Optimal Multi-antenna Transmission with Multiple Power Constraints

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***Abstract*—We determine the capacity-optimal transmission strategy for a multiple-input-multiple-output (MIMO) Gaussian channel under multiple power constraints, namely joint sum power constraint (SPC), per group power constraints (PGPC), and per antenna power constraints (PAPC). First, we focus on cases where we can analytically determine the optimal transmit strategy under Joint SPC-PGPC-PAPC. We obtain results for the following cases: (1) $n_t \times 1$ Multiple-Input-Single-Output (MISO), (2) MIMO channel with full column rank and full rank optimal covariance matrix, and (3) $2 \times n_r$ MIMO channel. These results generalize some recent results for the special cases of PAPC only and Joint SPC-PAPC. Then, we propose a Projected Factored Gradient Descent (PFGD) algorithm for the general MIMO Gaussian channel under Joint SPC-PGPC-PAPC including the possibility of additional rank constraints. This algorithm matches the solution of standard convex optimization tools with lower complexity. The algorithm also overcomes the limitations of existing algorithms in terms accuracy and applicability to low rank channels.**

I. Introduction

THE capacity of multiple-input multiple-output (MIMO) Gaussian channels under a sum-power constraint (SPC) was obtained in [3]. Gaussian signalling with a transmit covariance matrix determined using the singular value decomposition (SVD) of the channel matrix and a water-filling algorithm is optimal under the SPC.

In general, a multi-antenna system may have multiple simultaneous transmit power contraints. The SPC limits the total power used by the transmitter. Such a constraint is usually imposed by regulations and by the need to limit the total energy consumption. Per-antenna power contraints (PAPC) and per-group power constraints (PGPC) may arise due to hardware limitations in sharing the total available power across antennas. In distributed antenna systems, the transmit antennas are spread across multiple locations and are not driven by the same power amplifier. In such a setting, the total power cannot be arbitrarily allocated across the different geographically separated antennas. Such a situation arises in cellular systems using coordinated multipoint transmission (CoMP) [4] and in cell-free massive MIMO [5]. If the multiple transmission points in CoMP are single antenna transmitters, we get PAPC. If they are multi-antenna transmission points, we get PGPC. PAPC or PGPC also arise in the computation of the cutset bound in network information theory, eg., [6, Appendix B] and [7, Sec. III]. In the evaluation of the cutset bound for a network, the information flow across any cut, from the transmitters on one side to the receivers on the other side, is usually upper bounded by the cooperative MIMO capacity. However, when each node has its own power constraint in the network, we get MIMO capacity under PAPC or PGPC depending on whether the node has single or multiple transmit antennas. Therefore, it is useful to determine the optimal MIMO scheme under multiple simultaneous power constraints, i.e., SPC, PAPC and PGPC.

While Gaussian signaling is optimal even under multiple power constraints, there is no general analytical solution for the optimal transmit covariance matrix and the capacity as in the case of SPC. Exact analytical solutions are limited to the MISO and some full rank MIMO settings. The MISO case has been studied in [8–12]. In [8], the capacity and optimal covariance matrix are obtained in closed form for the multiple-input single-output (MISO) case under PAPC. In [9, 10], the MISO case under Joint SPC-PAPC has been solved. PGPC has been considered for the more general MISO broadcast channel in [11, 12], and *numerical* algorithms based on uplink-downlink duality have been proposed for transmitter optimization in this broadcast setting. In [13], a closed-form solution for MIMO capacity under PAPC is obtained when the channel matrix has full column rank and the optimal covariance matrix is also full rank.

Capacity of MIMO Gaussian channels under PAPC has been studied in [14, 15]. In [14], an iterative algorithm is proposed to compute the capacity for single-stream MIMO under PAPC and multi-stream MIMO with *per-stream* PAPC. No closed form solutions are

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provided in [14]. In [15], an algorithm to compute the MIMO capacity under PAPC is proposed for the case when the channel matrix has full column rank or full row rank. MIMO capacity under PGPC has been studied in [16]. However, the closed form solution in [16] is only an *approximate* solution and is exact only for the case considered in [13]. In [17], an iterative algorithm to determine MIMO capacity under Joint SPC-PAPC is proposed based on the approximate solution under PGPC in [16].

In this paper, we study MIMO capacity under multiple simultaneous power constraints - SPC, PAPC and PGPC. First, we focus on *analytical* solutions under Joint SPC-PGPC-PAPC. We provide solutions for the (1) MISO channel, (2) MIMO channel with full column rank when the optimal covariance matrix is also full rank, and (3) $2 \times n_r$ MIMO channel. For the MISO case, our results generalize the results in [9, 10] to the case of Joint-SPC-PGPC-PAPC. Then, for the general MIMO case, we propose a projected factored gradient descent (PFGD) algorithm to find the optimal transmission strategy under Joint SPC-PGPC-PAPC. In this method, instead of solving for the covariance matrix, the algorithm determines the precoding/beamforming matrix (i.e., square root of the covariance matrix) directly. Numerical results show that the solution from the PFGD algorithm matches with the solution provided by standard convex optimization packages like CVX [18, 19], but with lower complexity. The solution also matches the solution in [15] for the special case of PAPC for channel matrices with full column rank or full row rank. While [16] and [17] provide solutions for the PGPC and Joint SPC-PAPC cases, respectively, the solutions are approximate and do not match CVX. For the PFGD algorithm, we also include the possibility of a rank constraint in addition to Joint SPC-PGPC-PAPC. Such a constraint is motivated by systems where the number of transmit antennas is large but the number of spatial streams transmitted is limited by the number of receive antennas or the rank of the channel. The PFGD algorithm can take advantage of low rank structure for a reduced complexity solution. Both the Joint SPC-PGPC-PAPC and the rank-constrained Joint SPC-PGPC-PAPC problems have not been studied earlier in the literature.

The rest of the paper is organised as follows. In Section II, we discuss the system model and the different power and rank constraints. In Section III, we present the results for MISO under Joint SPC-PGPC-PAPC. In Section IV, we present the analytical results for some special cases of the MIMO channel under Joint SPC-PGPC-PAPC. In Section V, we propose the PFGD algorithm to find the optimal transmission strategy for the general MIMO case under Joint-SPC-PGPC-PAPC and rank constraints. In Section VI, we present numerical results, and summarize the work in Section VII.

II. System model

Consider a MIMO Gaussian channel with n_t transmitters and n_r receivers. The channel output vector at the receiver is given by the linear model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}, \tag{1}$$

where $\mathbf{H} \in \mathbb{C}^{n_r \times n_t}$ is the channel matrix which is assumed to be perfectly known at both transmitter and receiver, $\mathbf{y} \in \mathbb{C}^{n_r}$ is the received vector, $\mathbf{x} \in \mathbb{C}^{n_t}$ is the transmit vector, and $\mathbf{z} \in \mathbb{C}^{n_r}$ is zero mean circularly symmetric complex Gaussian noise with $E[\mathbf{z}\mathbf{z}^H] = \mathbf{I}_{n_r}$. Let $E[\mathbf{x}\mathbf{x}^H] \triangleq \mathbf{Q} = [Q_{ij}]$ be the transmit covariance matrix.

We consider MIMO capacity under three simultaneous power constraints, namely SPC, PAPC and PGPC. These constraints are briefly described here.

1) Sum Power Constraint (SPC): Under SPC, the total average transmit power across the n_t transmit antennas is limited to P_{tot}. Mathematically, the set of all feasible transmit covariance matrices $\mathbf{Q}$ that satisfy SPC P_{tot} is given by $S_{spc} := \{\mathbf{Q} \succeq 0 : \text{tr}(\mathbf{Q}) \le P_{tot}\}$.

2) Per-antenna Power Constraints (PAPC): Under PAPC, the average transmit power of the i^{th} transmit antenna is limited to $\hat{P}_i$. The set of feasible transmit covariance matrices $\mathbf{Q}$ that satisfy PAPC is $S_{papc} := \{\mathbf{Q} \succeq 0 : Q_{ii} \le \hat{P}_i,\ i = 1, 2, \cdots, n_t\}$.

3) Per-group Power Constraints (PGPC): The n_t transmit antennas are partitioned into g disjoint groups of antennas with n_k antennas in the k^{th} group. Let $I(k)$ be the set of indices of the antennas in k^{th} group. Under PGPC, the k^{th} group of antennas has an average sum transmit power constraint $\tilde{P}_k$. The feasible set for $\mathbf{Q}$ in this case is $S_{pgpc} := \{\mathbf{Q} \succeq 0 : \sum_{j \in I(k)} Q_{jj} \le \tilde{P}_k;\ k = 1, 2, \cdots, g\}$.

In summary, we solve the following Joint SPC-PGPC-PAPC problem.

$$\begin{aligned} \max_{\mathbf{Q}} \quad & \log |I_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^H| \\ \text{s.t.} \quad & \mathbf{Q} \in S_J \triangleq S_{spc} \cap S_{pgpc} \cap S_{papc}. \end{aligned} \tag{2}$$

We denote the optimal transmit covariance matrix for this problem as $\mathbf{Q}^{(J)}$.

