. This trajectory corresponds to that of an observer moving with a linear velocity (κ/λ) along a circle of radius

(κ/σ^2). The coordinates (t, r, θ, z) of an observer rotating about the z-axis with an angular frequency Ω are related to the Minkowski coordinates by the following transformations:

$$t = t \;\; ; \;\; x = r\cos(\theta + \Omega t) \;\; ; \;\; y = r\sin(\theta + \Omega t) \;\; ; \;\; z = z. \tag{26}$$

In the rotating coordinate system, flat spacetime is described by the line element

$$ds^2 = dt^2 - dr^2 - r^2\,\left(d\theta + \Omega\, dt\right)^2 - dz^2. \tag{27}$$

3.1.3. A cusped motion

On setting $\lambda = \kappa$ and $\rho = 0$, the Killing vector field $\xi^\mu(\tilde{x})$ reduces to

$$\xi^\mu(\tilde{x}) = \left(1 + \kappa x, \kappa t - \kappa y, \kappa x, 0\right). \tag{28}$$

This Killing vector field gives rise to a peculiar cusped motion with the trajectory

$$\tilde{x}(\tau) = \left(\tau + (\kappa^2\tau^3/6), (\kappa\tau^2/2), (\kappa^2\tau^3/6), 0\right). \tag{29}$$

A natural coordinate system corresponding to an observer in motion along such a trajectory is related to the Minkowski coordinates by the following transformations:

$$\begin{aligned} t &= \left(a^2\bar{t}^3/6\right) + \left[a\bar{x} + (1/2)\right]\,\bar{t} + \bar{y} \quad ; \quad x = \left(a\bar{t}^2/2\right) + \left[\bar{x} - (a/2)\right] \\ y &= \left(a^2\bar{t}^3/6\right) + \left[a\bar{x} - (1/2)\right]\,\bar{t} + \bar{y} \quad ; \quad z = z, \end{aligned} \tag{30}$$

where a is a constant. The flat spacetime line element in terms of the new coordinates $(\bar{t}, \bar{x}, \bar{y}, z)$ is given by

$$ds^2 = 2a\,\bar{x}\,d\bar{t}^2 + 2\,d\bar{y}\,d\bar{t} - d\bar{x}^2 - dz^2. \tag{31}$$

For want of a better name, we shall hereafter refer to the coordinates $(\bar{t}, \bar{x}, \bar{y}, z)$ as the 'cusped' coordinates.

3.2. Comparison

The quantum field we shall consider in this section is a *real* and massless scalar field Φ described by the action

$$\mathcal{S}[\Phi] = \left(\frac{1}{2}\right)\int d^4x\,\sqrt{-g}\,\left(g_{\mu\nu}\,\partial^\mu\Phi\,\partial^\nu\Phi\right), \tag{32}$$

where $g_{\mu\nu}$ is the metric tensor describing the classical gravitational background. Varying this action leads to an equation of motion such as (1) with m set to zero and the operator $\hat{H}$ given by

$$\hat{H} \equiv \frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu\right). \tag{33}$$

Let us now assume that the massless quantum scalar field $\hat{\Phi}$ is in the Minkowski vacuum state. For such a case, the Wightman function (12) in terms of the Minkowski coordinates is given by the following expression (see, for e.g., Ref. [1], Sec. 3.3):

$$G^+(\tilde{x}, \tilde{x}') = \left(\frac{-1}{4\pi^2}\right) \left(\frac{1}{(t - t' - i\epsilon)^2 - |\mathbf{x} - \mathbf{x}'|^2}\right), \tag{34}$$

where, as we had mentioned earlier, $\epsilon \to 0^+$. The transition probability rate of the Unruh-DeWitt detector in the Minkowski vacuum when it is in motion along the non-inertial trajectories we had discussed in the last subsection is then obtained by substituting these trajectories in the above Wightman function and evaluating the integral (13). These transition probability rates have already been evaluated in literature [17, 18, 19, 20]. The Bogolubov coefficients relating the modes in these non-inertial coordinate systems and the Minkowski modes have been obtained in literature as well [12, 20, 21].

In what follows, we shall first evaluate the quantum mechanical kernel $K(\tilde{x}, \tilde{x}'; s)$ (as defined in Eq. (15)) corresponding to the operator $\hat{H}$ (given by Eq. (33) above) in the non-inertial coordinate systems. Substituting this kernel in Eq. (16) we shall obtain the resulting Feynman propagator. (Since evaluating the kernel and the corresponding Feynman propagator involves lengthy algebra we shall relegate the details of the calculation to the Appendix.) Then, from the coincidence limit (i.e. when $\tilde{x}' = \tilde{x}$) of the kernel, we shall evaluate the effective Lagrangian using the expression (14) and compare these results with the response of Unruh-DeWitt detectors and the results from the Bogolubov transformations. We calculate the Feynman propagator using Schwinger's proper time formalism so that it can be compared with the Wightman function (34) evaluated along the trajectory of the detector. [The boundary condition and the resulting pole structure of the Wightman function is, of course, different from that of the Feynman propagator. In general, the correct boundary condition can always be identified by comparing the pole structure in the limit of free field theory. In this limit, the Wightman function should have the term $(t - t' - i\epsilon)^2$ (cf. Eq. (34)), whereas the Feynman propagator will contain the term $\left[(t - t')^2 - i\epsilon\right]$ (cf. Eq. (A.3)).] This check is to ensure that we are evaluating the effective Lagrangian corresponding to the same conditions under which the response of the Unruh-DeWitt detectors have been studied in literature.

Before we go on to discuss the case of the non-inertial trajectories, let us very briefly discuss the inertial case. (The arguments we shall present here will prove to be useful for our discussion later on.) Consider an inertial detector stationed at a point, say, **a**. Let us now evaluate the transition amplitude (in fact, its complex conjugate) of this detector in the Minkowski vacuum. It is easy to see from Eq. (10) that it is *only* the positive norm modes of the quantum field that contribute to the resulting integral. Therefore, the transition amplitude of the detector corresponding to a single mode **k** of the field is given by

$$\begin{aligned}\mathcal{A}_{\mathbf{k}}^{*}(\mathcal{E}) &= \left(\frac{e^{i\mathbf{k}\cdot\mathbf{a}}}{\sqrt{(2\pi)^3\, 2\omega_k}}\right) \int\limits_{-\infty}^{\infty} dt\, \exp - \left[i(\omega_k + \mathcal{E})t\right] \\ &= \left(\frac{e^{i\mathbf{k}\cdot\mathbf{a}}}{\sqrt{4\pi\omega_k}}\right) \delta^{(1)}(\mathcal{E} + \omega_k), \end{aligned} \tag{35}$$

where $\omega_k = |\mathbf{k}|$. In the Minkowski coordinates, the definition of positive norm modes match the definition of positive frequency modes. Therefore, the quantity ω_k appearing in the delta function above is always greater than (or equal to) zero. Since $\mathcal{E}$ is greater than zero as well, the argument of the delta function is a positive definite quantity and hence the transition amplitude $\mathcal{A}_{\mathbf{k}}^{*}(\mathcal{E})$ above reduces to zero for *all* modes **k**. In other words, an inertial detector does not respond in the Minkowski vacuum state.

The kernel (15) corresponding to the operator $\hat{H}$ (as defined in Eq. (33)) in the Minkowski coordinates can be easily evaluated. In the coincidence limit, this kernel reduces to (cf. Eq. (A.2))

$$K(\tilde{x}, \tilde{x}; s) = \left(\frac{1}{16\pi^2 i s^2}\right) \tag{36}$$

and the corresponding effective Lagrangian is given by

$$\mathcal{L}_{\rm corr}^{0} = -\left(\frac{1}{16\pi^2}\right) \int\limits_{0}^{\infty} \frac{ds}{s^3}. \tag{37}$$

This quantity diverges near $s = 0$ and, as we had pointed out in Section 2.3, all other effective Lagrangians have to be regularized by subtracting this divergent expression.

3.2.1. In the Rindler coordinates

The Wightman function in the frame of a uniformly accelerated observer is obtained by substituting the trajectory (21) in Eq. (34). It is given by

$$G^{+}(\tilde{x}, \tilde{x}') = \left(\frac{-1}{4\pi^2}\right) \sum_{n=-\infty}^{\infty} \left(\tau - \tau' - i\epsilon + 2\pi i n \kappa^{-1}\right)^{-2}. \tag{38}$$

(This Wightman function corresponds to the case wherein the quantities η and g in the Feynman propagator (A.10) are set to (τ/ξ) and $(\kappa\xi)$, respectively.) The resulting transition probability rate can be evaluated to be [17, 18]

$$\mathcal{R}(\mathcal{E}) = \left(\frac{1}{2\pi}\right) \left(\frac{\mathcal{E}}{e^{2\pi\mathcal{E}\kappa^{-1}} - 1}\right), \tag{39}$$

which is a thermal spectrum corresponding to a temperature $T = (\kappa/2\pi)$. The Bogolubov coefficient β relating the Rindler modes and the Minkowski modes turns out to be non-zero and, in fact, the expectation value of the Rindler number operator in the Minkowski vacuum yields the above thermal spectrum as well [12]. However, on evaluating the kernel (15) in the Rindler coordinates, we find that, in the coincidence limit, it reduces to the kernel (36) in the Minkowski coordinates (cf. Eq. (A.9)). Therefore, the effective Lagrangian in the Rindler coordinates vanishes on regularization.

3.2.2. In the rotating coordinates

On substituting the trajectory (25) in Eq. (34), we find that the Wightman function along the trajectory of a rotating detector is given by

$$G^{+}(\tilde{x}, \tilde{x}') = \left(\frac{-\sigma^2}{4\pi^2}\right) \left(\frac{1}{\lambda^2(\tau - \tau' - i\epsilon)^2 - 4\kappa^2\sigma^{-2}\sin^2\left[\sigma(\tau - \tau')/2\right]}\right). \tag{40}$$

(It is easy to see that this Wightman function corresponds to the case wherein we set $t = (\lambda\tau/\sigma)$, $r = (\kappa/\sigma^2)$ and $\Omega = (\sigma^2/\lambda)$ in the Feynman propagator (A.21).) The transition probability rate of the rotating detector turns out to be non-zero, but the resulting integral cannot be expressed in a closed form. However, it has been evaluated numerically [19, 22]. On the other hand, the Bogolubov coefficient β relating the modes in the rotating frame and the Minkowski modes vanishes identically [20, 21]. Also, the kernel corresponding to the operator $\hat{H}$ in the rotating frame reduces to (36) in the coincidence limit (cf. (A.20)) which then implies that the effective Lagrangian reduces to zero in the rotating coordinates on regularization.

