Preventing transition to turbulence: a viscosity stratification does not always help

Vijayakumar Chikkadi, A. Sameen, and Rama Govindarajan*

Engineering Mechanics Unit,

Jawaharlal Nehru Centre for Advanced Scientific Research,

Jakkur, Bangalore 560064, India

(Dated: October 9, 2018)

Abstract

In channel flows a step on the route to turbulence is the formation of streaks, often due to algebraic growth of disturbances. While a variation of viscosity in the gradient direction often plays a large role in laminar-turbulent transition in shear flows, we show that it has, surprisingly, little effect on the algebraic growth. Non-uniform viscosity therefore may not always work as a flow-control strategy for maintaining the flow as laminar.

PACS numbers: 47.27.Cn, 47.20.Ft, 47.50.+d

*Electronic address: rama@jncasr.ac.in

In many flow applications, preventing laminar flow from undergoing a transition to turbulence is highly desirable. One of the most popular methods has been to employ a stratification of viscosity in the direction normal to the wall. A reduction of near-wall viscosity by the addition of shear-thinning substances, or by heating/ cooling the walls, can have a large stabilising effect on the *linear* disturbance modes in a laminar flow [1–4]. However, in internal shear flows such as those through pipes and channels [5, 6], the flow becomes turbulent while it is still linearly stable. For example, the laminar flow through a two-dimensional channel is linearly stable upto a critical Reynolds number Re_{cr}, based on channel half-width and centerline velocity, of 5772. If the channel as well as the incoming flow are specially designed to be extremely quiet, the flow can be kept laminar beyond the critical Reynolds number for linear instability, as Nishioka *et al.*[7] demonstrated. However, in a typical channel, in the absence of such special care, transition to turbulence occurs at $Re \sim 1000$. The formation of streamwise streaks is usually one of the first steps towards turbulence [8, 9]. At low levels of external disturbance, these streaks initially form due to the algebraic growth of disturbances [10–13]. A nonlinear process then makes it possible for the streaks to sustain themselves. (At higher noise levels streak formation could itself be nonlinear.) If we are to control the flow, i.e., keep it laminar, by imposing a viscosity stratification, we must first know what it does to the initial departure from a laminar flow towards a turbulent state, i.e. to streak formation. Rather unexpectedly, we find here that it has very little effect on the algebraic growth mechanism. The effect on nonlinear processes in transition, such as the self-sustenance of streaks needs to be investigated in future. On the other hand, fully developed turbulence, especially the problem of drag reduction due to polymers, has been investigated by many.

We study the symmetric flow through a channel of (i) shear-thinning fluids with negligible visco-elasticity, such as dispersions, concentrated colloidal suspensions and carboxymethyl cellulose, and (ii) two miscible fluids of equal densities but different viscosities. Here, fluid 1 flows in the region $-p < y < p$ where the walls are at $y = \pm 1$. Fluid 2 flows in $|y| \geq p + q$, with a thin mixed layer of thickness q between the two where viscosity varies from μ_1 at p to $m\mu_1$ at $p + q$. For the purpose of isolating the effect of viscosity variation, and to make a better comparison with case (i), we have not included the effect of diffusivity in case (ii). We note that this work does not give a firm answer on the complex effect of polymers, but we are able to state firmly that a stratification of viscosity alone does not affect transient

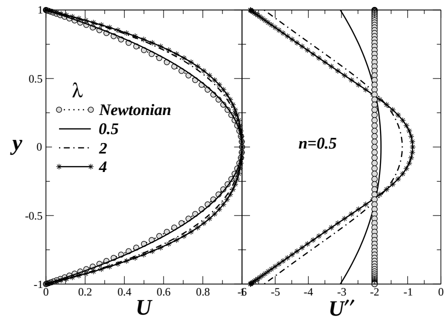

FIG. 1: Velocity (left) and its second derivative (right) profiles at $n = 0.5$ and different values of λ.

growth. Incidentally, with the exception of [14], who had a completely different objective, there is no computation to our knowledge of transient growth with varying viscosity.