In Section V, we also consider the possibility of a rank constraint on the transmit covariance matrix, i.e., $\text{rank}(\mathbf{Q}) \le r$, in addition to power constraints. If we choose a large enough r, for example $r = \min\{n_t, n_r\}$, then the rank constrained problem is equivalent to problem without rank constraint in (2). The rank constraint is motivated by applications where the number of transmit antennas are large, but the number of spatial streams is limited by the number of receivers or by the channel rank. The rank-constrained Joint SPC-PGPC-PAPC problem is:

$$\begin{aligned} \max_{\mathbf{Q}} \quad & \log |I_{n_r} + \mathbf{HQH}^H| \\ \text{s.t.} \quad & \mathbf{Q} \in S_{spc} \cap S_{pgpc} \cap S_{papc}, \text{rank}(\mathbf{Q}) \leq r. \end{aligned} \tag{3}$$

For this problem, we denote the constraint set S_R and the optimal transmit covariance matrix $\mathbf{Q}^{(R)}$.

III. MISO under Joint SPC-PGPC-PAPC

In this section, we solve the Joint SPC-PGPC-PAPC problem in (2) for a MISO channel. The channel matrix for the MISO channel is $\mathbf{H} = [h_1,\ h_2,\ \cdots,\ h_{n_t}]$. We will denote this $\mathbf{H}$ as $\mathbf{h}^T$. The MISO problem can be written as:

$$\max_{\mathbf{Q} \in S_J} \mathbf{h}^T \mathbf{Q} \mathbf{h}^*, \tag{4}$$

where $\mathbf{h}^*$ denotes the conjugate of $\mathbf{h}$. First, we note that it is sufficient to consider the case where $P_{tot} \leq \sum_{i=1}^{g} \tilde{P}_i$ and $\tilde{P}_i \leq \sum_{j \in I(i)} \hat{P}_j,\ \forall i = 1, 2, \cdots, g$. If $P_{tot} > \sum_{i=1}^{g} \tilde{P}_i$, then the SPC is redundant and we can set the sum power constraint $P_{tot} = \sum_{i=1}^{g} \tilde{P}_i$ without any loss in capacity. Similarly for any i, if $\tilde{P}_i > \sum_{j \in I(i)} \hat{P}_j$, we can set $\tilde{P}_i = \sum_{j \in I(i)} \hat{P}_j$ without any loss in capacity.

First, we observe in Proposition 1 that the optimal solution uses the full available power P_{tot}.

Proposition 1. *For Joint SPC-PGPC-PAPC, with $P_{tot} \leq \sum_{k=1}^{g} \tilde{P}_k$, and $\tilde{P}_i \leq \sum_{j \in I(i)} \hat{P}_j,\ \forall i = 1, 2, \cdots, g$, the optimal strategy $\mathbf{Q}^{(J)}$ uses all the sum power, i.e., tr($\mathbf{Q}^{(J)}$) $= P_{tot}$.*

Proof. See Appendix A. This proof relies on the observation that if $\mathbf{Q}_1 - \mathbf{Q}_2$ is positive definite, the rate achieved with $\mathbf{Q}_1$ is larger than the rate achieved with $\mathbf{Q}_2$. □

For the MISO channel, it is already known that beamforming, i.e., rank-one transmission, is optimal for under SPC alone [3], under PAPC alone [8] and under Joint SPC-PAPC [9]. In the following proposition, we show the optimality of rank-one transmission for general Joint SPC-PGPC-PAPC problem. The main assumption, without loss of generality, is that $h_i \neq 0$ for each i. In case any $h_i = 0$, we can always remove the corresponding antenna from the model and consider the other antennas alone.

Proposition 2. *(Optimality of beamforming) For a MISO channel $\mathbf{h} \in \mathbb{C}^{n_t \times 1}$ with $h_i \neq 0, \forall i \in \{1, 2, \cdots, n_t\}$, under Joint SPC-PGPC-PAPC, beamforming is the optimal transmission strategy, i.e., rank($\mathbf{Q}^{(J)}$) $= 1$.*

Proof. See Appendix B. We argue that the rank of the optimal transmission scheme is upper bounded by the rank of the channel, which is 1 in the MISO case. □

Now, given Proposition 2, let $\mathbf{q}^{(J)}$ be the optimal beamforming vector corresponding to optimal $\mathbf{Q}^{(J)}$, i.e., $\mathbf{Q}^{(J)} = \mathbf{q}^{(J)} \mathbf{q}^{(J)H}$. The next Lemma follows directly from [9, Lemma 1].

Lemma 1. *(Phase of the entries of the optimal beamformer) The phase of the j^{th} element $q_j^{(J)}$ of $\mathbf{q}^{(J)}$ is equal to negative of the phase of $h_j, j = 1, 2, \cdots, n_t$, i.e.,*

$$\mathbf{q}^{(J)} \in \mathbb{Q}^J := \Bigg\{ \mathbf{q} : \mathbf{q} = \Bigg[\frac{\sqrt{P_1} h_1^*}{|h_1|}, \frac{\sqrt{P_2} h_2^*}{|h_2|}, \cdots, \frac{\sqrt{P_{n_t}} h_{n_t}^*}{|h_{n_t}|} \Bigg], \mathbf{qq}^H \in S_J \Bigg\}.$$

Here P_j denotes the transmit power of the j^{th} antenna.

Proof. The proof in [9] is written for problem under Joint SPC-PAPC. However, the same steps apply for the Joint SPC-PGPC-PAPC problem. □

From Proposition 2 and Lemma 1, the problem of finding the optimal covariance matrix $\mathbf{Q}^{(J)}$ now reduces to finding the optimal power allocation for each antenna. We will denote this optimal power allocation $P_1^{(J)}, P_2^{(J)}, \ldots, P_{n_t}^{(J)}$. While solving for this optimal power allocation, we will consider two relaxations of the Joint SPC-PGPC-PAPC problem, namely the Joint SPC-PGPC problem and the SPC problem, in the intermediate steps. We will denote the solution to the Joint SPC-PGPC problem by $P_1^{(1)}, P_2^{(1)}, \ldots, P_{n_t}^{(1)}$ and the solution to the SPC problem by $P_1^{(2)}, P_2^{(2)}, \ldots, P_{n_t}^{(2)}$.

The optimal solution $P_1^{(J)}, P_2^{(J)}, \ldots, P_{n_t}^{(J)}$, for the Joint SPC-PGPC-PAPC problem is determined as shown in the flow chart in Fig. 1. The channel $\mathbf{H}$ and the power constraints are the inputs. First, we compute the optimal powers for the Joint SPC-PGPC problem as derived in Theorem 1 in this section. Then, we check if this solution $P_1^{(1)}, P_2^{(1)}, \ldots, P_{n_t}^{(1)}$ satisfies PAPC. If PAPC is

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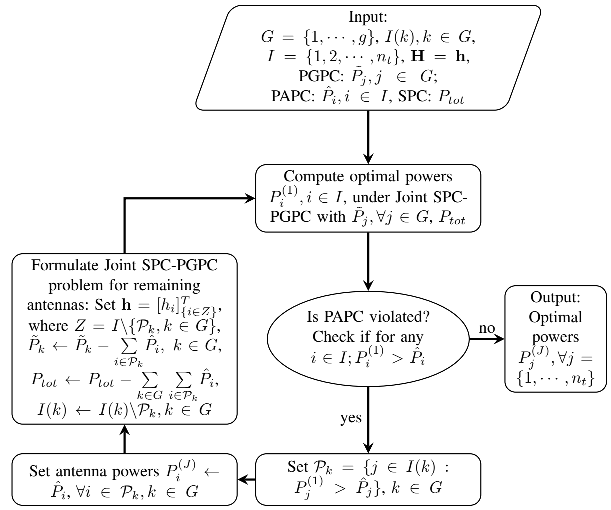

Fig. 1: Flow chart to find optimal powers for MISO under Joint-SPC-PGPC-PAPC

satisfied, then we have the optimal Joint SPC-PGPC-PAPC solution to be the same as the Joint SPC-PGPC solution. If PAPC is violated for some antennas, then we can show that these antennas are allocated the maximum power allowed by PAPC in the final Joint SPC-PGPC-PAPC solution. The justification for this step is based on the following Lemma similar to [9, Lemma 2].

Lemma 2. *Let* $C \subseteq I \triangleq \{1, 2, \cdots, n_t\}$, *and* $S(C) := \{\mathbf{Q} \succeq 0,\ tr(\mathbf{Q}) \le P_{tot},\ \sum_{j \in I(k)} Q_{jj} \le \tilde{P}_k,\ k \in G, Q_{ii} \le \hat{P}_i, i \in C\}$. *Let* $P_i^{(S(C))}$ *be the optimal power allocation under the power constraint set* $S(C)$. *Let* $D := \{i \in I \backslash C : P_i^{(S(C))} > \hat{P}_i\}$ *and* $C' = C \cup D$. *If* $D \neq \phi$, *then* $P_i^{(S(C'))} = \hat{P}_i,\ \forall i \in D$.