3.2.3. In the 'cusped' coordinates

The Wightman function in the Minkowski vacuum evaluated along the trajectory (29) is given by

$$G^{+}(\tilde{x}, \tilde{x}') = \left(\frac{-3}{\pi^2}\right)\left(\frac{1}{12(\tau-\tau'-i\epsilon)^2+\kappa^2(\tau-\tau')^4}\right). \tag{41}$$

(This Wightman function corresponds to the case wherein we choose $\bar{x} = (1/2a)$, $a = \kappa$ and $\bar{t} = \tau$ in the Feynman propagator (A.28).) On substituting this Wightman function in the integral (13), we find that the resulting transition probability rate of the detector is given by

$$\mathcal{R}(\mathcal{E}) = \left(\frac{\mathcal{E}^2}{8\sqrt{3}\pi^2\kappa^2}\right) \exp - \left(2\sqrt{3}\mathcal{E}\kappa^{-1}\right). \tag{42}$$

However, the Bogolubov coefficient β relating the modes in the 'cusped' coordinates and the Minkowski modes turns out to be zero [20, 21]. Also, it is easy to see from Eq. (A.27) that the kernel in the 'cusped' coordinates reduces to the kernel (36) in the coincidence limit. Therefore, as in the case of the Rindler and the rotating coordinates, the effective Lagrangian in the 'cusped' coordinates vanishes on regularization.

3.3. Detector response in terms of Bogolubov coefficients

It is clear from our discussion in the last section that the response of the Unruh-DeWitt detector matches the results from the Bogolubov transformations only in the case of the Rindler coordinates. In the rotating and the 'cusped' coordinate systems, the response of the detector turns out to be non-zero even when the Bogolubov coefficient β is identically zero.

In order to identify the origin of this difference, let us now write down the response of a non-inertial Unruh-DeWitt detector in terms of the Bogolubov coefficients. Let $\{u_i(\tilde{x})\}$ and $\{\bar{u}_k(\tilde{x})\}$ denote the complete set of positive norm modes corresponding to the operator $\hat{H}$ in the Minkowski and the non-inertial coordinate systems, respectively. Then, in terms of the modes $u_i(\tilde{x})$, the Wightman function (12) in the Minkowski vacuum is given by the expression

$$G^{+}\left[\tilde{x}, \tilde{x}'\right] = \sum_{i} u_i(\tilde{x})\, u_i^{*}(\tilde{x}'). \tag{43}$$

Earlier, we had obtained the Wightman function in the non-inertial frame by substituting the trajectory of the detector at the two different points $\tilde{x}(\tau)$ and $\tilde{x}(\tau')$ in the above expression. Instead, let us now express the modes $u_i(\tilde{x})$ in terms of the modes $\bar{u}_k(\tilde{x})$ in the frame of the detector using the Bogolubov transformations (3). We obtain that

$$\begin{aligned} G^{+}&\left[\tilde{x}(\tau), \tilde{x}(\tau')\right] \\ &= \sum_{i}\sum_{k}\sum_{l} \left[\alpha_{ki}^{*}\, \bar{u}_k(\tilde{x}) - \beta_{ki}\, \bar{u}_k^{*}(\tilde{x})\right] \left[\alpha_{li}\, \bar{u}_l^{*}(\tilde{x}') - \beta_{li}^{*}\, \bar{u}_l(\tilde{x}')\right] \\ &= \sum_{i}\sum_{k}\sum_{l} \biggl\{\alpha_{ki}^{*}\, \alpha_{li}\, \bar{u}_k(\tilde{x})\, \bar{u}_l^{*}(\tilde{x}') - \beta_{ki}\, \alpha_{li}\, \bar{u}_k^{*}(\tilde{x})\, \bar{u}_l^{*}(\tilde{x}') \\ &\qquad\qquad - \alpha_{ki}^{*}\, \beta_{li}^{*}\, \bar{u}_k(\tilde{x})\, \bar{u}_l(\tilde{x}') + \beta_{ki}\, \beta_{li}^{*}\, \bar{u}_k^{*}(\tilde{x})\, \bar{u}_l(\tilde{x}')\biggr\}. \end{aligned} \tag{44}$$

Since we had chosen the trajectory of the detector to be an integral curve of a timelike Killing vector field, the modes $\bar{u}_k(\tilde{x})$ can be decomposed as follows:

$$\bar{u}_k(\tilde{x}) = e^{-i\nu_k\tau}\, f_k(\bar{\mathbf{x}}), \tag{45}$$

where τ and $\bar{\mathbf{x}}$ denote the proper time and the spatial coordinates in the frame of the detector. Let us now assume that the detector is at the position $\bar{\mathbf{a}}$ in its own coordinate system. On substituting the modes (45) in the expression (44), then substituting the resulting Wightman function in Eq. (11) and finally integrating over τ and τ', we find that the transition probability of the detector is given by [22]

$$\begin{aligned} \mathcal{P}(\mathcal{E}) = (2\pi)^2 \sum_{i}\sum_{k}\sum_{l} &\biggl\{\alpha_{ki}^{*}\, \alpha_{li}\, f_k(\bar{\mathbf{a}})\, f_l^{*}(\bar{\mathbf{a}})\, \delta^{(1)}(\mathcal{E}+\nu_k)\, \delta^{(1)}(\mathcal{E}+\nu_l) \\ &- \beta_{ki}\, \alpha_{li}\, f_k^{*}(\bar{\mathbf{a}})\, f_l^{*}(\bar{\mathbf{a}})\, \delta^{(1)}(\mathcal{E}-\nu_k)\, \delta^{(1)}(\mathcal{E}+\nu_l) \\ &- \alpha_{ki}^{*}\, \beta_{li}^{*}\, f_k(\bar{\mathbf{a}})\, f_l(\bar{\mathbf{a}})\, \delta^{(1)}(\mathcal{E}+\nu_k)\, \delta^{(1)}(\mathcal{E}-\nu_l) \\ &+ \beta_{ki}\, \beta_{li}^{*}\, f_k^{*}(\bar{\mathbf{a}})\, f_l(\bar{\mathbf{a}})\, \delta^{(1)}(\mathcal{E}-\nu_k)\, \delta^{(1)}(\mathcal{E}-\nu_l)\biggr\}. \end{aligned} \tag{46}$$

Recall the fact that the modes $\bar{u}_k(\tilde{x})$ are positive *norm* modes. Let us now assume that the definition of positive norm modes match the definition of positive frequency modes in the frame of the detector for all frequencies (i.e. $\nu_k \geq 0\ \forall k$). In such a situation, only the last term in the expression above will contribute to $\mathcal{P}(\mathcal{E})$ with the result

$$\mathcal{P}(\mathcal{E}) = (2\pi)^2\, |f_{\mathcal{E}}(\bar{\mathbf{a}})|^2 \ \sum_i |\beta_{\mathcal{E}i}|^2. \tag{47}$$

Clearly, in such cases, the detector response will prove to be non-zero only when the Bogolubov coefficient β is not zero. Moreover, the detector response will actually match the expectation value of the number operator in the non-inertial frame evaluated in the Minkowski vacuum (compare Eq. (47) above with Eq. (8)). This is exactly what happens in the case of the Rindler coordinates.

On the other hand, if some of the negative frequency modes in the frame of the detector have a positive norm (i.e. $\nu_k < 0$ for some values of k), then it is easy to see from Eq. (46) that the first term can contribute to $\mathcal{P}(\mathcal{E})$ even when the Bogolubov coefficient β turns out to be zero. In such a case, the transition probability of the non-inertial detector reduces to

$$\mathcal{P}(\mathcal{E}) = (2\pi)^2\, |f_{-\mathcal{E}}(\bar{\mathbf{a}})|^2 \ \sum_i |\alpha_{-\mathcal{E}i}|^2. \tag{48}$$

It is known that there exists a range of frequencies for which negative frequency modes have a positive norm in the rotating as well as the 'cusped' coordinates [21]. It is these modes that excite the detector as a result of which the response of the Unruh-DeWitt detector along these trajectories proves to be non-zero even when the Bogolubov coefficient β is identically zero.

We had pointed out earlier that it is the norm of the modes (rather than their frequency) that has to be taken into account in decomposing the quantum field in terms of the creation and the annihilation operators. These operators in turn define the vacuum state of the field. Our discussion above points to the fact that non-trivial effects can arise in the vacuum (even in situations wherein the Bogolubov coefficient β proves to be zero) when the norm and the frequency of the modes do not match. Though such modes arise in flat spacetime due to the non-inertial motion of the observer, these modes occur even in *inertial* frames in backgrounds such as a time-independent electric field. As we shall see later, it is these modes that turn out to be responsible for exciting an inertial detector in such a background (under conditions wherein particle production is expected to occur).

4. IN THE PRESENCE OF BOUNDARIES IN FLAT SPACETIME

In this section, we shall consider the response of inertial and rotating Unruh-DeWitt detectors when boundaries are present in flat spacetime. We shall discuss two cases: (i) the response of an inertial detector in the Casimir vacuum and (ii) the response of a rotating detector when boundary conditions are imposed on the field at the horizon in the rotating frame. We shall compare the response of these detectors with the results from the effective Lagrangian approach. The system we shall consider here is a massless scalar field Φ described by the action (32).

4.1. In an inertial frame

Let us first consider the response of an inertial detector in the Casimir vacuum. Let us impose periodic boundary conditions on the quantum field $\hat{\Phi}$ along the x-axis. In other words, we shall assume that the field takes on the same value at, say, x and $(x+L)$. In such a case, the positive norm modes of the quantum field are given by

$$u_{\mathbf{k}}(t,\mathbf{x}) = \left(\frac{1}{\sqrt{(2\pi)^2\, 2\omega_k L}}\right)\, e^{-i\omega_k t}\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{49}$$

where $\omega_k = |\mathbf{k}|$, $k_x = (2n\pi/L)$ and $n = 0, \pm 1, \pm 2, \ldots$. Now, consider an inertial detector stationed at a point, say, $\mathbf{a}$. The transition amplitude $\mathcal{A}^*_{\mathbf{k}}(\mathcal{E})$ of such a detector in the Casimir vacuum is proportional to a delta function as in Eq. (35). Since $\omega_k \geq 0$ for all $\mathbf{k}$, an inertial detector does not respond in the Casimir vacuum for the same reasons an inertial detector does not respond in the Minkowski vacuum.

