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Optimal Placement and Shape Morphing Of Thin Plates Using Dynamic Inversion Design

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Abstract: This paper presents active vibration control and morphing of thin plates using an array of piezoelectric actuator-sensor system whose locations are determined by optimization. The sudden application of control input for morphing leads to unwanted vibrations which are suppressed using the piezoelectric actuator-sensor couples, which form a feedback control loop. Dynamic Inversion technique is used to determine the control inputs to morph the plate and to suppress the vibrations in the process. The Dynamic Inversion controlled system is compared to uncontrolled system and as a reference, the results are compared with that of Linear Quadratic Controller. The partial differential equations governing the behaviour of plate and piezoelectric actuation are solved using lower dimensional projection method, following Design-then-Approximate (DTA) method, which will reduce spillover effects. Two reference configurations are considered to perform simulations. The actuators are designed for both vibration control and morphing since thin plates have poor damping characteristics and need external damping. The displacement, velocity and error norm time histories are analysed and the configuration achieved by the system by both controllers are compared.

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Keywords: distributed systems, dynamic inversion, linear systems theory, piezoelectric actuators, optimal placement.

1. INTRODUCTION

Many industries are incorporating thin, light weight, and smart structures in their designs to improve the efficiency of structural components like morphing wing, satellite solar panels etc. The most important structural element in these structures are thin plates. One of the key issues in employing thin plates is the poor damping characteristics with a lighter mass which is more vulnerable to external disturbances. In order to suppress the unwanted vibrations, external damping should be provided through actuators. Due to the electro-mechanical coupling properties of piezoelectric materials, they have become the right candidates as sensors and actuators. This lead to the development of whole new field of smart thin structures with actuated damping along with morphing abilities. Vibration control of thin cantilever elastic plate excited by impulsive transversal force at free corner is addressed in Caruso et al. (2003). H_2 control laws are used in the controller design based on the finite element model with modal reduction of the system with three piezoelectric patches. Prior to this study, much work has been done with single piezoelectric actuators in the control schemes. This study emphasizes on usage of multiple sensors and actuation for controlling more complex structures. In Qiu et al. (2007), vibrations of a cantilever plate with piezoelectric sensors and actuators is controlled using positive position feedback and proportional-derivative control. The bending modes are separated from torsional modes by applying Ritz's method to the plate equation. The optimal position of piezoelectric patches are obtained based on the degree of observability and controllability indices. The experimental and the simulations show the effectiveness of the controller in suppressing the vibrations. Padhi and Balakrishnan (2007) presented a control design approach combining the principles of dynamic inversion and optimisation for a class of nonlinear distributed parameter systems. The control theory is applied to both continuous and discrete actuators based on design-then-approximate approach. A real life temperature problem is solved to demonstrate the potential of the proposed technique.

Decades of research have lead to a great deal of literature on control of distributed parameter systems. Padhi and Ali (2009) attempted to give a brief account on the chronological developments of this field of research with less mathematics to attract a wider audience. Ali et al. (2009) developed a two staged state feedback control design with first stage consisting of a primary controller which provides the force required to obtain desired closed loop response and second stage consisting of optimal dynamic inversion to control the forces applied through MR dampers. The proposed control design was simulated for the study of benchmark highway bridge problem. Ali and Padhi (2009) presented an active vibration control approach based on optimal dynamic inversion for nonlinear Euler-Bernoulli beams that utilises the nonlinear PDE and hence, it is free from approximation errors. The performance of the control system is demonstrated by simulation of beam with continuous and discrete controllers.