Proof. The proof by contradiction uses an argument similar to the proof of Lemma 2 in [9] and is skipped. It is shown that if $P_i^{(S(C'))} < \hat{P}_i$, for any $i \in D$, then $P_i^{(S(C))}$ could not have been the optimal power allocation under the power constraint set $S(C)$. This mainly relies on the fact that each PAPC constraint is on a different power variable. □

Therefore, once we check PAPC for the Joint SPC-PGPC solution, we know the optimal power under Joint SPC-PGPC-PAPC for at least one antenna. Now, we can remove the antennas whose optimal power have been determined and formulate a new Joint SPC-PGPC problem by modifying $\tilde{P}_k$, P_{tot} and $I(k)$ as described in Fig. 1. Repeating the above steps, we determine the optimal Joint SPC-PGPC-PAPC solution in at most n_t steps.

As mentioned above, the optimal solution for the Joint SPC-PGPC problem is given in the following Theorem.

Theorem 1. *(Closed form solution for Joint SPC-PGPC problem) Suppose that we order groups such that*

$$\frac{\sum_{j \in I(1)} |h_j|^2}{\tilde{P}_1} \ge \frac{\sum_{j \in I(2)} |h_j|^2}{\tilde{P}_2} \ge \cdots \ge \frac{\sum_{j \in I(g)} |h_j|^2}{\tilde{P}_g} \tag{5}$$

is satisfied. Let antenna j *belong to group* i, *i.e.,* $j \in I(i)$. *Then, the optimal powers* $P_j^{(1)}$, $j = \{1, 2, \cdots, n_t\}$ *are given by*

$$P_j^{(1)} = \begin{cases} \left(\dfrac{\tilde{P}_i}{\sum_{r \in I(i)} |h_r|^2} \right) |h_j|^2 & \text{if } i = 1, 2, \cdots, k \\ \left(\dfrac{P_{tot} - \sum_{j=1}^{k} \tilde{P}_j}{\sum_{\substack{r \in I(i), \\ \forall i \ge k+1}} |h_r|^2} \right) |h_j|^2 & \text{if } i = k+1, \cdots, g \end{cases},$$

where k *is the number of active PGPC constraints and is determined by the least solution of*

$$\frac{P_{tot} - \sum_{j=1}^{k} \tilde{P}_j}{\sum_{\substack{j \in I(i), \\ \forall i \ge k+1}} |h_j|^2} \le \frac{\tilde{P}_{k+1}}{\sum_{j \in I(k+1)} |h_j|^2}. \tag{6}$$

Proof. See Appendix C. □

We end this section with the following remarks.

- The above Joint SPC-PGPC result in Theorem 1 generalizes the closed form solution in [10] from the Joint SPC-PAPC to the Joint SPC-PGPC case.
- In the proof of Theorem 1, we explicitly show how the closed form solution is obtained by first ordering the groups and solving a sequence of SPC problems. One of the important steps here is to identify the criteria for ordering in (5).
- For the case where each group has just one antenna, i.e., Joint SPC-PAPC, this shows how the closed form result in [10] can be obtained by ordering the antennas and then using the procedure in [9].
- We have also shown how the Joint SPC-PGPC solution can be used to solve the more general Joint SPC-PGPC-PAPC problem.

IV. Special cases of MIMO channels

A. MIMO with full column rank and full rank optimal Q under Joint-SPC-PGPC-PAPC

In this section, we consider the MIMO channel where the $n_r \times n_t$ channel matrix $\mathbf{H}$ has full column rank and the optimal covariance matrix $\mathbf{Q}$ also has full rank. Such

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a MIMO channel has been considered under PAPC in [13], and is generally useful in high SNR settings. For the more general case, we propose an algorithm later in Section V. In this section, we find the optimal transmit covariance matrix under Joint SPC-PGPC-PAPC.

Let $\mathbf{H} = \mathbf{U}_h \mathbf{D} \mathbf{V}_h^H$ be the SVD of $\mathbf{H}$. The optimization problem (2) can now be rewritten as

$$\begin{aligned} \max_{\mathbf{Q}} \quad & \log |\mathbf{I}_{n_r} + \mathbf{D}\mathbf{V}_h^H \mathbf{Q} \mathbf{V}_h \mathbf{D}| \\ \text{s.t.} \quad & \text{tr}(\mathbf{Q}) \le P_{tot}, \sum_{j \in I(i)} Q_{jj} \le \tilde{P}_i, \ i = 1, 2, \cdots, g, \\ & Q_{jj} \le \hat{P}_i, \ j = 1, 2, \cdots, n_t, \mathbf{Q} \succeq 0. \end{aligned} \tag{7}$$

As in Section III, we consider the case where $P_{tot} \le \sum_{i=1}^{g} \tilde{P}_i$ and $\tilde{P}_i \le \sum_{j \in I(i)} \hat{P}_j, \ \forall i = 1, \ldots, g$. We define the following notation:

$$(\mathbf{H}^H \mathbf{H})^{-1} \triangleq \mathbf{A} \triangleq [A_{ij}], \sum_{j \in I(i)} (\mathbf{H}^H \mathbf{H})^{-1}_{jj} \triangleq a_i. \tag{8}$$

The Lagrangian for the above problem is

$$L(\mathbf{Q}, \mathbf{K}, \{\gamma_k\}, \{\lambda_k\}, \lambda) = \log |\mathbf{I}_{n_r} + \mathbf{D}\mathbf{V}_h^H \mathbf{Q} \mathbf{V}_h \mathbf{D}| + \text{tr}(\mathbf{K}\mathbf{Q} - \mathbf{\Gamma}\mathbf{Q}) + \sum_{i=1}^{n_t} \gamma_i \hat{P}_i - \text{tr}(\mathbf{\Lambda}\mathbf{Q}) + \sum_{i=1}^{g} \lambda_i \tilde{P}_i - \lambda(\text{tr}(\mathbf{Q})),$$

where $\mathbf{K}$, $\gamma_i, i = 1, 2, \cdots, n_t$, $\lambda_i, i = 1, 2, \cdots, g, \lambda$ are Lagrange multipliers for the positive semidefinite constraints PAPC, PGPC, and SPC, respectively.

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 \mathbf{I}_{n_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 \mathbf{I}_{n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_g \mathbf{I}_{n_g} \end{bmatrix} \tag{9}$$

and $\mathbf{\Gamma}$ is a diagonal matrix with diagonal entries γ_1, γ_2, ..., γ_{n_t}. At the optimal $\mathbf{Q}^{(J)}$, we have $\frac{\partial L}{\partial \mathbf{Q}^{(J)}} = 0$, i.e.,

$$\mathbf{V}_h(\mathbf{D}^{-2} + \mathbf{V}_h^H \mathbf{Q}^{(J)} \mathbf{V}_h)^{-1} \mathbf{V}_h^H - \mathbf{\Gamma} - \mathbf{\Lambda} - \lambda \mathbf{I} + \mathbf{K} = \mathbf{0}. \tag{10}$$

If $\mathbf{Q}^{(J)}$ is full rank, then $\mathbf{K} = 0$. Therefore, from (10), we have

$$\mathbf{V}_h(\mathbf{D}^{-2} + \mathbf{V}_h^H \mathbf{Q}^{(J)} \mathbf{V}_h)^{-1} \mathbf{V}_h^H = \mathbf{\Gamma} + \mathbf{\Lambda} + \lambda \mathbf{I}, \tag{11}$$

$$\begin{aligned} \text{i.e., } \mathbf{Q}^{(J)} &= (\mathbf{\Gamma} + \mathbf{\Lambda} + \lambda \mathbf{I})^{-1} - \mathbf{V}_h \mathbf{D}^{-2} \mathbf{V}_h^H \\ &= (\mathbf{\Gamma} + \mathbf{\Lambda} + \lambda \mathbf{I})^{-1} - (\mathbf{H}^H \mathbf{H})^{-1}, \end{aligned} \tag{12}$$

and $\log |\mathbf{I}_{n_r} + \mathbf{D}\mathbf{V}_h^H \mathbf{Q} \mathbf{V}_h \mathbf{D}| = \log |\mathbf{H}^H \mathbf{H}(\mathbf{\Gamma} + \mathbf{\Lambda} + \lambda \mathbf{I})^{-1}|$. Since, $\mathbf{\Gamma} + \mathbf{\Lambda} + \lambda \mathbf{I}$ is a diagonal matrix, the only unknowns in $\mathbf{Q}^{(J)}$ are its diagonal entries.