On the other hand, it is easy to show that the effective Lagrangian proves to be non-zero in such a situation [29]. The operator $\hat{H}$ in such a case corresponds to that of a free particle along the t, y and z directions. Whereas, along the x-direction, the eigen functions of the operator $\hat{H}$ should be assumed to take on the same value at x and $(x+L)$. Therefore, the kernel in such a case can be written as

$$K(\tilde{x},\tilde{x};s) = \left(\frac{i}{(4\pi i s)^{3/2}}\right)\, \langle x|e^{-i\hat{H}' s}|x\rangle, \quad \text{where} \quad \hat{H}' = -d_x^2. \tag{50}$$

On imposing the periodic boundary condition, the normalized eigen functions of the operator $\hat{H}'$ corresponding to an energy eigen value $E = (4n^2\pi^2/L^2)$ are given by

$$\Psi_E(x) = \left(\frac{1}{\sqrt{L}}\right) e^{(2in\pi x/L)}, \quad \text{where} \quad n = 0, \pm 1, \pm 2, \ldots . \tag{51}$$

The corresponding kernel in the coincidence limit can then be written using the Feynman-Kac formula as follows (see, for instance, Ref. [30], p. 88):

$$\langle x|e^{-i\hat{H}'s}|x\rangle = \left(\frac{1}{L}\right) \sum_{n=-\infty}^{\infty} \exp - \left(4in^2\pi^2 s/L^2\right) . \tag{52}$$

Using the Poisson sum formula, this sum can be rewritten as (cf. Ref. [31], p. 483):

$$\langle x|e^{-i\hat{H}'s}|x\rangle = \left(\frac{1}{\sqrt{4\pi is}}\right) \sum_{n=-\infty}^{\infty} \exp\left(in^2L^2/4s\right) . \tag{53}$$

Therefore, the complete kernel is given by

$$\begin{aligned} K(\tilde{x}, \tilde{x}; s) &= \left(\frac{1}{16\pi^2 is^2}\right) \sum_{n=-\infty}^{\infty} \exp\left(in^2L^2/4s\right) \\ &= \left(\frac{1}{16\pi^2 is^2}\right) \left\{1 + 2\sum_{n=1}^{\infty} \exp\left(in^2L^2/4s\right)\right\}. \end{aligned} \tag{54}$$

On substituting this kernel in Eq. (14) and subtracting the quantity $\mathcal{L}^0_{\rm corr}$, we obtain that [29]

$$\bar{\mathcal{L}}_{\rm corr} = \left(\frac{1}{\pi^2 L^4}\right) \sum_{n=1}^{\infty} n^{-4} = \left(\frac{1}{\pi^2 L^4}\right) \zeta(4) = \left(\frac{\pi^2}{90\, L^4}\right), \tag{55}$$

where we have made use of the fact that $\zeta(4) = (\pi^4/90)$ (cf. Ref. [32], p. 334). Clearly, this effective Lagrangian is a real quantity and, in fact, corresponds to the Casimir energy arising due to the boundaries (see, for e.g., Ref. [33], pp. 138–142).

4.2. In a rotating frame

In Section 3, we had found that a detector in a rotating frame responds non-trivially in the Minkowski vacuum. We had also shown that it is the negative frequency modes which have a positive norm that are responsible for exciting the rotating detector. It is easy to see from the line element (27) that the velocity of a observer stationed at a radius r greater than Ω^{-1} in the rotating frame exceeds the velocity of light. In other words, flat spacetime exhibits a horizon in the rotating frame at $r = \Omega^{-1}$. Due to this reason, it has been argued in literature that the quantum field has to be assumed to vanish on the horizon. Interestingly, imposing such a boundary condition at the horizon leads to a situation wherein there exists no negative frequency modes with a positive norm in the rotating frame and, as a result, the rotating detector ceases to respond [34].

Two important points need to be noted about this curious result. Firstly, imposing a boundary condition at the horizon alters the vacuum structure of the field and, hence, the field is not any more in the Minkowski vacuum but is in a Casimir vacuum. Secondly, we had seen earlier that the effective Lagrangian vanishes in the rotating frame. But, if we impose a boundary condition on the field at a particular radius, the effective Lagrangian for such a case would turn out to be non-zero and would, in fact, correspond to the Casimir energy of a cylinder (see, for e.g., Ref. [35]).

5. IN CLASSICAL ELECTROMAGNETIC BACKGROUNDS

The quantum field we shall consider in this section is a *complex* scalar field Φ described by the action

$$S[\Phi] = \int d^4x \left\{ (\partial_\mu \Phi + iqA_\mu \Phi) \left(\partial^\mu \Phi^* - iqA^\mu \Phi^*\right) - m^2 \Phi\Phi^* \right\}, \quad (56)$$

where A^μ is the vector potential describing the classical electromagnetic background and q and m are the charge and the mass of a single quanta of the scalar field. Varying this action leads to an equation of motion such as Eq. (1) with the operator $\hat{H}$ given by

$$\hat{H} \equiv (\partial_\mu + iqA_\mu) \left(\partial^\mu + iqA^\mu\right). \quad (57)$$

5.1. The non-linearly coupled detector

The Lagrangian (9) describes the interaction between the Unruh-DeWitt detector and a *real* scalar field. For the case of the complex scalar field we are considering here, the interaction Lagrangian (9) can be generalized to

$$\mathcal{L}_{\rm int} = \bar{c} \left(\mu(\tau)\, \Phi[\tilde{x}(\tau)] + \mu^*(\tau)\, \Phi^*[\tilde{x}(\tau)]\right). \quad (58)$$

Under a gauge transformation of the form: $A^\mu \to (A^\mu + \partial^\mu\chi)$, the complex scalar field transforms as: $\Phi \to \left(\Phi\, e^{-iq\chi}\right)$. Clearly, the interaction Lagrangian (58) will not be invariant under such a gauge transformation, unless we assume that the monopole moment transforms as follows: $\mu \to \left(\mu\, e^{iq\chi}\right)$. However, we would like to treat the detector part of the coupling, viz. the monopole moment $\mu(\tau)$, as a quantity that transforms as a scalar under gauge transformations. In such a case, the simplest of the Lagrangians that is explicitly gauge-invariant is the non-linear interaction [24]

$$\mathcal{L}_{\rm int} = \bar{c}\, \mu(\tau) \left(\Phi[\tilde{x}(\tau)]\, \Phi^*[\tilde{x}(\tau)]\right). \tag{59}$$

It is important to note that demanding gauge invariance naturally leads to non-linear interactions. A physical manifestation of gauge invariance is charge conservation. As we shall see later, the non-linear and gauge-invariant interaction (59) leads to the excitation of a particle-anti-particle pair thereby conserving charge.

In an electromagnetic background, the quantized complex scalar field $\hat{\Phi}$ can, in general, be decomposed as follows (see, for instance, Ref. [36]):

$$\hat{\Phi}(\tilde{x}) = \sum_i \left[\hat{a}_i\, u_i(\tilde{x}) + \hat{b}_i^\dagger\, v_i(\tilde{x})\right], \tag{60}$$

where $u_i(\tilde{x})$ and $v_i(\tilde{x})$ are positive and negative *norm* modes, respectively[2]. These modes are normalized with respect to the following gauge-invariant scalar product (see, for e.g., Ref. [3], p. 227)

$$(u_i, u_j) = -i \int\limits_{t=0} d^3x\, \left(u_i \left[\partial_t - iqA_t\right] u_j^* - u_j^* \left[\partial_t + iqA_t\right] u_i\right), \tag{61}$$

where A_t is the zeroth component of the vector potential A^μ. The vacuum state $|0\rangle$ of the quantum field $\hat{\Phi}$ is defined as the state that is annihilated by both the operators $\hat{a}_i$ and $\hat{b}_i$ for all i.

Let us now assume that the quantized complex scalar field $\hat{\Phi}$ is initially in the vacuum state $|0\rangle$. Then, up to the first order in perturbation theory, the amplitude of transition of the detector that is coupled to the field through

[2]The only non-trivial commutation relations satisfied by the two sets of operators $\left\{\hat{a}_i, \hat{a}_i^\dagger\right\}$ and $\left\{\hat{b}_i, \hat{b}_i^\dagger\right\}$ are: $\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}$. All other commutators vanish.

the interaction Lagrangian (59) is given by

$$\tilde{\mathcal{A}}(\mathcal{E}) = \left(\frac{1}{2}\right) \int_{-\infty}^{\infty} d\tau\, e^{i\mathcal{E}\tau}\, \langle\Psi| \biggl(\hat{\Phi}[\tilde{x}(\tau)]\, \hat{\Phi}^{\dagger}[\tilde{x}(\tau)] + \hat{\Phi}^{\dagger}[\tilde{x}(\tau)]\, \hat{\Phi}[\tilde{x}(\tau)]\biggr)|0\rangle, \quad (62)$$

where, as in the case of the Unruh-DeWitt detector, $\mathcal{E} = (\bar{E} - \bar{E}_0)$, $\bar{E}_0$ and $\bar{E}$ are the energy eigen values corresponding to the ground state and the excited state of the detector and $|\Psi\rangle$ is the state of the quantum field after its interaction with the detector. On substituting the decomposition (60) for the field $\hat{\Phi}$ in the transition amplitude (62), we obtain that

$$\tilde{\mathcal{A}}^{*}(\mathcal{E}) = \int_{-\infty}^{\infty} d\tau\, e^{-i\mathcal{E}\tau} \biggl\{\tilde{G}_1[\tilde{x}(\tau), \tilde{x}(\tau)]\, \langle 0|\Psi\rangle + \sum_i \sum_j u_i[\tilde{x}(\tau)]\, v_j^{*}[\tilde{x}(\tau)]\, \langle 0|\hat{a}_i \hat{b}_j|\Psi\rangle\biggr\}, \quad (63)$$

where $\tilde{G}_1[\tilde{x}, \tilde{x}']$ is the two-point function defined as

$$\tilde{G}_1\,[\tilde{x}, \tilde{x}'] = \left(\frac{1}{2}\right) \langle 0| \left[\hat{\Phi}(\tilde{x})\, \hat{\Phi}^{\dagger}(\tilde{x}') + \hat{\Phi}^{\dagger}(\tilde{x}')\, \hat{\Phi}(\tilde{x})\right] |0\rangle. \quad (64)$$