The stability operator is non-orthogonal, which enables linearly stable eigenmodes to grow algebraically to give high levels of transient growth. If no other process intervened, these would eventually decay, but nonlinearity takes over when sufficient amplitudes are attained. We begin by obtaining linear eigenmodes, as already done for case (ii) in [3]. Case (i) is described below, and affords no surprise, i.e., a shear-thinning viscosity stabilises linear perturbations. The basic flow velocity is $u = U(y), v = w = 0$ in the streamwise (x), normal to the wall (y) and spanwise (z) directions respectively. The apparent viscosity of shear-thinning fluids is a function of the scalar invariants of the shear rate $\dot{\gamma}$. The Carreau model [15],

$$\frac{\mu - \mu_\infty}{\mu_0 - \mu_\infty} = [1 + (\lambda\dot{\gamma})^2]^{\frac{(n-1)}{2}} \tag{1}$$

where μ_0 and μ_∞ are the viscosities at zero and infinite shear rate respectively, λ is the time constant of the fluid and n is the shear-thinning index, is known to be a good representation of the viscosity, $n = 1$ or $\lambda = 0$ correspond to a Newtonian fluid. The mean velocity profile is obtained from the steady x-momentum equation, given in non-dimensional form by

$$-P + \frac{d}{dy}(\mu\frac{dU}{dy}) = 0 \quad \text{where} \quad P \equiv Re\frac{dp}{dx} \tag{2}$$

The primes denote differentiation with respect to y, and μ is scaled by μ_0. The Reynolds number is based on viscosity averaged across the channel. Fig. 1 shows the velocity and its second derivatives with respect to y. Sample basic profiles for the two-fluid case are given in Fig. 2, details are available in [16].

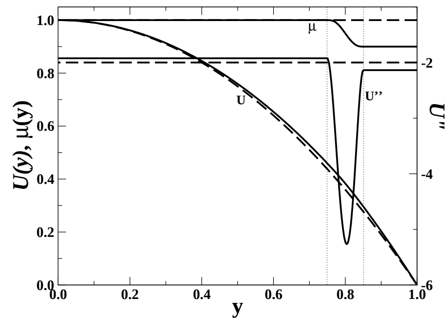

FIG. 2: The velocity and the viscosity profiles for the two-fluid case [16]. The second derivative of velocity is also shown. Solid lines: two fluids, $m = 0.9$; dashed lines: single fluid. The vertical lines show the extent of the mixed region.

Three-dimensional linear perturbations in the velocity, in normal mode form, e.g.,

$$(\hat{v}, \hat{\eta})(x, y, z, t) = (v, \eta)(y) \ \ \exp\{i(\alpha x + \beta z - \omega t)\}, \tag{3}$$

where α and β are their streamwise and spanwise wavenumbers respectively and ω is their frequency, satisfy the Orr-Sommerfeld and Squire's equations [11], which are modified here to include the effects of non-constant viscosity. The result is an eigenvalue problem described by

$$i\omega \begin{pmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{os} & 0 \\ i\beta DU & \mathcal{L}_{sq} \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix} \tag{4}$$

where the modified Orr-Sommerfeld and Squire operators are given respectively by

$$\mathcal{L}_{os} = i\alpha[U(k^2 - D^2) + U''] + \frac{1}{Re}[\mu(k^2 - D^2)^2 + 2\mu' D^3 + \mu'' D^2 - 2k^2\mu' D + k^2\mu''], \tag{5}$$

$$\mathcal{L}_{sq} = i\alpha U + \frac{1}{Re}\big[\mu(k^2 - D^2) + \mu' D\big], \tag{6}$$

$$k^2 = \alpha^2 + \beta^2, \tag{7}$$

$D \equiv d/dy$ and η is the normal vorticity of the disturbance. Equation (4) along with boundary conditions $v = Dv = \eta = 0$ at $y = \pm 1$ is solved using a Chebyshev collocation spectral method. The number of collocation points was taken to be 81. On using 161 collocation points, the growth rates changed only in the sixth significant decimal place in the worst

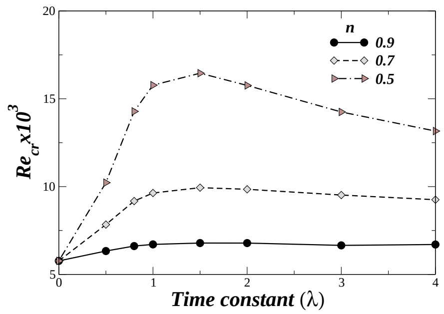

FIG. 3: Variation of the critical Reynolds number of a shear-thinning fluid, with λ, at n=0.5, 0.7 and 0.9, $\beta = 0$.

case. The difference in individual eigenvalues was much smaller. For specified Re, α and β, the imaginary part of the frequency gives the exponential decay rate.