Let the diagonal values of $\mathbf{D}^2$ or the eigen values of $\mathbf{H}^H \mathbf{H}$ be d_i^2, $i = 1, 2, \cdots, n_t$. Then, $\log |\mathbf{H}^H \mathbf{H}(\mathbf{\Gamma} + \mathbf{\Lambda} + \lambda \mathbf{I})^{-1}| = \log \left(\prod_{j=1}^{n_t} [(Q_{jj} + A_{jj}) d_j^2] \right)$. The optimization problem is now

$$\begin{aligned} \max_{\mathbf{Q}} \quad & \log \left(\prod_{j=1}^{n_t} [(Q_{jj} + A_{jj}) d_j^2] \right) \\ \text{s.t.} \quad & \sum_{i=1}^{n_t} Q_{jj} \le P_{tot}, \sum_{j \in I(i)} Q_{jj} \le \tilde{P}_i, \ i = 1, 2, \cdots, g, \\ & Q_{jj} \le \hat{P}_i, \ j = 1, 2, \cdots, n_t, \mathbf{Q} \succeq 0. \end{aligned} \tag{13}$$

The optimal entries $Q_{jj}^{(J)}$, $j = 1, 2, \cdots, n_t$, or $P_1^{(J)}, P_2^{(J)}, \ldots, P_{n_t}^{(J)}$, for the Joint SPC-PGPC-PAPC problem are now determined using similar steps as in the flow chart in Fig. 1 except for a change in the formulation of the reduced Joint SPC-PGPC problem. The matrix $\mathbf{A}$ and the power constraints are the inputs. First, we compute the optimal powers for the Joint SPC-PGPC problem as derived in Theorem 2 in this section. Then, we check if this solution $P_1^{(1)}, P_2^{(1)}, \ldots, P_{n_t}^{(1)}$ satisfies PAPC. If PAPC is satisfied, then we have the optimal Joint SPC-PGPC-PAPC solution to be the same as the Joint SPC-PGPC solution. If PAPC is violated for some antennas, then we can show that these antennas are allocated the maximum power allowed by PAPC in the final Joint SPC-PGPC-PAPC solution. The justification for this step is again based on Lemma 2. Therefore, once we check PAPC for the Joint SPC-PGPC solution, we know the optimal power under Joint SPC-PGPC-PAPC for at least one antenna. Now, we can remove the antennas whose optimal power have been determined and formulate a new Joint SPC-PGPC problem by modifying $\mathbf{A}$, $\tilde{P}_k$, P_{tot} and $I(k)$ as follows: $\mathbf{A} = [A_{ij}]_{\{i,j \in Z\}}$, where $Z = I \backslash \{\mathcal{P}_k, k \in G\}$, where $\mathcal{P}_k$ is the set of antennas violating PAPC in group k, and make $\tilde{P}_k \leftarrow \tilde{P}_k - \sum_{i \in \mathcal{P}_k} \hat{P}_i$, $\forall k \in G$, $P_{tot} \leftarrow P_{tot} - \sum_{k \in G} \sum_{i \in \mathcal{P}_k} \hat{P}_i$, $I(k) \leftarrow I(k) \backslash \mathcal{P}_k, k \in G$. Unlike the MISO case, note that the reduced problem here is obtained by modifying $\mathbf{A}$ and not $\mathbf{H}$. Repeating the above steps, we determine the optimal Joint SPC-PGPC-PAPC solution in at most n_t steps.

The optimal powers for Joint SPC-PGPC problem are found in the following Theorem. Here again, we identify a criteria for ordering the groups, and then obtain the solution using a sequence of SPC problems.

Theorem 2. *Assume $P_{tot} \le \sum_{k=1}^{g} \tilde{P}_k$. If the groups of $\mathbf{A}$ are arranged such that*

$$\frac{a_1 + \tilde{P}_1}{n_1} \le \frac{a_2 + \tilde{P}_2}{n_2} \le \cdots \le \frac{a_g + \tilde{P}_g}{n_g}. \tag{14}$$

is satisfied, and if $(\mathbf{\Lambda}+\lambda\mathbf{I})^{-1}-\mathbf{A}\succ 0$*, where* Λ *is as defined in (9) and* $\Lambda+\lambda\mathbf{I}$ *is found using*

$$(\lambda+\lambda_i)^{-1}=\begin{cases}\frac{\tilde{P}_i+a_i}{n_i} & \text{for } i=1,2,\cdots,k\\ \frac{P_{tot}+\sum\limits_{j>k}a_j-\sum\limits_{j=1}^{k}\tilde{P}_j}{\sum\limits_{j>k}n_j} & \text{for } i=k+1,\cdots,g\end{cases} \tag{15}$$

where k *is the smallest value in* $\{0,1,\ldots,g\}$ *for which*

$$\frac{\tilde{P}_{k+1}+a_{k+1}}{n_{k+1}}\geq\frac{P_{tot}+\sum\limits_{j>k}a_j-\sum\limits_{j=1}^{k}\tilde{P}_j}{\sum\limits_{j>k}n_j}, \tag{16}$$

then the optimal solution for Joint-SPC-PGPC is $\mathbf{Q}^{(1)}=(\mathbf{\Lambda}+\lambda\mathbf{I})^{-1}-\mathbf{A}$.

Proof. See Appendix D. □

Now, we make some observations about the Joint-SPC-PGPC solution in Theorem 2.

1) Note that the optimal $\mathbf{Q}$ under SPC (for full rank $\mathbf{Q}$) is

$$Q_{jj}^{(2)}=\frac{1}{\lambda}-A_{jj}$$

where $\lambda^{-1}=\frac{P_{tot}+\text{tr}(\mathbf{A})}{n_t}$.

2) k is the least solution of (16). We can find the optimal k by starting from $k=0$ and sequentially checking upto $k=g$ if (16) is satisfied.

3) For $k=0$, (16) reduces to $\frac{\tilde{P}_1+a_1}{n_1}\geq\frac{P_{tot}+\sum\limits_{j\geq 1}a_j}{\sum\limits_{j\geq 1}n_j}$. This condition can be rewritten as follows:

$$\tilde{P}_1\geq n_1\left(\frac{P_{tot}+\sum\limits_{j\geq 1}a_j}{\sum\limits_{j\geq 1}n_j}\right)-a_1$$

$$\Longrightarrow\ \tilde{P}_1\geq\sum_{j\in I(1)}Q_{jj}^{(2)}.$$

This is the same as checking if the SPC solution satisfies the group power constraint for the first group. If this is true, the other group power constraints are also satisfied by the SPC solution because of ordering and the SPC solution becomes the optimal solution for the Joint-SPC-PGPC problem.

4) For $k=i-1$, (16) reduces to $\frac{\tilde{P}_i+a_i}{n_i}\geq\frac{P_{tot}+\sum\limits_{j\geq i}a_j-\sum\limits_{j=1}^{i-1}\tilde{P}_j}{\sum\limits_{j\geq i}n_j}$. This condition can be rewritten as follows:

$$\tilde{P}_i\geq n_i\left(\frac{\left(P_{tot}-\sum\limits_{j=1}^{i-1}\tilde{P}_j\right)+\sum\limits_{j\geq i}a_j}{\sum\limits_{j\geq i}n_j}\right)-a_i$$

$$\Longrightarrow\ \tilde{P}_i\geq\sum_{j\in I(1)}Q_{jj}^{(2)mod}.$$

This is the same as checking if the SPC solution of a modified problem (denoted by the superscript mod above) – with only groups i to g, sum power constraint $\left(P_{tot}-\sum\limits_{j=1}^{i-1}\tilde{P}_j\right)$, and channel parameters A_j as defined in (8) based on the original $\mathbf{H}$ – satisfies the group power constraint for the i^{th} group. If this is true, the other group power constraints are also satisfied by this SPC solution for the reduced problem. For groups 1 to $i-1$, full group power should be used.

In summary, groups $i=1$ to k use full available group power, and optimal powers for other groups $i>k$ are found by solving a modified SPC problem.

B. $2\times n_r$ MIMO

Note that for $n_t=2$, the only possible PGPC constraints are the SPC and PAPC constraints. Therefore, the Joint SPC-PGPC-PAPC problem is the same as the Joint-SPC-PAPC problem. We consider the case when $P_{tot}\leq\hat{P}_1+\hat{P}_2$. Optimal powers $P_1^{(J)}$ and $P_2^{(J)}$ can be determined as follows. First, we observe that all the available transmit power is used in the optimal solution, i.e., $P_1^{(J)}+P_2^{(J)}=P_{tot}$. For calculating $P_1^{(J)}$, $P_2^{(J)}$, we first find the optimal $P_1^{(2)}$, $P_2^{(2)}$ for the SPC problem. Since $P_{tot}\leq\hat{P}_1+\hat{P}_2$, the PAPC constraints will be violated for only one i, either $i=1$ or $i=2$. For that i, set $P_i^{(J)}=\hat{P}_i$ and calculate the power for the other value of i using $P_1^{(J)}+P_2^{(J)}=P_{tot}$. Let $\mathbf{H}^H\mathbf{H}$ be $\begin{bmatrix}k_1 & k_2\\ k_2^* & k_3\end{bmatrix}$ (here $k_i's$ are known).