This two-point function can be expressed in the terms of the modes $u_i(\tilde{x})$ and $v_i(\tilde{x})$ as follows:

$$\tilde{G}_1\,[\tilde{x}, \tilde{x}'] = \left(\frac{1}{2}\right) \sum_i \left[u_i(\tilde{x})\, u_i^{*}(\tilde{x}') + v_i(\tilde{x})\, v_i^{*}(\tilde{x}')\right]. \quad (65)$$

The first term in the transition amplitude (63) contributes even when $|\Psi\rangle = |0\rangle$. But, since the two-point function $\tilde{G}_1[\tilde{x}, \tilde{x}]$ is an infinite quantity, we shall hereafter drop this term and assume that the transition amplitude $\tilde{\mathcal{A}}^{*}(\mathcal{E})$ above is given only by the second term[3]. The second term contributes only when $|\Psi\rangle = \hat{a}_i^{\dagger} \hat{b}_j^{\dagger}|0\rangle = |1_i, 1_j\rangle$. This implies that the interaction of the field with the detector leads to the excitation of a particle-anti-particle pair. Since the quantum field we are considering here is a

[3] We can formally justify this procedure by saying that we are normal ordering the creation and the annihilation operators in the matrix element in the transition amplitude (62). The divergent first term in Eq. (63) would not arise in such a case.

charged scalar field, the excitation of such a pair in essential for charge conservation. As we had pointed out above, it is the non-linear and gauge-invariant nature of the interaction Lagrangian (59) that ensures that such a pair is indeed excited.

Let us now consider the response of an inertial detector stationed at a point $\mathbf{a}$ in the Minkowski vacuum. In the absence of an electromagnetic background, the positive and negative norm modes are related as follows: $v_i(\tilde{x}) = u_i^*(\tilde{x})$. Moreover, as we have pointed out earlier, the definition of positive norm modes match the definition of positive frequency modes in the Minkowski coordinates. It is then clear from Eq. (63) that it is only the positive frequency modes $u_i(\tilde{x})$ that contribute to the transition amplitude $\tilde{\mathcal{A}}^*(\mathcal{E})$ in such a situation. Therefore, the transition amplitude of the detector corresponding to a pair of modes, say, $\mathbf{k}$ and $\mathbf{l}$ of the quantum field is given by

$$\tilde{\mathcal{A}}_{\mathbf{k},\mathbf{l}}^*(\mathcal{E}) = \left(\frac{e^{i(\mathbf{k}+\mathbf{l})\cdot\mathbf{a}}}{\sqrt{(2\pi)^4\, 4\omega_k\omega_l}}\right)\, \delta^{(1)}(\mathcal{E}+\omega_k+\omega_l), \tag{66}$$

where, for a given mode $\mathbf{k}$, $\omega_k = \left(|\mathbf{k}|^2+m^2\right)^{1/2}$. The quantities ω_k and ω_l are always $\geq m$ and, since $\mathcal{E} > 0$ as well, the argument of the delta function above is a positive definite quantity and, hence, the transition amplitude $\tilde{\mathcal{A}}_{\mathbf{k},\mathbf{l}}^*(\mathcal{E})$ reduces to zero for *all* $\mathbf{k}$ and $\mathbf{l}$. In other words, just like the Unruh-DeWitt detector, the non-linearly coupled detector does not respond in the Minkowski vacuum when in inertial motion.

The transition probability of the detector to all possible final states $|\Psi\rangle$ of the field is given by the expression

$$\tilde{\mathcal{P}}(\mathcal{E}) = \sum_{|\Psi\rangle} \left|\tilde{\mathcal{A}}(\mathcal{E})\right|^2 = \int\limits_{-\infty}^{\infty} d\tau \int\limits_{-\infty}^{\infty} d\tau'\, e^{-i\mathcal{E}(\tau-\tau')}\, \tilde{\mathcal{G}}\left[\tilde{x}(\tau),\tilde{x}(\tau')\right], \tag{67}$$

where $\tilde{\mathcal{G}}\left[\tilde{x}(\tau),\tilde{x}(\tau')\right]$ is a four-point function defined as

$$\begin{aligned}
&\tilde{\mathcal{G}}\left[\tilde{x}(\tau),\tilde{x}(\tau')\right] \\
&\quad = \left(\frac{1}{4}\right) \langle 0| \left(\hat{\Phi}[\tilde{x}(\tau)]\,\hat{\Phi}^\dagger[\tilde{x}(\tau)] + \hat{\Phi}^\dagger[\tilde{x}(\tau)]\,\hat{\Phi}[\tilde{x}(\tau)]\right) \\
&\qquad\qquad \times \left(\hat{\Phi}[\tilde{x}(\tau')]\,\hat{\Phi}^\dagger[\tilde{x}(\tau')] + \hat{\Phi}^\dagger[\tilde{x}(\tau')]\,\hat{\Phi}[\tilde{x}(\tau')]\right) |0\rangle.
\end{aligned} \tag{68}$$

Using the decomposition (60), this four-point function can be expressed as follows:

$$\tilde{\mathcal{G}}[\tilde{x},\tilde{x}'] = \tilde{G}_1[\tilde{x},\tilde{x}]\,\tilde{G}_1[\tilde{x}',\tilde{x}'] + \sum_i \left[u_i(\tilde{x})\,u_i^*(\tilde{x}')\right]\,\sum_j \left[v_j^*(\tilde{x})\,v_j(\tilde{x}')\right]. \quad (69)$$

The first term in this expression is a product of two two-point functions evaluated at the *same* spacetime point and hence is an infinite quantity[4]. Therefore, we shall drop this term and assume that the four-point function $\tilde{\mathcal{G}}[\tilde{x},\tilde{x}']$ above is given only by the second term.

We had pointed out above that, in the absence of an electromagnetic background, the positive and the negative norm modes are related by the following expression: $v_i(\tilde{x}) = u_i^*(\tilde{x})$. It is then useful to note that, in such a case, the second term in the four-point function $\tilde{\mathcal{G}}[\tilde{x},\tilde{x}']$ above will be given by the square of the Wightman function in the Minkowski vacuum. Therefore, when in inertial motion, the transition probability rate of the non-linearly coupled detector in the Minkowski vacuum would be identically zero (for exactly the same reasons) as it is in the case of the Unruh-DeWitt detector.

In the following three sections, we shall study the response of the non-linearly coupled detector in: (i) a time-dependent electric field, (ii) a time-independent electric field and (iii) a time-independent magnetic field, backgrounds. We had seen earlier that detectors on non-inertial trajectories respond non-trivially even in the Minkowski vacuum. Therefore, in order to avoid the effects due to non-inertial motion and to isolate the effects that arise due to the electromagnetic background, we shall restrict our attention to *inertial* trajectories here. We shall compare the response of the inertial detector with the results expected from the Bogolubov transformations and the effective Lagrangian approach.

5.2. In time-dependent electric field backgrounds

A time-dependent electric field background can be described by following vector potential:

$$A^\mu = (0, A(t), 0, 0), \quad (70)$$

where $A(t)$ is an arbitrary function of t. This vector potential gives rise to the electric field $\mathbf{E} = -(dA/dt)\,\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the unit vector along the

[4]It should be pointed out here that this term would not arise had we normal ordered the creation and the annihilation operators in the matrix element in the transition amplitude (62).

positive x-direction. The modes of a quantum field evolving in such a time-dependent electric field background can be decomposed as

$$u_{\mathbf{k}}(t, \mathbf{x}) = g_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{71}$$

In general, modes at early and late times will be related by a non-zero Bogolubov coefficient β and the expectation value of the number operator (corresponding to a given mode of the quantum field) at late times in the in-vacuum will be given by Eq. (8) (see, for e.g., Ref. [4], Sec. 2.1).

Now, consider a detector that is stationed at a particular point. Along the world line of such a detector, the second term in four-point function (69) corresponding to the modes (71) is given by

$$\tilde{\mathcal{G}}(t, t') = \sum_{\mathbf{k}} \sum_{\mathbf{l}} \left[g_{\mathbf{k}}(t)\, g_{\mathbf{l}}(t)\, g_{\mathbf{k}}^{*}(t')\, g_{\mathbf{l}}^{*}(t')\right] \tag{72}$$

and the transition probability of the detector reduces to

$$\tilde{\mathcal{P}}(\mathcal{E}) = \sum_{\mathbf{k}} \sum_{\mathbf{l}} |g_{\mathbf{kl}}(\mathcal{E})|^2, \tag{73}$$

where

$$g_{\mathbf{kl}}(\mathcal{E}) = \int_{-\infty}^{\infty} dt\, e^{-i\mathcal{E}t}\, \left[g_{\mathbf{k}}(t)\, g_{\mathbf{l}}(t)\right]. \tag{74}$$

Clearly, the response of the inertial detector will, in general, be non-zero.