In Sadek et al. (2014), active vibration control of rectangular plates integrated with piezoelectric sensors and actuators is studied. An explicit control law is arrived using maximum principle theory, a Hamiltonian functional, in two spatial dimensions. The system of equations are solved using modal approximation. In formulating maximum principle, a Hamiltonian functional is introduced. Due to its uniqueness, the admissible

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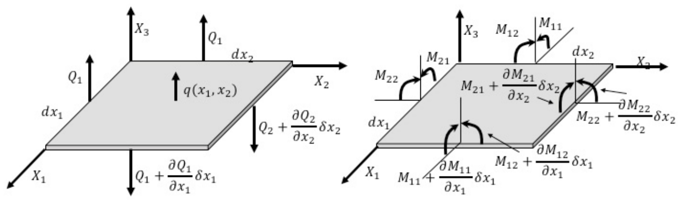


Fig. 1. Differential element to derive equilibrium equations.

control function that maximises Hamiltonian forms the optimal control. Active vibration control of circular plate, sandwiched within piezoelectric layers was studied in Khorshidi et al. (2015). The sandwiched plate is excited by planar sound waves and the transverse vibrations are controlled using Fuzzy Logic Controller and Linear Quadratic Regulator. The system is solved analytically along with Maxwell's electricity equation to get closed form solutions. Active vibration control of geometrically nonlinear simply supported thin plates using continuous and discrete controller based on Optimal Dynamic Inversion technique is studied in Pradeesh and Ali (2016). Continuous optimization, as well as discrete optimization with constraint laws, are used to find the control inputs for the system.

In much of the references, the control theories are used on the numerically approximated system equations or modal reduced systems which can lead to spillover effects where the higher modes of excitation are left out during approximation. This paper investigates morphing of a thin plate, using an array of piezoelectric sensors and actuators whose positions are optimized, to attain a predefined reference displacement profile. The input voltages to various patches are obtained using Dynamic Inversion (DI) control technique. The controller is designed from the PDE and then approximated (DTA) to reduce spillover effects. Linear Quadratic Regulator control scheme is also used to compare the results of the DI controller. The sensors and actuators have a dual role in morphing the cantilever plate as well as providing external damping towards the vibrations of the structure.

2. MATHEMATICAL FORMULATION

A thin simply supported plate P is considered in $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$ where $\Omega \subset \mathbb{R}^2$. According to the classical plate theory, Timoshenko and Woinowsky-Krieger (1959), the displacement field, $u(x_1,x_2,x_3)$, where x_i are the coordinate axes of the plate at a given point in spatial domain, $\bar{x} = (x_1,x_2,x_3)$ is given as,

$$u = \{u_1,u_2,u_3\} = \{x_3 w_{,1}, x_3 w_{,2}, w\} \tag{1}$$

where w is the out of plane displacement at a point in plate (x_1,x_2,x_3), $w_{,i}$ is differentiation of w with respect to x_i. $\underline{()}$ denotes tensor quantity of 2^{nd} or higher order. The stress tensor σ can be obtained from the curvature $\underline{\kappa}$ of displacement field.

$$\underline{\sigma} = \underline{C} : \underline{\varepsilon} = \underline{C} : x_3\, \underline{\kappa} \tag{2}$$

The bending moment tensor due to the stresses in the cross section of plate are derived as,

$$\underline{M} = \int_{x_3} x_3\, \underline{\sigma}\, dx_3 \tag{3}$$

$$\underline{M} = \underline{D} : \underline{\kappa}$$

where, $\underline{D} = \int_{x_3} x_3^2 \underline{C}\, dx_3$ is the bending stiffness of the plate, $\underline{C}$ is the constitutive matrix of the material. The external, shear forces and bending moments acting on a differential plate element is shown in the Fig. 1. The transverse shear equilibrium 4 and moment equilibrium 5 equations are derived by equating the shear forces Q and bending moments $\underline{M}$ in the differential plate element.

$$\nabla \cdot Q = -q \tag{4}$$

$$\nabla \cdot \underline{M} = -Q \tag{5}$$

By combining 4 & 5, yields the relation between the internal moments and the external force applied to the plate.

$$\nabla \cdot \nabla \cdot \underline{M} = q \tag{6}$$

2.1 Piezoelectric Constitutive Relations

The constitutive relation for the piezoelectric material is given by

$$\underline{\sigma} = \underline{C}^E : \underline{\varepsilon} - \underline{e} : E \tag{7}$$

$$D = \underline{e}^T : \underline{\varepsilon} + \underline{\zeta}^S : E$$

where, the superscript $(\,)^E$ and $(\,)^S$ denotes measurement at constant electric field and at constant strain respectively, $(\,)^T$ denotes transpose of a matrix, D is the electric displacement vector, E is the electric field vector, $\underline{e}$ is the piezoelectric tensor and $\underline{\zeta}$ is the dielectric tensor.