The effect of shear-thinning viscosity on the linear instability is shown in Fig. 3, it is evident that shear-thinning stabilises the flow. The increase in the critical Reynolds number for instability (Re_{cr}) can be a factor of 3 over that for a Newtonian fluid. For inviscid flow [17] a necessary and sufficient condition for flow instability is the existence of an inflexion point in the mean velocity profile. In a finite Reynolds number flow no such theorem exists, but it is normally the case that a flow stabilisation results when the velocity profile becomes fuller, i.e., goes further away from containing a point of inflexion. The second derivatives of the velocity in Fig. 1 indicate that the observed effect is the expected one. The effect on linear stability was seen [3, 16] to be much more dramatic in the case of two-fluid flow, when the viscosity-stratified layer overlapped the production layer of disturbance kinetic energy. The critical Reynolds number can be higher by an order of magnitude. In the light of this, it is surprising that the dominant algebraic growth of disturbances is unmoved by the viscosity stratification.

For studying transient growth, the disturbance equation (4) is viewed as an initial value problem:

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ \eta \end{bmatrix} = \mathbf{M}^{-1}\mathbf{L}\begin{bmatrix} v \\ \eta \end{bmatrix} \tag{8}$$

$$\mathbf{M} = \begin{bmatrix} -D^2 + k^2 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} \mathcal{L}_{os} & 0 \\ -i\beta U' & \mathcal{L}_{sq} \end{bmatrix} \tag{9}$$

where $\mathcal{L}_{os}$ and $\mathcal{L}_{sq}$ are defined by equations (5) & (6). We choose to monitor the growth of the disturbance kinetic energy, given by

$$E_t = \frac{1}{2}\int_{-1}^{+1} (|Dv|^2 + k^2|v|^2 + |\eta|^2)dy. \tag{10}$$

The total kinetic energy of the perturbation can be found by integrating equation (10) at a given time over the $\alpha - \beta$ plane. The amplitude of disturbance kinetic energy at a given time depends, of course, on the initial amplitudes of various modes. Here, G is scaled by its initial value. For a given viscosity stratification, equation (11) gives the maximum possible amplitude G at any instant of time, optimised for each instant of time, over all initial conditions:

$$G(t|\alpha, \beta, R) \equiv G(t) = \max \frac{E(t|\alpha, \beta, R)}{E(0, \alpha, \beta, R)}. \tag{11}$$

The optimisation method followed here is outlined in [11]. The maximum of G over time is denoted as G_{max}, which is shown in Fig. 4. It is seen that a variation of viscosity does not make much difference to the transient growth of disturbances of this wavelength. The maximum energy contour for two-fluid flow, $m = 0.9$, is shown in Fig. 5. The corresponding contour for a single Newtonian fluid is available in [11]. There is very little change anywhere in the $\alpha - \beta$ due to viscosity variation. A similar conclusion was reached about the effect of shear-thinning. It is known [10, 11] that in the channel flow of a Newtonian fluid, a streamwise vortex, with $\alpha = 0$ and $\beta \simeq 2$, is the optimal disturbance, i.e., gives the highest G_{max}. As seen in Fig. 5, we find the same to be true of viscosity-stratified flow. The optimal disturbance is in the form of rolls, as shown in Fig. 6. We now choose these disturbances, that remain constant in the streamwise direction, and quantify the effect of a varying viscosity in Fig. 7. We see that there is at most a 15% change, for cases where the linear stability changed by a factor of 3 (shear-thinning fluid) and by an order of magnitude (two-fluid). In particular, when $m = 0.9$ in (b), the change is only about 1%, whereas linear stability changes by an order of magnitude. It is apparent that viscosity stratification is not effective in suppressing the growth of optimal disturbances.