Let us now assume that: (i) the function $A(t)$ behaves such that the electric field vanishes in the past and future infinity, (ii) the detector is switched on for a finite time interval in the future asymptotic domain and (iii) the effects that arise due to switching [37, 38, 39] can be neglected. Then, by relating the modes at future and past infinity using the Bogolubov transformations, we can express the response of the detector (in the in-vacuum) in terms of the Bogolubov coefficients (as we have done in Section 3.3). We find that the transition probability of the detector is given by [24]:

$$\begin{aligned}\tilde{\mathcal{P}}(\mathcal{E}) \;=\; (2\pi)^2 \sum_{\mathbf{k}} \sum_{\mathbf{l}} \Big(& |\alpha_{\mathbf{k}}|^2\, |\beta_{\mathbf{l}}|^2\, \delta^{(2)}\left(\mathcal{E} + \omega_-\right) \\ & + 2\, \left[\mathrm{Re.}(\alpha_{\mathbf{k}}\alpha_{\mathbf{l}}^{*}\beta_{\mathbf{k}}^{*}\beta_{\mathbf{l}})\right]\, \delta^{(1)}\left(\mathcal{E} + \omega_-\right)\, \delta^{(1)}\left(\mathcal{E} - \omega_-\right) \\ & + 4\, |\beta_{\mathbf{k}}|^2\, \left[\mathrm{Re.}(\alpha_{\mathbf{l}}\beta_{\mathbf{l}}^{*})\right]\, \delta^{(1)}\left(\mathcal{E} - \omega_+\right)\, \delta^{(1)}\left(\mathcal{E} - \omega_-\right)\end{aligned}$$

$$+ |\beta_{\mathbf{k}}|^2 \, |\alpha_{\mathbf{l}}|^2 \, \delta^{(2)} \left(\mathcal{E} - \omega_-\right)$$
$$+ |\beta_{\mathbf{k}}|^2 \, |\beta_{\mathbf{l}}|^2 \, \delta^{(2)} \left(\mathcal{E} - \omega_+\right) \Bigg), \tag{75}$$

where $\omega_\pm = (\omega_{\mathbf{k}} \pm \omega_{\mathbf{l}})$, with $\omega_{\mathbf{k}}$ and $\omega_{\mathbf{l}}$ being the positive definite (in fact $\geq m$) frequencies corresponding to the modes $\mathbf{k}$ and $\mathbf{l}$ in the out-region. Clearly, the detector responds *only* when the Bogolubov coefficient β is non-zero (i.e. only when particle production takes place). However, it is evident that the transition probability rate of detector we have obtained above is *not* proportional to the number of particles produced by the time-dependent electric field background.

The feature that the response of the detector does not turn out to be proportional to the number of particles produced by the background should not come as a surprise and, in fact, it can be attributed to the non-linearity of the interaction Lagrangian (59) for the following two reasons. Firstly, it can be easily shown that in a time-dependent gravitational background with asymptotically static domains, the response of the Unruh-DeWitt detector in the out-region will be given by an expression such as Eq. (47). In other words, the response of the Unruh-DeWitt detector in such a situation will be proportional to the number of particles produced by the background (cf. Ref. [1], pp. 57-59). Secondly, it is known that the response of a detector that is coupled to the stress-energy tensor of the quantum field (which is evidently a non-linear interaction) does *not* reflect the particle content of the field [40]. As we have discussed earlier, demanding gauge invariance naturally leads to non-linear interaction Lagrangians. Therefore, quite generically, we can expect that the response of detectors in classical electromagnetic backgrounds will *not* be proportional to the amount of particles produced by the background.

The imaginary part of the effective Lagrangian for a time-dependent electric field background is, in general, expected to be non-zero implying that such backgrounds always produce particles. However, it should be added that evaluating the effective Lagrangian for an arbitrary time-dependent electric field proves to be a difficult task and the effective Lagrangian has been obtained in a closed form only in a few cases (for efforts on evaluating the effective Lagrangian for non-trivial backgrounds, see Ref. [41] and references therein).

5.3. In time-independent electric field backgrounds

Consider the vector potential

$$A^{\mu} = (A(x), 0, 0, 0), \tag{76}$$

where $A(x)$ is an arbitrary function of x. Such a vector potential gives rise to a time-independent electric field along the x-direction given by $\mathbf{E} = -(dA/dx)\,\hat{\mathbf{x}}$. In such a case, the modes of the quantum field $\hat{\Phi}$ can be decomposed as follows:

$$u_{\omega\mathbf{k}_\perp}(t, \mathbf{x}) = e^{-i\omega t}\, f_{\omega\mathbf{k}_\perp}(x)\, e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}, \tag{77}$$

where $\mathbf{k}_\perp$ is the wave vector along the perpendicular direction. Due to lack of time dependence, the Bogolubov coefficient β relating these modes at two different times is trivially zero. Though the Bogolubov coefficient β is zero, particle production takes place in such backgrounds due to a totally different phenomenon. It is well known that if the depth of the potential $[qA(x)]$ is greater than $(2m)$, then the corresponding electric field will produce particles due to Klein paradox (see Ref. [36] and references therein; also see Ref. [42] for a recent discussion). It is then interesting to examine whether an inertial detector in a time-independent electric field background will respond under the same condition.

Consider a detector that is stationed at a particular point. It is easy to see from the form of the modes (77) that the transition amplitude $\tilde{\mathcal{A}}^*_{\mathbf{k},\mathbf{l}}(\mathcal{E})$ of such a detector will be proportional to a delta function as in the case of an inertial detector in the Minkowski vacuum (cf. Eq. (66)). But, unlike the Minkowski case wherein the definition of positive frequency modes match the definition of positive norm modes, in a time-independent electric field background, there exist negative frequency modes which have a positive norm whenever the depth of the potential $[qA(x)]$ is greater than $(2m)$. In other words, when Klein paradox occurs in an electric field background, ω_k and ω_l appearing in the argument of the delta function in Eq. (66) can be negative and, hence, there exists a range of values of these two quantities for which this argument can be zero. These modes excite the detector as a result of which the response of an inertial detector proves to be non-zero in such a background.

We shall now show (for the special case of the step potential) as to how there exist negative frequency modes which have a positive norm when the depth of the potential $[qA(x)]$ is greater than $(2m)$. In order to show that, let us evaluate the norm of the mode $u_{\omega\mathbf{k}_\perp}(t, \mathbf{x})$. On substituting the mode (77) and the vector potential (76) in the scalar product (61), we obtain that

$$(u_{\omega\mathbf{k}_\perp}, u_{\omega\mathbf{k}_\perp}) = 2\,(2\pi)^2\,\delta^{(2)}(0)\int\limits_{-\infty}^{\infty} dx\;[\omega - qA(x)]\;|f_{\omega\mathbf{k}_\perp}(x)|^2. \quad (78)$$

Let us now assume that $A(x) = -\,(\Theta(x)\,V)$, where $\Theta(x)$ is the step-function and V is a constant. For such a case, the function $f_{\omega\mathbf{k}_\perp}$ is given by

$$f_{\omega\mathbf{k}_\perp}(x) = \Theta(-x)\left(e^{ik_L x} + R_{\omega\mathbf{k}_\perp}\,e^{-ik_L x}\right) + \Theta(x)\,T_{\omega\mathbf{k}_\perp}\,e^{ik_R x}, \quad (79)$$

where

$$k_R = \left[(\omega + qV)^2 - |\mathbf{k}_\perp|^2 - m^2\right]^{1/2} \quad \text{and} \quad k_L = \left[\omega^2 - |\mathbf{k}_\perp|^2 - m^2\right]^{1/2}. \quad (80)$$

The quantities $R_{\omega\mathbf{k}_\perp}$ and $T_{\omega\mathbf{k}_\perp}$ are the usual reflection and tunneling amplitudes. They are given by the expressions

$$R_{\omega\mathbf{k}_\perp} = \left(\frac{k_L - k_R}{k_L + k_R}\right) \quad \text{and} \quad T_{\omega\mathbf{k}_\perp} = \left(\frac{2k_L}{k_L + k_R}\right). \quad (81)$$

If we now assume that k_R and k_L are real quantities, then it is easy to show that, for the case of the step potential we are considering here, the scalar product (78) is given by

$$(u_{\omega\mathbf{k}_\perp}, u_{\omega\mathbf{k}_\perp}) = (2\pi)^3\,\delta^{(3)}(0)\,\left[\omega\left(1 + R^2_{\omega\mathbf{k}_\perp}\right) + (\omega + qV)\,T^2_{\omega\mathbf{k}_\perp}\right]. \quad (82)$$

Let us now set $\mathbf{k}_\perp = 0$. Also, let us assume that $\omega = -(m + \varepsilon)$ and $(qV) = (2m + \varepsilon)$, where ε is a positive definite quantity. For such a case, $R_{\omega 0} = 1$, $T_{\omega 0} = 2$ and the scalar product (82) reduces to

$$(u_{\omega 0}, u_{\omega 0}) = 2\,(m - \varepsilon)\;(2\pi)^3\;\delta^{(3)}(0) \quad (83)$$

which is a positive definite quantity if we choose ε to be smaller than m. We have thus shown that there exist negative frequency modes (i.e. modes with $\omega \le -m$) which have a positive norm. Moreover, this occurs only when (qV) is greater than $(2m)$ (note that $(qV) = (2m + \varepsilon)$) which is exactly the condition under which Klein paradox is expected to arise. As we had discussed in the last paragraph, it is this feature of the Klein paradox that is responsible for exciting the detector.

As in the case of a time-dependent electric field background, evaluating the effective Lagrangian for an arbitrary time-independent electric field proves to be a difficult task and the effective Lagrangian in such cases has been evaluated only for a few specific examples. We had pointed out above that a time-independent electric field is expected to produce particles only if the depth of the potential $[qA(x)]$ is greater than $(2m)$. It will be a worthwhile exercise to show that the effective Lagrangian has an imaginary part only under such a condition.

5.4. In time-independent magnetic field backgrounds

A time-independent magnetic field background can be described by the vector potential

$$A^{\mu} = (0, 0, A(x), 0), \tag{84}$$

where $A(x)$ is an arbitrary function of x. This vector potential gives rise to the magnetic field $\mathbf{B} = (dA/dx)\,\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector along the positive z-axis. The modes of the quantum scalar field $\hat{\Phi}$ in such a background can be decomposed exactly as we did in Eq. (77) in the case of the time-independent electric field background. Hence, the transition amplitude $\tilde{\mathcal{A}}^{*}_{\mathbf{k},\mathbf{l}}(\mathcal{E})$ of an inertial detector in a time-independent magnetic field background will also be proportional to a delta function as in Eq. (66). However, on substituting the mode (77) and the vector potential (84) in the scalar product (61), we find that

$$(u_{\omega\mathbf{k}_\perp}, u_{\omega\mathbf{k}_\perp}) = (2\omega)\ (2\pi)^2\, \delta^{(2)}(0) \int\limits_{-\infty}^{\infty} dx\, |f_{\omega\mathbf{k}_\perp}(x)|^2, \tag{85}$$

which is clearly a positive definite quantity whenever $\omega \geq m$. In other words, unlike the case of the time-independent electric field background, in a time-independent magnetic field background, the definition of positive frequency modes *always* matches the definition of positive norm modes. Therefore, as in the case of an inertial detector in the Minkowski vacuum, an inertial detector will not respond in the vacuum state in a time-independent magnetic field background.