Assuming a linear variation of electric potential along the thickness of piezoelectric patch, the electric field can be derived as,

$$E = -\nabla \Phi_z = -\{0,0,1/h\}^T \Phi_z \tag{8}$$

The piezoelectric patches, attached to the plate, exerts a bending moment on the plate due to the potential difference in the piezoelectric material.

The addition of piezoelectric patches to the plate can be mathematically defined by,

$$\rho \ddot{w} + c\dot{w} + D_p \Delta^2 w = u(\bar{x},t) \tag{9}$$

$$u(\bar{x},t) = \sum_{m=1}^{N} \bar{u}_m\, \mathbb{H}$$

$\mathbb{H}$ defines the domain of the m^{th} patch, $\bar{u}_m$ is the force generated by n^{th} patch.

2.2 Dynamic Inversion Control

The control forces are applied through the actuators attached to the plate to suppress the vibrations and to reach the reference configuration i.e. $w(x,t) \to w_{ref}$ & $\dot{w(x,t)} \to 0$ as $t \to \infty$, where w_{ref} is the reference output. In the Dynamic Inversion technique, the control force is obtained by enforcing the system to follow a sliding surface of stable error dynamics defined by the same order of equation as the system. The second order plate dynamics with output is given as,

$$\rho \ddot{w} + c\dot{w} + D_p \Delta^2 w = \mathbb{H}\bar{u}_m \tag{10}$$

$$Y = \mathbb{G} w$$

$$w \in \mathbb{R}^n, \bar{u}_m \in \mathbb{R}^m, Y \in \mathbb{R}^p$$

where Y is the output of the system, say sensor potential, m and p are the number of actuators and sensors respectively, w is the configuration, u are the control variables, $\mathbb{G}$ defines domains of sensors. To obtain the control variables, the error vector is defined as,

$$E(t) = Y(t) - Y^*(t); \tag{11}$$

where Y^* is the reference output. The controller is designed such that as $t \to \infty$, $E(t) \to 0$ and $Y(t) \to Y^*$. To ensure reaching the target response, positive gains K_v and K_p are selected and the error dynamics is defined as

$$\ddot{E} + K_v\dot{E} + K_pE = 0 \tag{12}$$

Keeping the reference outputs $\dot{Y}^*$ and $\ddot{Y}^*$ as 0, the control variable is obtained as

$$\mathbb{GH}\bar{u} = -K_p\mathbb{G}w + \mathbb{G}D_p\Delta^2 w - (K_v\mathbb{G} - \mathbb{G}c)\dot{w} + K_pY^* \tag{13}$$
$$Au = b$$

Selection of K_v and K_p is particular to the problem and designer. Solving for $Au = b$, the control variable can be obtained. If the matrix A is not a square matrix, the system of equations can be solved in two cases.

Case 1: p>m The problem can be solved by least squares method,

$$u = A^+b \tag{14}$$

This provides optimum values for u. A^+ denotes Moore-Penrose pseudoinverse.