Theorem 3. *The optimal* $\mathbf{Q}^{(J)}$ *is as follows:*
If rank$(\mathbf{H})=2$ *and the channel matrix* $\mathbf{H}$ *satisfies* $P_1^{(J)}P_2^{(J)}>\frac{|k_2|^2}{k_1k_3-|k_2|^2}$*, then*

$$\mathbf{Q}^{(J)}=\begin{bmatrix}P_1^{(J)} & \frac{k_2}{k_1k_3-|k_2|^2}\\ \frac{k_2^*}{k_1k_3-|k_2|^2} & P_2^{(J)}\end{bmatrix},$$

else

$$\mathbf{Q}^{(J)} = \begin{bmatrix} P_1^{(J)} & \sqrt{P_1^{(J)}P_2^{(J)}}\frac{k_2}{|k_2|} \\ \sqrt{P_1^{(J)}P_2^{(J)}}\frac{k_2^*}{|k_2|} & P_2^{(J)} \end{bmatrix}.$$

Proof. The problem can be rewritten as

$$\begin{aligned} \max \quad & k_2^*Q_{12} + k_2Q_{12}^* + (k_1k_3 - |k_2|^2)(P_1^{(4)}P_2^{(4)} - |Q_{12}|^2) \\ \text{s.t} \quad & P_1^{(J)}P_2^{(J)} \geq |Q_{12}|^2. \end{aligned}$$

Now, we will solve this problem separately for each possible rank of $\mathbf{H}$, i.e., rank$(\mathbf{H}) = 1$ and rank$(\mathbf{H}) = 2$.

Case (a): *rank*$(\mathbf{H}) = 1$*:* If rank($\mathbf{H}$)=1, then $k_1k_3 = |k_2|^2$. Therefore, we have to solve

$$\max \quad (k_2^*Q_{12} + k_2Q_{12}^*), \quad \text{s.t.} \quad P_1^{(J)}P_2^{(J)} \geq |Q_{12}|^2,$$

or, equivalently, solve $\max\ |k_2|.|Q_{12}|.\cos\theta$ s.t. $P_1^{(J)}P_2^{(J)} \geq |Q_{12}|^2$, where θ is the angle between k_2 and Q_{12}. The maximum of $|k_2|.|Q_{12}|.\cos\theta$ occurs when $|Q_{12}| = \sqrt{P_1^{(J)}P_2^{(J)}}$ and $\cos\theta = 1$, i.e., Q_{12} should be in the direction of k_2. Thus, we obtain optimal value of Q_{12} as $Q_{12} = \sqrt{P_1^{(J)}P_2^{(J)}}\frac{k_2}{|k_2|}$. Note that the rank of the optimal $\mathbf{Q}^{(J)}$ is always 1 when rank($\mathbf{H}$)=1.

Case (b): *rank*$(\mathbf{H}) = 2$*:* In this case, rank of $\mathbf{Q}^{(J)}$ can be either one or two. Suppose its rank is 1, then the solution is the same as the case when rank$(\mathbf{H}) = 1$. Suppose that rank of $\mathbf{Q}^{(J)}$ is 2, then the solution is already found in Theorem 2 and the optimal transmit covariance matrix is $\mathbf{Q}^{(J)} = \begin{bmatrix} P_1^{(J)} & \frac{k_2}{k_1k_3-|k_2|^2} \\ \frac{k_2^*}{k_1k_3-|k_2|^2} & P_2^{(J)} \end{bmatrix}$. □

V. Rank-Constrained MIMO Capacity under Joint SPC-PGPC-PAPC

In this section, we propose a Projected Factored Gradient Descent algorithm (PFGD) algorithm to find the optimal transmission scheme to maximize MIMO capacity under Joint SPC-PGPC-PAPC constraints and an additional rank constraint on $\mathbf{Q}$ as defined in (3). As in Section III, we will consider the case $P_{tot} \leq \sum_{i=1}^{g}\tilde{P}_i$ and $\tilde{P}_i \leq \sum_{j\in I(i)}\hat{P}_j,\ \forall i = 1, 2, \cdots, g$. If r is chosen to be large enough, e.g., $r = \min\{n_t, n_r\}$, then the problem is equivalent to problem under Joint SPC-PGPC-PAPC in (2).

In projected gradient descent, after each gradient descent update the constraints are enforced by a projection onto the constraint set. In the PFGD algorithm, we use the fact that $\mathbf{Q} \succeq 0$ iff $\exists$ a matrix $\mathbf{U}$ such that $\mathbf{Q}$ can be factored as $\mathbf{Q} = \mathbf{U}\mathbf{U}^H$. Then, the problem is reformulated in terms of the factor $\mathbf{U}$. We choose the PFGD approach because:

- The matrix $\mathbf{U}$ is the optimal precoder for multi-antenna transmission that is actually needed to achieve a transmit covariance matrix $\mathbf{Q} = \mathbf{U}\mathbf{U}^H$, and we can directly solve for $\mathbf{U}$ without finding $\mathbf{Q}$.
- The $\mathbf{Q} \succeq 0$ constraint is easily enforced by the factorization $\mathbf{Q} = \mathbf{U}\mathbf{U}^H$.
- A rank constraint of r on $\mathbf{Q}$ can be easily enforced by simply choosing the size of $\mathbf{U}$ to be $n_t \times r$. Such a rank constraint would be very difficult to enforce in an iterative algorithm to directly determine $\mathbf{Q}$.
- Recently, in [20], the PFGD approach has been shown to be a good low complexity approach for solving convex covariance optimization problems even though the problem is non-convex in terms of the factor $\mathbf{U}$.

The effectiveness of our approach will be presented later in this section and by comparing with standard convex optimization using CVX and other approaches to the MIMO capacity problem in [15–17].

The transformed non-convex optimization problem is

$$\begin{aligned} &\max_{\mathbf{U}} \log|\mathbf{I}_{n_r} + \mathbf{H}\mathbf{U}(\mathbf{H}\mathbf{U})^H| \\ &\text{s.t} \sum_{l\in I(k)} (\mathbf{U}\mathbf{U}^H)_{ll} \leq \tilde{P}_k,\ \forall k = 1, \ldots, g, \\ &\sum_{l=1}^{n_t}(\mathbf{U}\mathbf{U}^H)_{ll} \leq P_{tot}, (\mathbf{U}\mathbf{U}^H)_{ii} \leq \hat{P}_i,\ \forall i = 1, \ldots, n_t. \end{aligned}$$

The constraint $\sum_{l\in I(k)} (\mathbf{U}\mathbf{U}^H)_{ll} \leq \tilde{P}_k$ can be rewritten as $\sum_{l\in I(k)} \|\mathbf{u}_l\|^2 \leq \tilde{P}_k$, which is a norm ball, and where $\mathbf{u}_l$ denotes the l^{th} row of matrix $\mathbf{U}$. The ability to easily incorporate the rank constraint is useful when (1) the channel is sparse, and (2) when the number of spatially multiplexed streams is limited by the number of receive antennas.

The proposed PFGD algorithm is given below. Here, $f(\mathbf{Q}) = \log|\mathbf{I}_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^H|$.

---

1. *Initialization:* Start with initial point $\mathbf{U}_0$.
2. *Projected Gradient Descent*: Compute the gradient
   $\nabla_{\mathbf{U}_k} f(\mathbf{U}_k\mathbf{U}_k^H) = 2\mathbf{H}^H(\mathbf{I}_{n_r} + \mathbf{H}\mathbf{U}_k(\mathbf{H}\mathbf{U}_k)^H)^{-1}\mathbf{H}\mathbf{U}_k$.
   Then, $\mathbf{U}_{k+1} = \Pi_e(\mathbf{U}_k + \eta\ \nabla_{\mathbf{U}_k} f(\mathbf{U}_k\mathbf{U}_k^H))$, where $\Pi_e(\mathbf{V})$ is the projection of $\mathbf{V}$ onto the constraint set $\mathcal{C}^{(J)} = \{\mathbf{U} : \sum_{l=1}^{n_t}(\mathbf{U}\mathbf{U}^H)_{ll} \leq P_{tot},\ \sum_{l\in I(k)} (\mathbf{U}\mathbf{U}^H)_{ll} \leq \tilde{P}_k,\ k = 1, 2, \cdots, g,\ (\mathbf{U}\mathbf{U}^H)_{ii} \leq \hat{P}_i,\ \forall i = 1, 2, \cdots, n_t\}$, and η is a step size parameter.

3. *Stopping condition:* Stop when $\|\mathbf{U}_{k+1}-\mathbf{U}_k\|_F < \epsilon$.
4. *Solution:* Optimal $\mathbf{Q}^{(R)} = \mathbf{U}_{k+1}\mathbf{U}_{k+1}^H$ and $C = \log|\mathbf{I}_{n_r} + \mathbf{H}\mathbf{U}_{k+1}(\mathbf{H}\mathbf{U}_{k+1})^H|$.