Let us now try to evaluate the effective Lagrangian for an arbitrary time-independent magnetic field background [43]. The operator $\hat{H}$ corresponding to the vector potential (84) is given by

$$\hat{H} \equiv {\partial_t}^2 - \nabla^2 + 2iqA(x)\,\partial_y + q^2 A^2(x). \tag{86}$$

Using the translational invariance of the operator $\hat{H}$ along the time coordinate t and the spatial coordinates y and z, the kernel corresponding to this operator can be written as

$$K(\tilde{x}, \tilde{x}, s) = \left(\frac{1}{4\pi s}\right) \int_{-\infty}^{\infty} \frac{dp_y}{2\pi} \, \langle x | e^{-i\hat{H}' s} | x \rangle, \tag{87}$$

where

$$\hat{H}' \equiv -d_x^2 + [p_y - qA(x)]^2 \,. \tag{88}$$

The quantity $\langle x | e^{-i\hat{H}' s} | x \rangle$ can now expressed using the Feynman-Kac formula as follows (see, for e.g., Ref. [30], p. 88):

$$\langle x | e^{-i\hat{H}' s} | x \rangle = \sum_E |\Psi_E(x)|^2 \, e^{-iEs}, \quad \text{where} \quad \hat{H}' \Psi_E = E \Psi_E, \tag{89}$$

so that $K(\tilde{x}, \tilde{x}, s)$ is given by

$$K(\tilde{x}, \tilde{x}, s) = \left(\frac{1}{4\pi s}\right) \int_{-\infty}^{\infty} \frac{dp_y}{2\pi} \sum_E |\Psi_E(x)|^2 \, e^{-iEs}. \tag{90}$$

(It is assumed here that the summation over E stands for integration over the relevant range when E varies continuously.) Since the potential term, viz. $[p_y - qA(x)]^2$, in the operator $\hat{H}'$ above is a positive definite quantity, the eigen value E can only lie in the range $(0, \infty)$. Substituting the above expression for $K(\tilde{x}, \tilde{x}, s)$ in Eq. (14), we find that $\mathcal{L}_{\rm corr}$ is given by

$$\mathcal{L}_{\rm corr} = -\left(\frac{i}{4\pi}\right) \int_{-\infty}^{\infty} \frac{dp_y}{2\pi} \sum_E |\Psi_E(x)|^2 \int_0^\infty \frac{ds}{s^2} \, e^{-i(m^2+E)s}. \tag{91}$$

(It should be noted here that for the case of the complex scalar field we are considering here, $\mathcal{L}_{\rm corr}$ is, in fact, twice the quantity defined in Eq. (14).) On carrying out the integral over s, we finally obtain that

$$\mathcal{L}_{\rm corr} = \left(\frac{1}{4\pi}\right) \int_{-\infty}^{\infty} \frac{dp_y}{2\pi} \sum_E |\Psi_E(x)|^2 \, (m^2 + E) \, \left[\ln(m^2 + E) - 1\right]. \tag{92}$$

Since $(m^2 + E) > 0$, it is easy to see from this expression that $\mathcal{L}_{\rm corr}$ is a real quantity. Though we are unable to express the effective Lagrangian for an

TABLE 1.

Comparison

	Bogolubov coefficient β	Detector response $\mathcal{P}(\mathcal{E})$	Effective Lagrangian Re. $\mathcal{L}_{\rm corr}$	Effective Lagrangian Im. $\mathcal{L}_{\rm corr}$
In inertial coordinates	0	0	0	0
In Rindler coordinates	$\neq 0$	$\neq 0$	0	0
In rotating coordinates	0	$\neq 0$	0	0
In 'cusped' coordinates	0	$\neq 0$	0	0
Between Casimir plates	0	0	$\neq 0$	0
In a time-dependent electric field	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$
In a time-independent electric field	$\neq 0$ [a]	$\neq 0$	$\neq 0$	$\neq 0$
In a time-independent magnetic field	0	0	$\neq 0$	0

[a] Actually, the Bogolubov coefficient β is trivially zero in a time-independent electric field background. We refer here to particle production that can occur in such a background due to Klein paradox (see Section 5.3).

arbitrary time-independent magnetic field in a closed form, we have been able to show that it does not have an imaginary part which then implies that such a background will not produce particles.

6. DISCUSSION

In this concluding section, we shall first briefly summarize the results of the analysis we have carried out in this paper and then go on to discuss the implications of our analysis for classical gravitational backgrounds.

6.1. What do detectors detect?

In order to clearly illustrate the conclusions we wish to draw from our analysis, we have tabulated the results we have obtained in the last three sections in Table 1.

To begin with, we would like to emphasize the point we had discussed in detail earlier, viz. that the response of a detector can be non-zero even when the Bogolubov coefficient β is zero. The cases of the rotating detector and that of the detector in motion along the 'cusped' trajectory clearly support this statement (see rows three and four, columns one and two in Table 1). Also, it is important to note that the detector response can be non-zero even when the effective Lagrangian vanishes identically—evidently, all the non-inertial cases support this point (cf. rows and columns two, three and four). It should be stressed here that this is true even in case of the Rindler coordinates, a non-inertial frame in which the Bogolubov coefficient β proves to be non-zero. Clearly, a non-zero response of a detector does not necessarily imply particle production.

Having said that, it is important to note that irrespective of its motion the response of a detector will be non-zero whenever there is particle production taking place. In that sense a detector *is* sensitive to particle production. Moreover, if we restrict the motion of the detector to inertial trajectories, then we can avoid the non-inertial effects and, in such cases, the detector response will be non-zero *only* when particle production takes place. The fact that an inertial detector does not respond either in the Casimir vacuum or in a time-independent magnetic field (wherein the effective Lagrangian had no imaginary part, cf. rows five and eight, columns two and four); whereas such a detector responds non-trivially both in time-dependent as well as time-independent electric fields (wherein the imaginary part of the effective Lagrangian is, in general, expected to be non-zero, cf. rows six and seven, columns two and four) support this point. However, as the case of the time-dependent electric field background suggests, the response of an inertial detector will not necessarily be proportional to the number of particles produced by the background.

6.2. Implications for classical gravitational backgrounds

Unlike in flat spacetime or classical electromagnetic backgrounds, there exists no special frame of reference in a classical gravitational background and all coordinate systems have to be treated equivalently. This feature severely restricts the utility of a detector to study the phenomenon of particle production in a classical gravitational background. Until now, we had discussed as to how the response of a detector compares with the results from the Bogolubov transformations and the effective Lagrangian approach. In what follows, we shall attempt to understand as to how the effective Lagrangian would behave under arbitrary coordinate transformations.

Consider a massless and real quantum scalar field evolving in a gravitational background described by the metric tensor $g_{\mu\nu}$. This scalar field will satisfy an equation of motion such as Eq. (1) with m set to zero and the operator $\hat{H}$ given by Eq. (33). The quantity $\mathcal{L}_{\rm corr}$ obtained by integrating out the degrees of freedom of the quantum scalar field can then be expressed in terms of the determinant of the operator $\hat{H}$ (see, for e.g., Ref. [16]). The determinant of the operator $\hat{H}$ can in turn be expressed as a product of its eigen values, say, λ_i, where these eigen values are obtained by solving the differential equation $\hat{H} w_i = \lambda_i w_i$ with respect to a complete set of modes $\{w_i(\tilde{x})\}$. Let us now perform a coordinate transformation on the metric tensor $g_{\mu\nu}$. Let the operator and its eigen values in the new coordinate system be $\bar{\hat{H}}$ and $\bar{\lambda}_i$, where the eigen values are now obtained by solving the eigen value equation $\bar{\hat{H}} \bar{w}_i = \bar{\lambda}_i \bar{w}_i$ with respect to a new set of modes $\{\bar{w}_i(\tilde{x})\}$. If we now assume that the new modes $\bar{w}_i$ are obtained from the old ones (viz. w_i) by explicitly substituting the corresponding coordinate transformation, then it is easy to show that the eigen values λ_i will remain unchanged (i.e. $\bar{\lambda}_i = \lambda_i$). In such a case, the effective Lagrangian will remain invariant under coordinate transformations and will therefore behave as a scalar quantity.

Though the effective Lagrangian thus obtained will be invariant under coordinate transformations, it will be a divergent quantity and we will need to regularize this expression. Now, a complete set of modes can be used to evaluate the kernel and, the kernel in turn, can be used to obtain the corresponding Green function. Therefore, choosing to work with a particular set of modes $\{w_i(\tilde{x})\}$ (from which all the other sets $\{\bar{w}_i(\tilde{x})\}$ are obtained by explicitly substituting the coordinate transformations) corresponds to choosing a particular vacuum state for the quantum field. In flat spacetime, divergent expressions are always regularized by subtracting the contribution due to the Minkowski vacuum. So, if we choose to work with those set of modes that lead to the Green function in the Minkowski vacuum, then the regularized effective Lagrangian will be trivially zero in all coordinates in flat spacetime. In fact, this is exactly what we have found from our analysis. We found that the kernel that leads to the Green function in the Minkowski vacuum is invariant under coordinate transformations in the coincident limit (cf. Eqs. (A.2), (A.9), (A.20), (A.27)) and, hence, the corresponding effective Lagrangian identically reduced to zero in all the non-inertial coordinates on regularization.

However, these arguments do not still imply that the effective Lagrangian will be unique in a given gravitational background. Instead of choosing to work with modes that led to the Green function in the Minkowski vacuum, we could have chosen to work with modes that lead to the Green function in the Rindler vacuum. If we now use these modes to evaluate the kernel, then the effective Lagrangian corresponding to this kernel will be different from the effective Lagrangian that corresponds to the Minkowski vacuum and, hence, will lead to a non-zero value on regularization. In fact, when the contribution due to the Minkowski vacuum is subtracted from the effective

Lagrangian in the Rindler vacuum, one obtains a non-zero and real quantity which has a thermal nature (see Ref. [44]; also see Ref. [45] in this context).