Case 2: p<m Optimal Dynamic Inversion technique can be used. For the same system in (10), the error dynamics (12) and the constraint (13), a cost function is constructed to minimize the control variable.

$$J = \frac{1}{2}(u^TRu) \tag{15}$$
$$\text{Subjected to } Au = b$$

So, the performance index is constructed as,

$$\bar{J} = \frac{1}{2}(u^TRu) + \lambda^T(Au - b) \tag{16}$$

Optimising the cost function, the control variable can be obtained by finding λ. The control variable is given as,

$$u = R^{-1}A^T(AR^{-1}A^T)^{-1}b \tag{17}$$

Asymptotic tracking is guaranteed for Optimal Dynamic Inversion technique based control.
Dynamic Inversion technique is very sensitive to the control parameters K_p & K_v. An approximate estimation of the parameters can be obtained by applying the control variable to the equation and approximating the PDE using projection method.

$$\ddot{w} + [C']\dot{w} + [K']w = [F']Y^* \tag{18}$$

The modified damping, $[D']$, and stiffness, $[S']$, of the system, as given in (18), should be positive definite in order for the system to reach equilibrium. By insisting this condition, the optimal estimate of K_p and K_v values can be obtained.

2.3 Linear Quadratic Regulator

The aim of optimal control is operating a dynamical system with minimum cost. The system is defined using linear differential equations and the cost function is quadratic. For an infinite horizon continuous system, the state space form can be constructed from the differential equations.

$$\dot{X} = AX + Bu \tag{19}$$

And the cost function is defined as,

$$J_{LQR} = \int_0^\infty \left(X^TQX + u^TRu\right)dt \tag{20}$$

The feedback control law that minimizes the cost function is given as

$$u = -KX \tag{21}$$
$$\text{where } K = R^{-1}B^TP$$

P is found by solving the contnuous time Riccati differential equation.

$$A^TP + PA - PBR^{-1}B^TP + Q = 0 \tag{22}$$

The new system dynamics will be,

$$\dot{X} = A_cX \tag{23}$$
$$A_c = A - BK$$

2.4 Solution of PDE

The configuration space of distributed parameter systems is infinite dimensional. An optimal solution is obtained by projecting the infinite dimensional configuration to a finite dimensional vector space.
Let the configuration of the system be denoted by Ψ. The configuration of the system, in the space Ω, is derived from the differential equation,

$$\mathbf{L}^2(\Psi) - f = 0 \tag{24}$$

where $\mathbb{L}^2$ is a differential operator defined over the domain Ω and f a forcing function. Let $\mathbb{V}$ be a finite dimensional vector space defined by independent vectors $\{V_1, V_2, .., V_n\}$. The system, defined in (24), is projected onto the space $\mathbb{V}$. This leads to bilinear symmetric system of equations.

$$\sum_{k=1}^{N}\int_{\delta\Omega_k}\mathbb{V}\left(\mathbf{L}^2(\Psi) - f\right)d\delta\Omega_k = 0$$
$$\sum_{k=1}^{N}\int_{\delta\Omega_k}(\mathbf{L}(\mathbb{V})\mathbf{L}(\Psi) - \mathbb{V}f)\,d\delta\Omega_k = 0 \tag{25}$$

The configuration of the system can be written as a linear combination of the functions V_n.

$$\Psi = \sum_{n=1}^{N}\psi_nV_n(x_1, x_2) \tag{26}$$
$$V_n(x_1, x_2) = X(x_1)Y(x_2) \tag{27}$$

where V_n is a set of decoupled shape functions which are admissible for the system boundary conditions. Substituting (26) in (25), the algebraic system of equations are arrived.

$$\left[\sum_{i,j}\int_{\delta\Omega_k}\mathbf{L}^2(V_i)\mathbf{L}^2(V_j)\,d\delta\Omega_k\right]\{\psi\} - \sum_i\int_{\delta\Omega_k}V_if\,d\delta\Omega_k = 0 \tag{28}$$