Note that $\nabla_{\mathbf{U}} f(\mathbf{Q})$ where $\mathbf{Q} = \mathbf{U}\mathbf{U}^H$ is derivative with respect to $\mathbf{U}$ and $\nabla_{\mathbf{Q}} f(\mathbf{Q})$ is derivative with respect to $\mathbf{Q}$ and $\nabla_{\mathbf{U}} f(\mathbf{Q})$ =2 $\nabla_{\mathbf{Q}} f(\mathbf{Q})\mathbf{U}$.

Now, we will describe how to compute the projection of a matrix $\mathbf{V}$ onto the Joint SPC-PGPC-PAPC constraint set $\mathcal{C}^{(J)}$, i.e., to find $\Pi_e(\mathbf{V})$.

A. Projection onto Joint SPC-PGPC-PAPC constraint set

The projection is obtained as follows. First, we find the projection onto the Joint SPC-PGPC constraint set $\mathcal{C}^{(1)}$ and then check if any row i violates PAPC. If it does, we scale that row i such that its norm is equal to the PAPC constraint $\hat{P}_i$. This is justified by the same reasoning as in Lemma 2 but with $C \subseteq G$, $S(C) := \{\text{tr}((\mathbf{U}\mathbf{U}^\mathbf{H})) \leq P_{tot}, \sum_{i\in I(k)} (\mathbf{U}\mathbf{U}^H)_{ii} \leq \tilde{P}_k,\ k \in \{1,2,\cdots,g\},\ (\mathbf{U}\mathbf{U}^H)_{ii} \leq \hat{P}_i, i \in C\}$, and $D := \{k \in \{1,2,\cdots,n_t\}\backslash C : \|\mathbf{u}_i\|^2_{(S(C))} > \hat{P}_i\}$. Then, we remove these rows that have already been scaled and solve a reduced size problem. The number of steps required will be less than or equal to n_t.

The projection onto Joint SPC-PGPC constraint set $\mathcal{C}^{(1)}$ is derived in the following theorem. We note here that each row of a matrix after projection is just a scaled version of the row before projection.

Theorem 4. *(Projection onto Joint SPC-PGPC constraint set with respect to Frobenius norm) Let $\mathbf{U}^{(*)}$ be the projection of $\mathbf{V}$ onto $\mathcal{C}^{(1)} = \{\mathbf{U} : \sum_{l=1}^{n_t} (\mathbf{U}\mathbf{U}^H)_{ll} \leq P_{tot}, \sum_{l\in I(k)} (\mathbf{U}\mathbf{U}^H)_{ll} \leq \tilde{P}_k,\ k = 1,2,\cdots g\}$, i.e., $\Pi_e(\mathbf{V}) = \mathbf{U}^{(*)}$. We arrange groups (indexed by k) in ascending order of $\frac{\tilde{P}_k}{\sum_{l\in I(k)} \|\mathbf{v}_l\|^2}$, and place them in an ordered set R. Let $R(j),\ j \in \{1,2,\cdots,g\}$ denote the j^{th} element of R. Then, each row $\mathbf{u}_l^{(*)}$ is a scaled version of original row $\mathbf{v}_l$, i.e., $\mathbf{u}_l^{(*)} = \xi_l.\mathbf{v}_l$, where*

$$\xi_l = \begin{cases} \sqrt{\frac{\tilde{P}_i}{\sum_{l\in I(i)} \|\mathbf{v}_l\|^2}},\ l \in I(i),\ i \in \{R(1),\cdots,R(k)\} \\ \sqrt{\frac{P_{tot} - \sum_{i=R(1)}^{R(k)} \tilde{P}_i}{\sum_{i=R(k+1)}^{R(g)} \left(\sum_{l\in I(i)} \|\mathbf{v}_l\|^2\right)}},\ else \end{cases},$$

where k is the least element in $\{0,1,2,\cdots,g\}$, such that

$$\frac{\tilde{P}_{R(k+1)}}{\sum_{l\in I(R(k+1))} \|\mathbf{v}_l\|^2} \geq \frac{P_{tot} - \sum_{i=R(1)}^{R(k)} \tilde{P}_i}{\sum_{i=R(k+1)}^{R(g)} \left(\sum_{l\in I(i)} \|\mathbf{v}_l\|^2\right)}. \quad (17)$$

Proof. See Appendix E. □

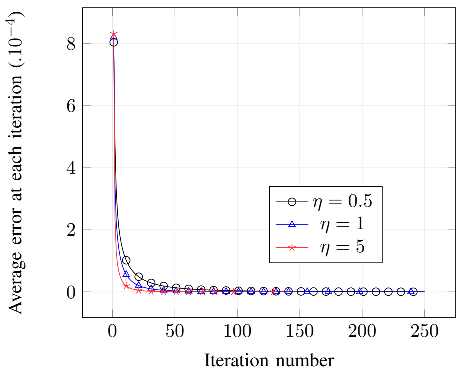
B. Initialization and Convergence

Initialization of $\mathbf{U}$ is done as suggested in [20]. Let $\mathbf{X}_0 = \frac{1}{\|\nabla_Q f(\mathbf{0}) - \nabla_Q f(\mathbf{e}_1\mathbf{e}_1^H)\|_F}\Pi_+(\nabla_{\mathbf{Q}} f(\mathbf{0}))$, where $\Pi_+(R)$ is the projection of R onto the set of positive semi-definite (PSD) matrices. In our setting, $\Pi_+(\nabla_Q f(\mathbf{0})) = \mathbf{H}^\mathbf{H}\mathbf{H}$. Compute $\widetilde{\mathbf{U}}_0$ such that $\mathbf{X}_0 = \widetilde{\mathbf{U}}_0\widetilde{\mathbf{U}}_0^H$, and find the initial point by projecting $\widetilde{\mathbf{U}}_0$ onto the power constraint region, i.e., $\mathbf{U}_0 = \Pi_e(\widetilde{\mathbf{U}}_0)$.

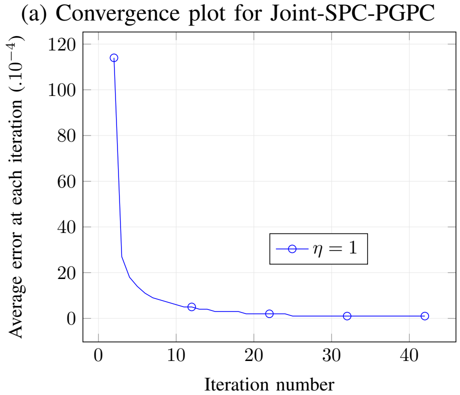
Comments on Convergence: In [20, Theorem 3.1], the local convergence of the PFGD algorithm is proved under the following assumptions: (i) function f is restricted strongly convex and smooth, (ii) constraint set $\mathcal{C}$ is convex, compact and faithful, (iii) projection operation is an entry-wise scaling, i.e., $\Pi_e(\mathbf{V}) = \mathbf{V}$ if $\mathbf{V} \in \mathcal{C}$, and $\Pi_e(\mathbf{V}) = \xi.\mathbf{V}$ if $\mathbf{V} \notin \mathcal{C}$, where $\xi \geq 0.78$ and step size η_k at k^{th} iteration η satisfying $\eta \leq \frac{1}{128(L\|\mathbf{Q}_k\|_2 + \|\nabla_Q f(\mathbf{Q}_k)\|_2)}$, where $L = \|\nabla_Q f(\mathbf{0}) - \nabla_Q f(\mathbf{e}_1\mathbf{e}_1^H)\|_F$.

For our problem, the function f is $\log|\mathbf{I}_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^H|$ and the constraint set is $\mathcal{C}^{(J)}$. It can be shown that conditions (i) and (ii) are satisfied. As determined earlier, the projection step in our problem is a row-wise scaling, i.e., $\Pi_e(\mathbf{V}) = \mathbf{\Xi}.\mathbf{V}$, where $\mathbf{\Xi}$ is a diagonal matrix with diagonal entries $\mathbf{\Xi}_{ii} = \xi_i,\ i = 1,2,\cdots,n_t$. It satisfies (iii) if all the row-wise scaling constants are equal. One case where this is true is the problem under SPC only. Following steps similar to the proof of [20, Theorem 3.1], we can prove the local convergence for the more general row-wise scaling case if $\min\{\xi_i,\ i = 1,2,\cdots,n_t\} \triangleq \xi_{min} \geq 0.78$ and step size η_k at k^{th} iteration satisfying $\eta \leq \frac{1}{128(L\|\mathbf{Q}_k\|_2 + \|\nabla_Q f(\mathbf{Q}_k)\|_2)}$, where $L = \|\nabla_Q f(\mathbf{0}) - \nabla_Q f(\mathbf{e}_1\mathbf{e}_1^H)\|_F$. We skip the detailed proof since it is only a minor modification of the proof in [20, Theorem 3.1].