This result can be stated in a more formal and general terms along the following lines, which will prove to be useful. We notice that the essential physics of a free field theory is contained in the two-point function $G(\tilde{x}, \tilde{x}') = \langle 0|\mathrm{T}\,[\Phi(\tilde{x})\,\Phi(\tilde{x}')]\,|0\rangle$ (where T denotes time-ordering), which satisfies an inhomogeneous differential equation. But the same differential equation will be satisfied by a function $\mathcal{F}(\tilde{x}, \tilde{x}') = \langle\bar{\Psi}|\mathrm{T}\,[\Phi(\tilde{x})\,\Phi(\tilde{x}')]\,|\bar{\Psi}\rangle$ defined with respect to any normalizable quantum state $|\bar{\Psi}\rangle$. In particular, if there exist two different vacuum states, then the corresponding two-point functions will differ, i.e. they will not be related by a coordinate relabelling appropriate for a biscalar (which is precisely what happens in the case of Rindler and Minkowski vacuum states). But, since the functions $G(\tilde{x}, \tilde{x}')$ and $\mathcal{F}(\tilde{x}, \tilde{x}')$ satisfy the same inhomogeneous differential equation, they will, in general, differ by a solution to the homogeneous equation. Alternatively, they will differ by the boundary conditions both at the asymptotic regimes as well as near horizons that the spacetime may contain. (One popular way of choosing the boundary condition in standard quantum field theory is through Euclidean continuation which, of course, will not work in a general curved spacetime.)

The above discussion highlights the key difficulty: unless external criteria are imposed to choose a particular boundary condition, the class of functions $\mathcal{F}(\tilde{x}, \tilde{x}')$ are valid two-point functions of the theory, a priori. In order to choose one (or a few) of them as special, we need to study their general behavior and impose some boundary conditions. In fact, not all of them will lead to quantum field theories which are unitarily equivalent. It has been shown in literature that there exists no unitary transformation relating the Fock space constructed from the Minkowski vacuum and the Fock space determined by the Rindler vacuum [46]. (Evidently, it is this feature that leads to the inequivalent quantization and, as a result, the non-zero effective Lagrangian in the Rindler vacuum.) This result points to the fact that in an arbitrary gravitational background not all coordinate transformations can be implemented unitarily. This implies that, in general, there exist families of inequivalent Fock spaces in a curved spacetime (see, for e.g., Ref. [47]). In the case of flat spacetime, the Fock space associated with the Minkowski vacuum provides us with a natural basis. But, no such special Fock space seems to be available to us in a curved spacetime. In such a situation, which of the inequivalent Fock spaces should we choose to work with? Will we be able to choose one of these Fock spaces on our own or will it be chosen automatically when we set up an experiment?

An important aspect of the modes associated with the Minkowski coordinates in flat spacetime are that they are well defined and regular in the entire region of the spacetime. Recently, it has been argued that this feature should be utilized to propose a possible criterion for selecting a particular vacuum state (and the associated Fock space) from amongst the different possibilities in a curved spacetime [48]. The requirement that the modes be regular *throughout* the spacetime (so that the states can evolve from data in the infinite past) has been suggested as the physical criterion to distinguish between the different states. (This criterion immediately picks out the Minkowski vacuum state in flat spacetime as other states such as the Rindler vacuum lead to divergences on the horizons.)

It is, however, not clear whether this condition may turn out to be overly restrictive. In spacetimes with horizons, two-point functions for different vacua will have different—and sometimes singular—behavior at the horizon. For example, the Minkowski coordinates of flat spacetime is similar to the Kruskal coordinates of Schwarzschild spacetime; the analogue of the Minkowski vacuum in the Schwarzschild spacetime will be the Hartle-Hawking vacuum. We, however, have physical situations which are best described with respect to the Unruh vacuum (corresponding to a collapsing star) or even the Boulware vacuum (around a static star) in the Schwarzschild spacetime. It is probably better to classify the boundary conditions and try to identify a class of vacuum states rather than impose regularity throughout the spacetime.

A closely related issue is the distinction between a change of reference frame and coordinate relabelling. If one deals with tensorial quantities, coordinate relabelling does not lead to any new physical effects. One can certainly use the Minkowski modes in the Rindler frame (after expressing the Minkowski coordinates in the modes in terms of the Rindler coordinates) to define the Minkowski vacuum state and carry out quantum field theory in the Rindler frame. The results will be completely equivalent to field theory in the Minkowski coordinates—albeit expressed in a strange language. When one uses the terminology "change of reference frame" one has something different in mind—though its exact definition is difficult to express in general terms. In simple contexts like the Minkowski and Rindler coordinates, one implies changing over the description to a language which is *natural* to the coordinates that have been chosen (such as, for e.g., choosing to work with positive norm modes defined with respect to the new time coordinate). There is an important distinction which arises between the electromagnetic and gravitational fields in this context, which we shall now briefly describe.

Let us consider a laboratory experiment in which a pair of parallel capacitor plates are set up with a given potential difference between them, corresponding to an electric field. If the field is strong enough, we should see pair production between the plates. The pairs produced will move towards the plates and will try to reduce the charge densities on the plates thereby reducing the strength of the electric field between them. To maintain the original strength of the electric field, the external source has to do work which will supply the energy of the particles produced by the electric field. Note that, nowhere in this description did we need to specify the gauge used to describe the electric field, even though to set up the quantum field theory and obtain the pair production rate in the electric field, one might choose to work in a specific gauge. The key reason for this result—which is not often emphasized—is that the source of electromagnetic field, viz. the electric current J^μ, is *gauge-invariant rather than merely being gauge-covariant*; i.e. J^μ is a "scalar" under gauge transformations (unlike, for example, the charged scalar field, which is only gauge-covariant and picks up a phase factor under a gauge transformation). We can therefore specify the experimental set up in terms of charges and currents and ask what will happen in the laboratory without ever concerning ourselves about the gauge.

The situation is quite different in the case of gravity. The analogue of the gauge transformation in gravity is the coordinate transformation. But, the source of the classical gravitational field, viz. the stress-energy tensor $T^{\alpha\beta}$, is only a covariant quantity rather than an invariant one. This feature, of course, makes no difference in classical theory. We may choose any coordinate system we like to specify the components of the stress-energy tensor and solve the Einstein's equations to obtain the metric; if we change the coordinates, then, both the stress-energy tensor and the metric will change suitably, maintaining general covariance at the classical level. The situation is different in quantum theory where the vacuum state, for instance, can be different based on the modes which are chosen. Since, different sets of modes may be natural for different coordinate frames—corresponding to different metric and stress-energy tensor components—we cannot phrase questions in the case of gravity in a manner similar to the electromagnetic case (unless one could reformulate Einstein's equations entirely in terms of scalar invariants which seems to be an impossible task).

Thus, we come to the inevitable conclusion that an extra prescription—say, in terms of the boundary conditions on the two-point function—is required in an arbitrary spacetime to define and deal with issues such as particle production. We conjecture that in spacetimes without horizons, this could be achieved with asymptotic boundary conditions, whereas in spacetimes with horizons, we may also need to specify the behavior of the modes on the horizon as well. In fact, it should be possible to arrive at some general conclusions regarding the behavior of two-point functions in arbitrary spacetimes along these lines. We hope to address these issues further in a future publication.

APPENDIX: EVALUATING THE FEYNMAN PROPAGATOR

In this appendix, we shall evaluate the Feynman propagator for the case of a massless scalar field in the three non-inertial coordinate systems we had discussed in Section 3.

Before we go on to evaluate the Feynman propagator in the non-inertial coordinate systems, let us first consider the case of the Minkowski coordinates. In these coordinates the operator $\hat{H}$ as defined in Eq. (33) is given by

$$\hat{H} \equiv \left(\partial_t^2 - \nabla^2\right). \tag{A.1}$$

This is the time evolution operator of a free quantum mechanical particle and the kernel (15) corresponding to such an operator is given by (see, for e.g., Ref. [30], p. 42)

$$K(\tilde{x}, \tilde{x}'; s) = \left(\frac{1}{16\pi^2 i s^2}\right) \exp - \left(\frac{i}{4s}\right) \left[(t-t')^2 - |\mathbf{x} - \mathbf{x}'|^2\right]. \tag{A.2}$$

On substituting this kernel in Eq. (16) and evaluating the resulting integral, we find that the Feynman propagator in the Minkowski coordinates is given by

$$G_{\rm F}(\tilde{x}, \tilde{x}') = \left(\frac{i}{4\pi^2}\right) \left(\frac{1}{[(t-t')^2 - i\epsilon] - |\mathbf{x} - \mathbf{x}'|^2}\right) \tag{A.3}$$

which, apart from a factor of i, is the same as the Wightman function (34) *provided* we modify the quantity $\left[(t-t')^2 - i\epsilon\right]$ to $(t - t' - i\epsilon)^2$. (The factor i arises because the Feynman propagator is $(-i)$ times the vacuum expectation value of the time-ordered product of the quantum field.)