By applying the technique to (10), the dynamical system is obtained in matrix set of equations. The inertial and stiffness forces of the system with plate and piezoelectric patches are modelled directly using projection method. Derivation of piezoelectric and dielectric stiffness matrices are given in Allik and Hughes (1970). The combined set of equations are given as,

$$[M]\{\ddot{\psi}\} + [K_{\psi\psi}]\{\psi\} + [K_{\psi\phi}]\{\phi\} = F(t)$$
$$[K_{\phi\psi}]\{\psi\} + [K_{\phi\phi}]\{\phi\} = Q(t) \tag{29}$$

where $[M]$ is the combined mass matrix, $[K_{\psi\psi}]$ is the elastic stiffness matrix, $[K_{\psi\phi}] = [K_{\phi\psi}]^T$ is the piezoelectric stiffness matrix, $[K_{\psi\psi}]$ is the dielectric stiffness matrix, F & Q are the force and charge vectors respectively. The system of equations in (29) is common for both sensor potentials ϕ_s and actuator potentials ϕ_a. These equations are used to simulate uncontrolled plate behaviour when actuation is applied.
The system of equations after rearranging for actuator inputs is given as,

$$[M]\{\ddot{\psi}\} + [K_{\psi\psi}]^*\{\psi\} = [B]\left(\{\phi\} - \{\phi\}_{ref}\right) \tag{30}$$

where $[K_{\psi\psi}]^*$ is net effective stiffness matrix of the system after applying the control variables. The system of equations with the control forces is of the form,

$$[M]\ddot{\psi} + [K_{\psi\psi}]^*\psi = \sum_{i=1}^{N} B_i\phi_i \tag{31}$$

where B_i is the force vector corresponding to i^{th} actuator input. In the eigenspace, the B consists of the modal participation factors for each mode and actuator.

2.5 Optimization of Patch Location

An optimal position method of piezoelectric sensors and actuators for the plate system to attain the defined shape is given according to the maximum modal participation factor of the sensors and actuators. Based on the sizes of sensors and actuators, an optimal location is derived for collocated sensors and actuators. The cost function is a squared sum of the modal participation factors of an actuator with its arbitrary position for different modes. The cost function is defined as,

$$J(x_a, y_a) = 1/2 * B(x_a, y_a)^T B(x_a, y_a) \tag{32}$$

where (x_a, y_a) is the centre coordinate of the piezoelectric patch. The width of the patches are fixed in the model itself. The cost function in 32 is nonlinear and is used to find the optimal locations for the centre of actuator-sensor couple for different modes as well as combination of modes. The cost function gives the effectiveness of actuator for various positions. The number of local maxima of the function provides the minimum number of actuators to be provided to achieve a shape. The hierarchy of importance of actuators can be determined by the value of the cost function at different maxima.

3. NUMERICAL SIMULATION

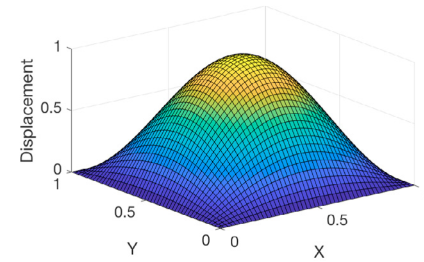
3.1 Location Optimization

the simulation studies of optimal placement of piezoelectric actuator-sensor couples are carried out. A unit square plate is considered to determine the optimal locations for various modes. The size of piezoelectric patch is taken as 25% area of the plate. Fig. 2 and Fig. 3 show 1^{st} four modes and corresponding optimal positions of patches respectively. Using the optimal position coordinates, the piezoelectric patches are modelled along with the plate.

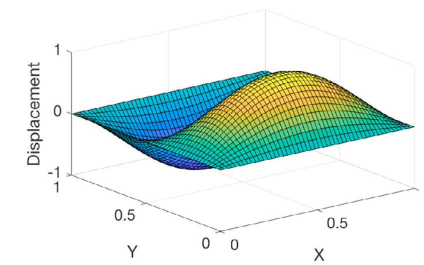
For the simulation of simply supported thin plate for vibration control and shape morphing, a combined mode shape consisting of modes 1, 2 & 3 is taken as reference displacement profile as shown in Fig. 4. Based on the cost function of first four modes, four locations of maximum performance is considered to attach piezoelectric patches.