In our numerical simulations, we observed good convergence for the PFGD algorithm. In Figure 2, we show the average error (averaged over 10000 channel realizations) vs iteration number for different choices of step sizes. In Figure 2a, we plot the average error vs iteration number for Joint-SPC-PGPC problem. In Figure 2b, we plot the average error vs iteration number for Joint-SPC-PGPC-PAPC problem. For both plots, we consider $n_t = n_r = 16$ channels and we assume there are two groups of antennas 1-8, 9-16. We use the power constraints, $\hat{P}_i = 2,\ i = 1, 2, \cdots, 16,\ \tilde{P}_j = \frac{4}{1.1},\ j = 1, 2,\ P_{tot} = \frac{8}{(1.1)^2}$. We observe that the convergence behaviour is good even though the theoretical guarantee is only a local convergence guarantee.



(a) Convergence plot for Joint-SPC-PGPC



(b) Convergence plot for Joint-SPC-PGPC-PAPC

Fig. 2: Convergence of PFGD

Complexity of PFGD algorithm: In every iteration of the PFGD algorithm, we have to find the gradient and do a projection step. The dominant computation is the gradient computation since the project is mainly a scaling operation. Gradient, $\nabla_{\mathbf{U}_k} f(\mathbf{U}_k\mathbf{U}_k^H) = 2\mathbf{H}^H(\mathbf{I}_{n_r} + \mathbf{H}\mathbf{U}_k(\mathbf{H}\mathbf{U}_k)^H)^{-1}\mathbf{H}\mathbf{U}_k$ can be rewritten as $\nabla_{\mathbf{U}_k} f(\mathbf{U}_k\mathbf{U}_k^H) = 2(\mathbf{H}^H\mathbf{H})\mathbf{U}_k(\mathbf{I}_r + \mathbf{U}_k^H(\mathbf{H}^H\mathbf{H})\mathbf{U}_k)^{-1}$ using matrix inversion lemma to reduce complexity. Finding the gradient involves finding inverse of a $r \times r$ matrix and multiplication of $n_t \times n_t$, $n_t \times r$, $r \times r$ matrices. Complexity of inverse operation for an $n \times n$ matrix is $6(2n^3)$ flops and multiplication of $m \times n$ matrix, $n \times p$ matrix is $6(2mnp)$ flops [21]. Here, we count every complex operation as 6 real flops. Therefore, complexity of PFGD algorithm is $6(2r^3 + 2n_t r(n_t + r))$ per iteration. Note that the complexity of the algorithm in [15], for the case where $n_r > n_t$ (complexity of the algorithm for the case $n_r \leq n_t$ is greater than the case of $n_r > n_t$, since it requires one extra inverse and some more multiplications) is higher at $6(20n_t^3)$ flops per iteration, since an eigenvalue decomposition of a $n_t \times n_t$ matrix is required.

VI. Numerical Results

A. Joint SPC-PGPC-PAPC

First, we compare the accuracy and runtime of the PFGD algorithm with CVX. In case of a MISO channel, we also compare the PFGD algorithm with the analytical solution derived in Section III. In Figure 3a, we compare capacity results of PFGD with CVX under Joint SPC-PGPC-PAPC for a randomly generated 4×4 channel given in (18).

Figure 3b is for a randomly generated 4-transmit antenna MISO channel with $\mathbf{H} = [0.3802 + 0.2254i, 1.2968 - 0.9247i, -1.5972 - 0.3066i, 0.6096 + 0.2423i]$. For both these figures, we consider two groups with antennas 1-2, 3-4 in each group with same PAPC $\hat{P}$ at all antennas, a PGPC of $\tilde{P} = \frac{2\hat{P}}{1.1}$ for both the groups and a SPC equal to $\frac{4\hat{P}}{1.1^2}$. We plot capacities found using PFGD, CVX and analytical results by varying PAPC $\hat{P}$, and find that they match well. Note that the analytical solution for the 4×4 MIMO channel in Fig. 3a is valid only when the optimal covariance matrix has full rank of 4. This is satisfied only when $\hat{P} \geq 1$W.

Next, we compare the runtime (in MATLAB) of the PFGD algorithm with the SeDuMi and MOSEK solvers in CVX. In this experiment, we consider the Joint SPC-PGPC problem with fixed number of receiver antennas $n_r = 2$ and vary number of transmit antennas n_t. We consider 2 groups of antennas with equal number of antennas in each group, for example if $n_t = 4$, antennas 1-2 are in first group and antennas 3-4 are in second group. The PGPC constraint for each group $\tilde{P}_k$ is taken as number of antennas in that group, and total power constraint is taken as $P_{tot} = \frac{\tilde{P}_1 + \tilde{P}_2}{1.1}$. Table I shows the average run time of PFGD, SeDuMi and MOSEK over 1000 i.i.d channel realizations. It can be observed that the proposed PFGD algorithm has much less runtime.

TABLE I: Average run time (in seconds)

	$n_t = 4$	$n_t = 8$	$n_t = 16$	$n_t = 32$
PFGD	0.0018	0.0021	0.0026	0.0048
SeDuMi	0.3245	0.3537	0.5236	0.9415
MOSEK	0.1299	0.1404	0.1805	0.3335

Algorithms for some special cases of the Joint SPC-PGPC-PAPC problem have been proposed in [15] (for PAPC and full rank $\mathbf{H}$), [16] (for PGPC) and [17] (for Joint SPC-PAPC). In Figure 4, we compare the capacity

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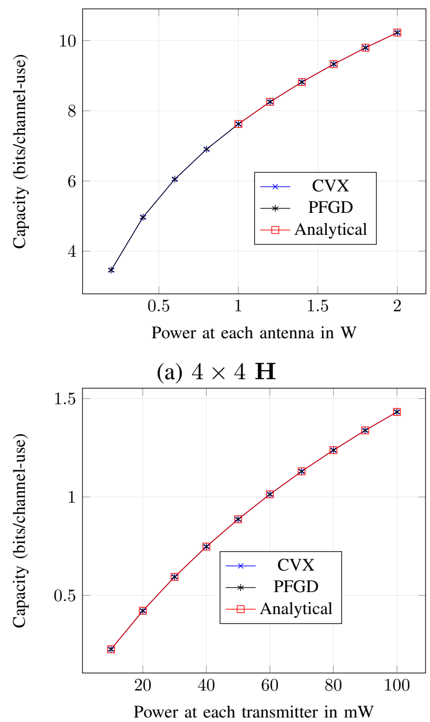
$$\mathbf{H} = \begin{bmatrix} 0.1964 - 0.1395i & -0.8928 + 0.8113i & 0.9790 - 0.4039i & -0.7217 - 0.7936i \\ 0.4522 + 0.2868i & 0.7852 + 0.4228i & -0.0444 + 0.1513i & -2.1729 + 0.2165i \\ -0.0573 - 1.0036i & -0.6997 - 0.9060i & 0.3174 + 0.6664i & 0.4428 - 0.8290i \\ 0.3825 - 0.5158i & -1.2932 - 1.5579i & -0.2569 + 0.0663i & -0.2027 - 0.6795i \end{bmatrix} \tag{18}$$



(a) 4×4 $\mathbf{H}$

(b) MISO $\mathbf{H}$

Fig. 3: Comparison of MIMO capacity under Joint-SPC-PGPC-PAPC

calculated by the PFGD algorithm with CVX, [16], and [15] under PAPC. We take a 4×4 channel with same PAPC $\hat{P}$ at all the antennas and plot for different values of $\hat{P}$. Figure 4a, we use the same full rank channel as used for generating Figure 3a and Figure 4b is for a randomly generated rank 3 channel given in (19).

Note that our algorithm matches CVX for both scenarios. The algorithm in [15] cannot be used if $\mathbf{H}$ is not full column rank or full row rank. The algorithm in [16] is only approximate and does not match CVX.

In Figure 5, we compare the capacity calculated by our PFGD algorithm with CVX and [17] under Joint-SPC-PAPC. We take a 4×4 channel with same PAPC $\hat{P}$ for each antenna and SPC equal to $\frac{4\hat{P}}{1.1}$. We plot the capacities by varying PAPC $\hat{P}$. For Figure 5a, we use the same channel as used for generating Figure 4a and for Figure 5b, we use the same channel used for generating Figure 4b. Note that our algorithm matches CVX in both scenarios. The algorithm in [17] is approximate since it uses the algorithm in [16].

B. Rank-constrained Joint SPC-PGPC-PAPC

Now, we present some simulation results for the rank-constrained capacity Joint-SPC-PGPC-PAPC. We consider two cases: one where the rank is constrained because of the channel rank, and the other where the rank is constrained because the number of transmit streams required is small compared to the number of transmit antennas.