A.1. IN THE RINDLER FRAME

The operator $\hat{H}$ (as defined in Eq. (33)) corresponding to the Rindler metric (23) is given by

$$\hat{H} \equiv \left(\frac{1}{\xi^2}\partial_\eta^2 - \frac{g^2}{\xi}\partial_\xi\left(\xi\,\partial_\xi\right) - \partial_y^2 - \partial_z^2\right). \tag{A.4}$$

This operator is invariant under translations along the y and the z directions. In other words, along these two directions, the kernel corresponds to that of a free particle. Exploiting this feature, we can write the quantum mechanical kernel corresponding to the case $y = y'$ and $z = z'$ as

$$K(\tilde{x}, \tilde{x}'; s) = \left(\frac{1}{4\pi i s}\right)\ \langle \eta, \xi | e^{-i\hat{H}' s} | \eta', \xi' \rangle, \tag{A.5}$$

where the operator $\hat{H}'$ is given by

$$\hat{H}' \equiv \left(\frac{1}{\xi^2}\partial_\eta^2 - \frac{g^2}{\xi}\partial_\xi\left(\xi\,\partial_\xi\right)\right). \tag{A.6}$$

On rotating the time coordinate η to the negative imaginary axis (i.e. on setting $\eta = -i\eta_E$) and changing variables to $u = \left(g^{-1}\xi\right)$, we find that

$$\hat{H}' \equiv \left(-\frac{1}{g^2 u^2}\partial_{\eta_E}^2 - \frac{1}{u}\partial_u\left(u\,\partial_u\right)\right). \tag{A.7}$$

If we now identify u as a radial variable and $(g\eta_E)$ as an angular variable, then the operator $\hat{H}'$ is similar in form to the Hamiltonian operator of a free particle in polar coordinates (in 2-dimensions) [49, 50, 51]. The kernel corresponding to this operator can then be written as

$$\langle \eta, \xi | e^{-i\hat{H}' s} | \eta', \xi' \rangle = \left(\frac{1}{4\pi s}\right)\ \exp\left(\frac{i}{4g^2 s}\right)\left(\xi^2 + \xi'^2 - 2\xi\xi' \cosh\left[g(\eta - \eta')\right]\right). \tag{A.8}$$

Therefore, when $\xi = \xi'$, the complete kernel is given by

$$K(\tilde{x}, \tilde{x}'; s) = \left(\frac{1}{16\pi^2 i s^2}\right)\ \exp - \left(\frac{i\xi^2}{g^2 s}\right)\left(\sinh^2\left[g(\eta - \eta')/2\right]\right) \tag{A.9}$$

and the Feynman propagator corresponding to this kernel can be easily evaluated to be

$$G_{\rm F}(\tilde{x}, \tilde{x}')\ =\ \left(\frac{i g^2}{16\pi^2 \xi^2}\right)\ \left(\sinh^{-2}\left[g(\eta - \eta')/2\right] + i\epsilon\right)$$

$$= \left(\frac{i}{4\pi^2\xi^2}\right)\sum_{n=-\infty}^{n=\infty}\left[\left(\eta-\eta'+2\pi i n g^{-1}\right)^2 - i\epsilon\right]^{-1}. \quad \text{(A.10)}$$

A.2. IN THE ROTATING COORDINATES

The operator $\hat{H}$ corresponding to the metric (27) in the rotating coordinates is given by

$$\hat{H} \equiv \left(\left(\partial_t - \Omega\,\partial_\theta\right)^2 - \frac{1}{r}\partial_r\left(r\,\partial_r\right) - \frac{1}{r^2}\partial_\theta^2 - \partial_z^2\right). \quad \text{(A.11)}$$

Exploiting the translational invariance of this operator along the t, z and the θ directions, we can write the kernel corresponding to this operator for the case $r = r'$, $\theta = \theta'$ and $z = z'$ as follows:

$$\begin{aligned} K(\tilde{x}, \tilde{x}'; s) \;=\; & \left(\frac{1}{\sqrt{4\pi i s}}\right)\left(\frac{1}{2\pi}\right) \\ & \times \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\; e^{i(\omega+m\Omega)^2 s}\;\langle r|e^{-i\hat{H}' s}|r\rangle, \end{aligned} \quad \text{(A.12)}$$

where $\hat{H}'$ is given by

$$\hat{H}' \equiv \left(-d_r^2 - \frac{1}{r}d_r + \frac{m^2}{r^2}\right). \quad \text{(A.13)}$$

On carrying out the integral over ω, we obtain that

$$\begin{aligned} K(\tilde{x}, \tilde{x}'; s) \;=\; & \left(\frac{1}{8\pi^2 s}\right)\, e^{-\left[i(t-t')^2/4s\right]} \\ & \times \sum_{m=-\infty}^{\infty} e^{im\Omega(t-t')}\;\langle r|e^{-i\hat{H}' s}|r\rangle. \end{aligned} \quad \text{(A.14)}$$

The normalized modes of the operator $\hat{H}'$ corresponding to an energy eigen value $E = q^2$ are given by (cf. Ref. [32], p. 591)

$$\Psi_q(r) = \sqrt{q}\; J_m(qr), \quad \text{(A.15)}$$

where J_m is a Bessel function of integral order and q runs continuously from zero to ∞. The kernel $\langle r|e^{-i\hat{H}' s}|r\rangle$ can now be expressed in terms of these modes using the Feynman-Kac formula as follows (see, for instance, Ref. [30], p. 88):

$$\langle r|e^{-i\hat{H}'s}|r\rangle = \int_0^\infty dq\, q\, J_m^2(qr)\, e^{-iq^2 s}. \tag{A.16}$$

On substituting this expression in Eq. (A.14), we obtain that

$$\begin{aligned} K(\tilde{x},\tilde{x}';s) &= \left(\frac{1}{8\pi^2 s}\right) e^{-\left[i(t-t')^2/4s\right]} \sum_{m=-\infty}^{\infty} e^{im\Omega(t-t')} \int_0^\infty dq\, q\, J_m^2(qr)\, e^{-iq^2 s} \\ &= \left(\frac{1}{8\pi^2 s}\right) e^{-\left[i(t-t')^2/4s\right]} \\ &\quad \times \left\{ \int_0^\infty dq\, q\, J_0^2(qr)\, e^{-iq^2 s} \right. \\ &\qquad \left. + 2\sum_{m=1}^{\infty} \cos\left[m\Omega(t-t')\right] \int_0^\infty dq\, q\, J_m^2(qr)\, e^{-iq^2 s} \right\}. \end{aligned} \tag{A.17}$$

The integrals over q can be expressed in terms of modified Bessel functions I_m as follows (see, for e.g., Ref. [52], p. 223):

$$\begin{aligned} K(\tilde{x},\tilde{x}';s) &= \left(\frac{1}{16\pi^2 i s^2}\right) \exp - \left(\frac{i}{4s}\right) \left[(t-t')^2 - 2r^2\right] \\ &\quad \times \left\{ I_0(r^2/2is) + 2\sum_{m=1}^{\infty} \cos\left[m\Omega(t-t')\right]\, I_m(r^2/2is) \right\}. \end{aligned} \tag{A.18}$$

On using the identity (cf. Ref. [52], p. 695)

$$\sum_{k=1}^{\infty} \cos(ka)\, I_k(z) = \frac{1}{2}\left[e^{z\cos(a)} - I_0(z)\right], \tag{A.19}$$

we finally obtain that

$$K(\tilde{x},\tilde{x}';s) = \left(\frac{1}{16\pi^2 i s^2}\right)\, \exp - \left(\frac{i}{4s}\right) \left[(t-t')^2 - 4r^2 \sin^2\left[\Omega(t-t')/2\right]\right]. \tag{A.20}$$

The corresponding Feynman propagator is given by

$$G_{\rm F}(\tilde{x},\tilde{x}') = \left(\frac{i}{4\pi^2}\right)\left(\frac{1}{(t-t')^2 - 4r^2\sin^2\left[\Omega(t-t')/2\right] - i\epsilon}\right). \quad \text{(A.21)}$$

A.3. IN THE 'CUSPED' COORDINATES

The operator $\hat{H}$ corresponding to the line element (31) is given by

$$\hat{H} \equiv \left(2\,\partial_{\bar{t}}\,\partial_{\bar{y}} - \partial_{\bar{x}}^2 - 2a\,\bar{x}\,\partial_{\bar{y}}^2 - \partial_z^2\right). \quad \text{(A.22)}$$

Exploiting the translational invariance along the $\bar{t}$, $\bar{y}$ and $\bar{z}$ directions, we can write the kernel corresponding to this operator for the case $\bar{x} = \bar{x}'$, $\bar{y} = \bar{y}'$ and $z = z'$ as

$$K(\tilde{x},\tilde{x}';s) = \left(\frac{1}{\sqrt{4\pi i s}}\right)\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{-i\omega(\bar{t}-\bar{t}')}\int\limits_{-\infty}^{\infty}\frac{dp_{\bar{y}}}{2\pi}\, e^{-2i\omega p_{\bar{y}} s}\,\langle\bar{x}|e^{-i\hat{H}' s}|\bar{x}\rangle, \quad \text{(A.23)}$$

where the operator $\hat{H}'$ is given by

$$\hat{H}' \equiv -d_{\bar{x}}^2 + 2\,a\,p_{\bar{y}}^2\,\bar{x}. \quad \text{(A.24)}$$

On integrating over ω, we obtain that

$$K(\tilde{x},\tilde{x}';s) = \left(\frac{1}{\sqrt{4\pi i s}}\right)\int\limits_{-\infty}^{\infty}\frac{dp_{\bar{y}}}{2\pi}\,\delta^{(1)}\left[2p_{\bar{y}}s - (\bar{t}-\bar{t}')\right]\ \langle\bar{x}|e^{-i\hat{H}' s}|\bar{x}\rangle. \quad \text{(A.25)}$$

The operator $\hat{H}'$ above corresponds to that of a particle in a linear potential. The kernel corresponding to this case is well-known and is given by (see, for instance, Ref. [53], p. 194)

$$\langle\bar{x}|e^{-i\hat{H}' s}|\bar{x}\rangle = \left(\frac{1}{\sqrt{4\pi i s}}\right)\exp -\left(\frac{i}{3}\right)\left(6ap_{\bar{y}}^2\bar{x}s + a^2p_{\bar{y}}^4 s^3\right). \quad \text{(A.26)}$$

On substituting this expression in Eq. (A.25), we find that the complete kernel is given by

$$
\begin{aligned}
&K(\tilde{x}, \tilde{x}'; s) \\
&= \left(\frac{1}{4\pi i s}\right) \int_{-\infty}^{\infty} \frac{dp_{\bar{y}}}{2\pi}\, \delta^{(1)}\left[2p_{\bar{y}} s - (\bar{t} - \bar{t}')\right]\, \exp - \left(\frac{i}{3}\right)\left(6ap_{\bar{y}}^2 \bar{x} s + a^2 p_{\bar{y}}^4 s^3\right) \\
&= \left(\frac{1}{16\pi^2 i s^2}\right)\, \exp - \left(\frac{i}{48 s}\right)\left[24a\bar{x}(\bar{t} - \bar{t}')^2 + a^2(\bar{t} - \bar{t}')^4\right]. \qquad \text{(A.27)}
\end{aligned}
$$

The resulting Feynman propagator can then be easily evaluated to be

$$
G_{\mathrm{F}}(\tilde{x}, \tilde{x}') = \left(\frac{3i}{\pi^2}\right)\left(\frac{1}{24a\bar{x}(\bar{t} - \bar{t}')^2 + a^2(\bar{t} - \bar{t}')^4 - i\epsilon}\right). \qquad \text{(A.28)}
$$

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