This result is counter-intuitive, especially given the major stabilisation achieved by the linear modes, and a pre-conditioning to expect stabilisation from a fuller velocity profile. Our attempts to provide a complete mathematical explanation have not been successful so far. We offer a partial explanation based on the observation that the largest growing transients are those that do not vary in the downstream direction, i.e., modes with $\alpha = 0$. Out of the terms containing derivatives of the viscosity in equation (12) it can be verified

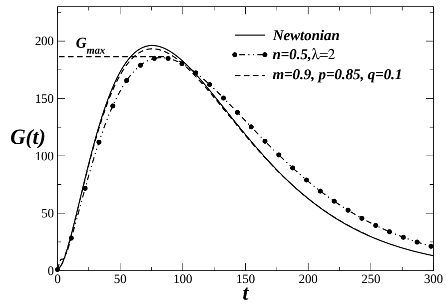

FIG. 4: Energy amplification $G(t)$ of Newtonian (solid line), shear-thinning fluid (filled circle) and two-fluid flow (dash line) at $Re = 1000$, $\alpha = 0.0$, $\beta = 2.05$

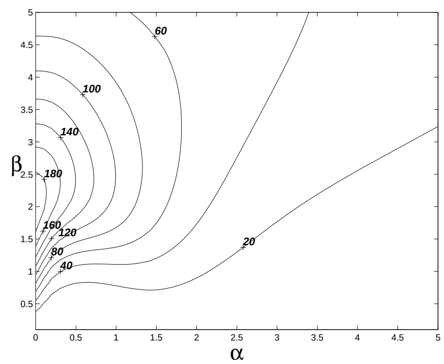

FIG. 5: Contours of the maximum over time of optimal disturbance G_{max} for two-fluid flow. The viscosity ratio is m=0.9, $p = 0.85$, $q = 0.1$ and Re=1000. The number mentioned on the contours is the value of G_{max}. The figure is very similar to that for a single Newtonian fluid in [11].

numerically that the only term which contributes any noticeable effect is that containing U''. Note that U'' may be written as

$$U'' = -\frac{P}{\mu} + \frac{Py}{\mu^2}\frac{d\mu}{dy}. \tag{12}$$

At $\alpha = 0$ the term containing U'' does not appear in equation (5), and so the major effect of viscosity stratification is absent in the case of growth of optimal disturbances. The reduced

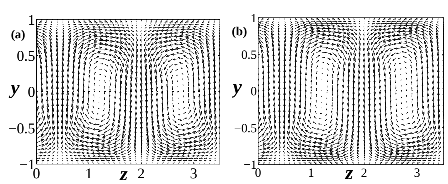

FIG. 6: Optimal disturbances at $\alpha = 0.0$, $\beta = 2.05$, $Re = 1000$. (a) Newtonian fluid; (b) Shear-thinning fluid, $n = 0.5$, $\lambda = 2.0$.

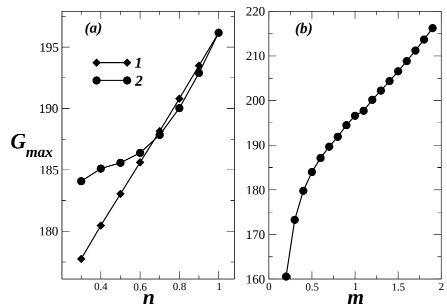

FIG. 7: Growth of streamwise-independent (α=0) disturbances at Re=1000, β=2.05. (a) Non-Newtonian fluid, with different levels of shear-thinning. (b) Two-fluid flow, m is the ratio of outer to inner fluid viscosity. G_{max} for a single Newtonian fluid is $\sim$ 196.

problem has been solved, and it has been verified that the eigenvalues and eigenfunctions are practically unchanged by viscosity stratification. This fact is evident from Fig. 6 as well.

A viscosity reduction near the wall is usually associated with stabilisation and with drag reduction. We conclude however, that while the linear modes are indeed significantly stabilised, the transient growth of disturbances, including that of streamwise vortices, is practically unaffected. Flow control using a stratification of viscosity is unlikely to work in this case, at least in suppressing the first deviation from a laminar state.

Acknowledgements: We thank Prof. Dan Henningson for discussions.

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