3.2 Morphing and Vibration Control

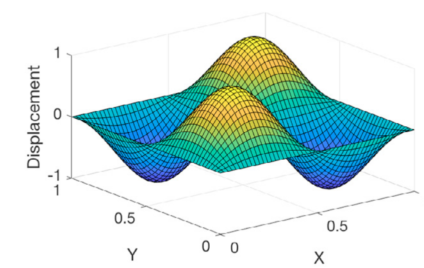
A thin simply supported plate is considered fo numerical simulations. The dimensions of the plate are $200 \times 200 \times 0.1mm$. The material and piezoelectric properties are given in the table 1. Based on the optimal placement locations obtained from Fig. 4, four piezoelecrtic patches of size $5 \times 5mm$ is attached to the base plate in bi-morph configuration. Damping is not considered for the current system in order to study the characteristics of the control scheme. The controller for the plate system is designed from the equations and are applied. The control voltages are applied to the actuators depending on the position and



(a) Mode 1.



(b) Mode 2.



(c) Mode 4.

Fig. 2. Mode shapes of unit square plate.

the difference in the cuurent and reference configurations. The bottom patches act as sensors for feedback.

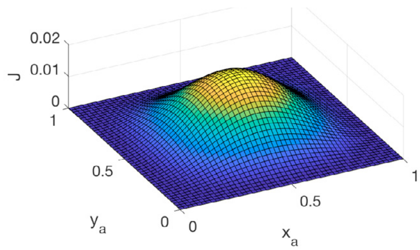
Table 1. Material Properties

Description	Parameter	Value
Young's Modulus (alum)	E_{alum}	$69GPa$
Young's Modulus (piezo)	E_{piezo}	$69GPa$
Poisson's ratio	$\nu(alum, piezo)$	0.3
Density (alum)	ρ_{alum}	$2700kg/m^3$
Density (piezo)	ρ_{piezo}	$7500kg/m^3$
Piezo Dielectric	ζ^S	1.6×10^{-8}
Piezoelectric strain	e	-12.5
Capacitance	C	6.3×10^{-7}

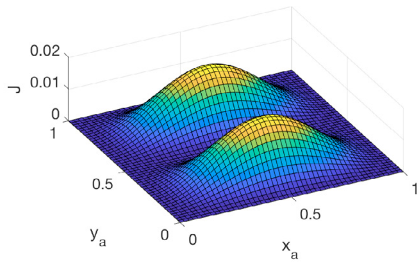
4. DISCUSSIONS

The time evolution of the states is obtained by solving the equations. The time histories of the displacement, velocity and displacement error norms with and without controllers are shown in Fig. 6, Fig. 7 and Fig. **??** respectively. The reference shape is chosen as a combination of modes 2 & 3 in order to verify working of the controller. The optimal placement method can be extensively used to obtain the patch locations.

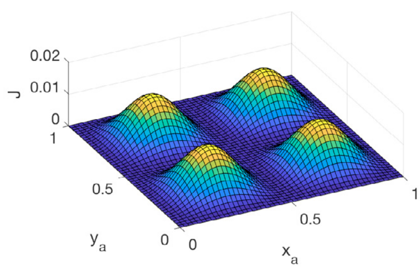
The displacement norms time history in Fig. 6 shows effectiveness of DI controller. The vibrations are suppressed by both the DI and LQR controllers. Compared to LQR, which has very little vibrations at the end. The velocity norm in Fig. 7 also shows that both the controlled systems damped out much



(a) Cost function J for Mode 1.

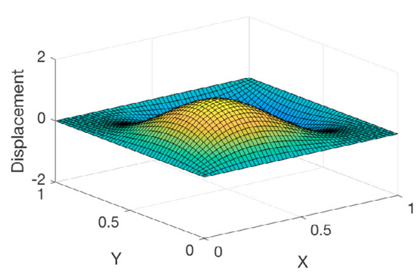


(b) Cost function J for Mode 2.

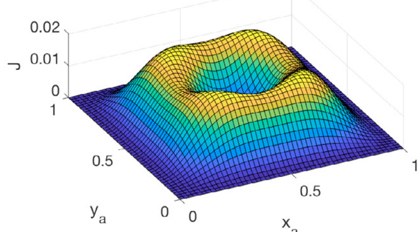


(c) Cost function J for Mode 4.

Fig. 3. Cost functions J for different modes.

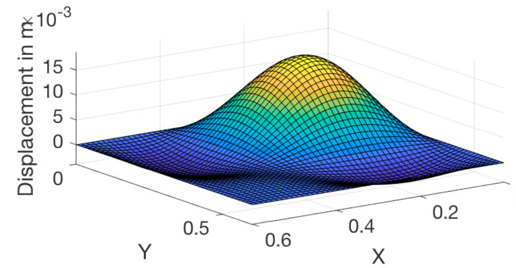


(a) Combined modes 1, 2 & 3.

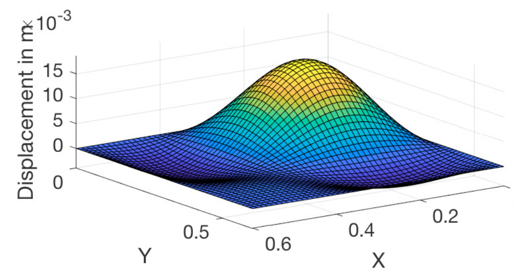


(b) Cost function J for combined modes 1, 2 & 3.

Fig. 4. Combined mode and its cost function.



(a) Reference Configuration



(b) Morphed Configuration

Fig. 5. Reference and morphed configurations.

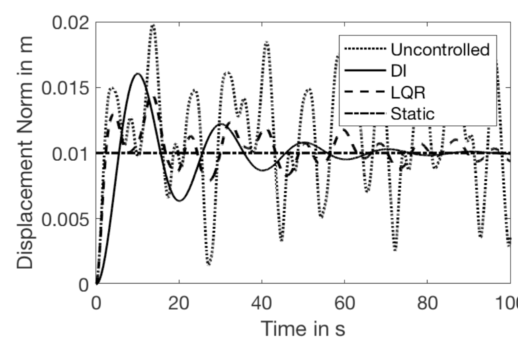


Fig. 6. Displacement Norm of the plate.

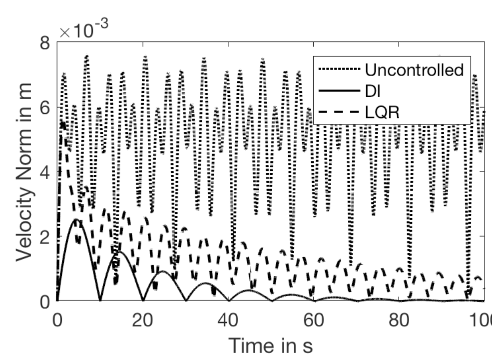


Fig. 7. Velocity Norm of the plate.

quickly than the uncontrolled system. The controller inputs for each actuator

The target shape is achieved with very minimal error by both controllers which is evident from the difference betwwen the target displacement norm (dash-dot line) and the displacement norm at any time instant of both controllers. Since the piezo-electric actuators and sensors are placed at optimal locations,

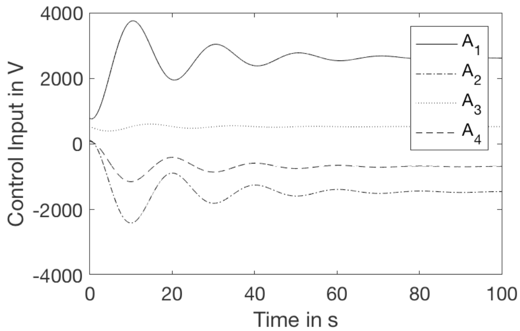


Fig. 8. DI controller input.

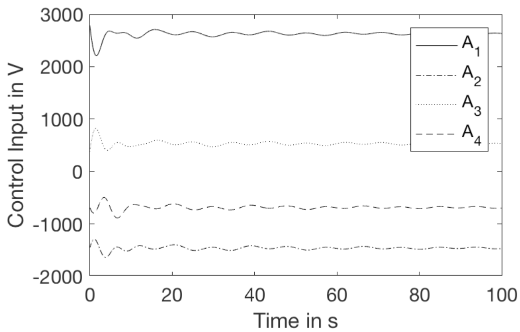


Fig. 9. LQR controller input.

the patches are able to morph the plate with minimum possible error. If the number of patches are reduced from its optimum required number, the controller will not be able to morph to the reference profile without errors. The same configuration of patches cannot be used for more than four modes because it may lead to controllability and observability issues.

The advantage of using the DI controller is that by back calculating the control parameters, the system can be made to behave like a first order dynamical system. Also, this controller is very simple to implement for both linear and nonlinear systems.

5. CONCLUSION

In this present paper, the thin plate system is modelled using projection method. By constructing a cost function from the state space matrices, the optimal location of piezoelectric patches for various mode shapes are obtained. A reference configuration is considered for morphing and optimum number of piezoelectric patches and their locations are determined. The array of piezoelectric patches are attached to the plate in bimorph configuration and are used as sensors and actuators to morph the thin plate and suppress the vibrations due to the application of control inputs. The stiffness and mass of the system consist both the plate and piezoelectric sensors and actuators. The control inputs are obtained by dynamic inversion technique. A brief description and derivation of the technique for the problem are carried out. For comparison, linear quadratic regulator control scheme is also used.

The control inputs are derived such that it dictates the plate to attain prescribed configuration. The controller is designed at first and the whole system is approximated to solve to reduce spillover issues. The simulation results reveal that the controller is effective in suppressing the vibrations and also to achieve the given configurations. This methodology plays important role help in developing technologies like morphing structures using a thin plate or shell like structures.

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REFERENCES

Ali, S., Ramaswamy, A., et al. (2009). Optimal dynamic inversion-based semi-active control of benchmark bridge using mr dampers. *Structural Control and Health Monitoring*, 16(5), 564–585.

Ali, S.F. and Padhi, R. (2009). Active vibration suppression of non-linear beams using optimal dynamic inversion. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, 223(5), 657–672.

Allik, H. and Hughes, T.J. (1970). Finite element method for piezoelectric vibration. *International journal for numerical methods in engineering*, 2(2), 151–157.

Caruso, G., Galeani, S., and Menini, L. (2003). Active vibration control of an elastic plate using multiple piezoelectric sensors and actuators. *Simulation Modelling Practice and Theory*, 11(5), 403–419.

Khorshidi, K., Rezaei, E., Ghadimi, A., and Pagoli, M. (2015). Active vibration control of circular plates coupled with piezoelectric layers excited by plane sound wave. *Applied Mathematical Modelling*, 39(3), 1217–1228.

Padhi, R. and Ali, S.F. (2009). An account of chronological developments in control of distributed parameter systems. *Annual Reviews in Control*, 33(1), 59 – 68. doi: https://doi.org/10.1016/j.arcontrol.2009.01.003.

Padhi, R. and Balakrishnan, S. (2007). Optimal dynamic inversion control design for a class of nonlinear distributed parameter systems with continuous and discrete actuators. *IET control theory & applications*, 1(6), 1662–1671.

Pradeesh, L.V. and Ali, S.F. (2016). Active vibration control of thin plate using optimal dynamic inversion technique. *IFAC-PapersOnLine*, 49(1), 326–331.

Qiu, Z.c., Zhang, X.m., Wu, H.x., and Zhang, H.h. (2007). Optimal placement and active vibration control for piezoelectric smart flexible cantilever plate. *Journal of Sound and Vibration*, 301(3), 521–543.

Sadek, I., Kucuk, I., and Adali, S. (2014). Active open-loop control of plates with multiple piezoelectric patches via the maximum principle. *Mechanics of Advanced Materials and Structures*, 21(9), 772–779.

Timoshenko, S.P. and Woinowsky-Krieger, S. (1959). *Theory of plates and shells*. McGraw-hill.