Rank-constrained capacity for the low rank Extended Saleh-Valenzuela Channel Model: In this model, the channel matrix is modelled as [22, 23] $\mathbf{H} = \sqrt{\frac{n_r n_t}{L}} . \sum_{l=1}^{L} \alpha_l . \mathbf{a_r}(\phi_l).\mathbf{a}_t^H(\theta_l)$, where $\mathbf{a_t}(\theta_l)$ and $\mathbf{a_r}(\phi_l)$ are the antenna array steering vectors at transmitter and receiver respectively, and $\alpha_l \sim \mathcal{CN}(0,1)$ is the complex gain of l^{th} path. We assume uniform linear arrays (ULAs). We consider the following parameters: 60 GHz carrier frequency, $n_t = n_r = 8$, $L = 4$ with typical inter antenna spacing, $d_r = d_t = \frac{\lambda}{2}$. The antennas are grouped into 4 groups: antennas 1-2, 3-4, 5-6, and 7-8 . The result is averaged over 1000 realizations of α, ϕ_l's, and θ_l's, where the ϕ_l and θ_l are taken to be i.i.d. in a 120^0 coverage area. Fig. 6 shows the rank-constrained capacity versus P for various choices of transmit rank constraint r. We use PAPC constraints $\hat{P}_i = P/8$, $i = 1, 2, \cdots, 8$, PGPC constraints for each group $\tilde{P}_k = \frac{\hat{P}_1 * 2}{1.2}$, $k = 1, 2, 3, 4$, total power constraint $P_{tot} = \sum_{i=1}^{4} \frac{\tilde{P}_k}{1.2}$. Since the maximum channel rank in this case is 4, we show the plots for $r = 1, 2, 3,$ and 4. The capacity increases as the rank constraint r increases. Increasing r beyond 4 will not increase capacity since the channel rank is 4.

Performance of cell-free massive MIMO: Here, we compare the performance of cell-free massive MIMO using conjugate beamforming to bounds computed using rank-constrained capacity under PAPC and Joint-SPC-PGPC. Cell-free massive MIMO [5] has a large number of distributed access points (APs) which are connected to a network controller (NC). Here, each user is served

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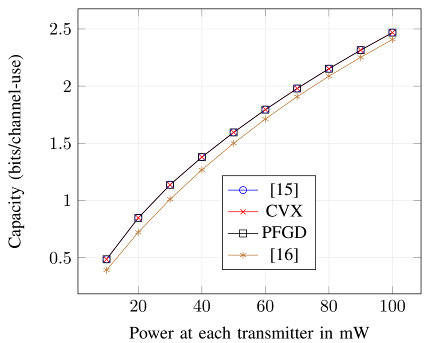
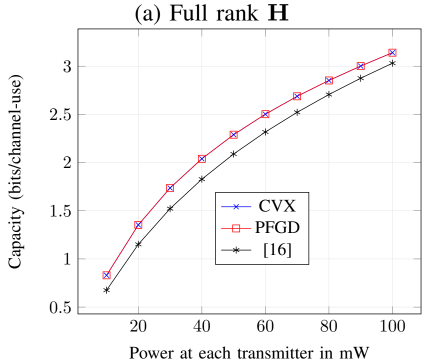
$$\mathbf{H} = \begin{bmatrix} -0.1926 - 0.1199i & 0.5327 + 1.3086i & 1.3846 - 0.6712i & -1.0193 + 0.6724i \\ 0.5653 + 0.5449i & -0.5461 - 0.1795i & -0.1892 + 0.6320i & 0.5356 - 0.7318i \\ -0.9071 + 0.5386i & 0.9302 + 0.5462i & 0.9911 - 1.1586i & -0.1081 + 1.9809i \\ 0.2396 + 0.3392i & -1.3570 - 0.1814i & -0.6045 + 1.2950i & 0.5442 - 1.5730i \end{bmatrix}. \quad (19)$$

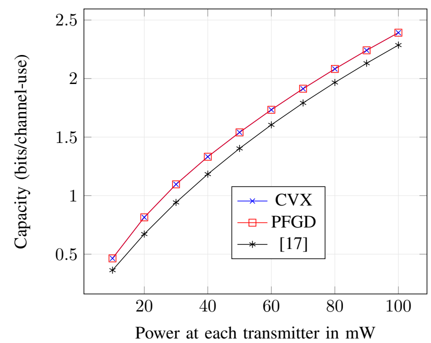


(a) Full rank **H**

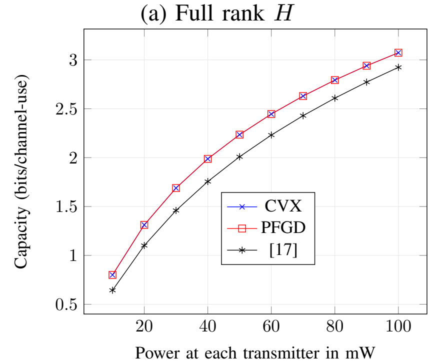


(b) Low rank **H**

Fig. 4: Comparison of MIMO capacity under PAPC



(a) Full rank H



(b) Low rank H

Fig. 5: Comparison of MIMO capacity under Joint-SPC-PAPC

by all APs simultaneously. Usually, each AP has its own power constraint. If each AP has a single antenna, we get PAPC. If each AP has multiple antennas, we get PGPC.

We assume a cell-free massive MIMO downlink with the following parameters. There are $M = 50$ APs serving a single user ($K = 1$) with 2 receive antennas. The channel coefficient between AP m and antenna k is taken as $h_{km} = \sqrt{\beta_{km}} h'_{km}$, where β_{km} is the large scale fading coefficient which changes very slowly and hence can be accurately estimated. Also, it is assumed that NC knows coefficients β_{km}. $h'_{km} \sim \mathcal{N}(0, 1)$ is the small scale fading coefficient. These h'_{km} are i.i.d random variables which stay constant during a coherent interval and are independent in different coherent intervals. Using pilot signals, APs estimate h'_{km}. We assume that APs estimate h'_{km} accurately. For Figs. 7a ,7b, we considered M randomly placed APs and randomly placed user in 2×2 km^2 area. For large scale fading coefficients, COST Hata model is used as in [5], i.e., $10 \log 10(\beta_{mk}) = -136 - 35 \log 10(d_{mk}) + X_{mk}$, where d_{mk} is the distance between AP m and antenna k of the user in kilometers, $X_{mk} \sim \mathcal{N}(0, \sigma^2_{shad})$ with $\sigma_{shad} = 8dB$, and noise variance at the receiver antennas is $\sigma^2_w = 290 \times \kappa \times B \times NF$, where κ, B, NF are the Boltzmann constant, bandwidth (20MHz), and noise figure (90 dB), respectively.

In Figs. 7a and 7b, we compare the CDF of rates achieved under PAPC and Joint-SPC-PGPC, respectively, for the following schemes: (1) Conjugate beamforming (CB) [5] to one of the two receive antennas randomly, (2) CB to the best receive antenna, and (3) rank-constrained capacity with rank $r = 1$. The rank-constrained capacity provides a useful upperbound on the

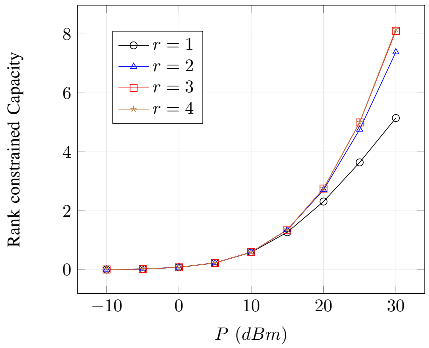


Fig. 6: Rank constrained capacity (bits/channel-use) versus P for the extended Saleh-Valenzuela model

performance of the CB-based cell-free massive MIMO system. For PAPC, we use a power constraint of 200 mW for each antenna, and for Joint-SPC-PGPC we assume 2 antennas in each group, a SPC of 8W, and PGPC of 400 mW for each group. Note that rank-constrained capacity assumes that the receiver also knows the channel coefficients h_{km} unlike the CB scheme. Despite this, we observe that the CB scheme is not much worse than the best possible scheme with rank one.

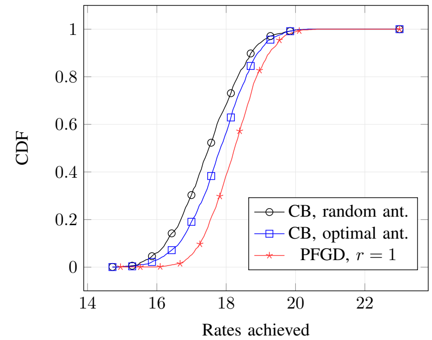
VII. Summary

In this paper, we obtained the optimal transmission scheme for MIMO under multiple simltaneous power constraints, namely sum, per-group, and per antenna power constraints. The solution was derived analytically for the MISO case and some special cases of MIMO. Since analytical solutions are not possible for the general MIMO case, we propose a projected factored gradient descent (PFGD) algorithm for the general case. This algorithm can also incorporate rank constraints in addition to Joint SPC-PGPC-PAPC and is, therefore, very general. The PFGD algorithm directly updates the beamforming/precoding matrix instead of the covariance matrix. From the numerical results, we observe that the proposed PFGD algorithm (1) provides accurate results that match with the CVX solution and analytical results for the cases where they are available, (2) has good convergence properties, and (3) has lower complexity than standard CVX solvers, especially for low-rank transmission.

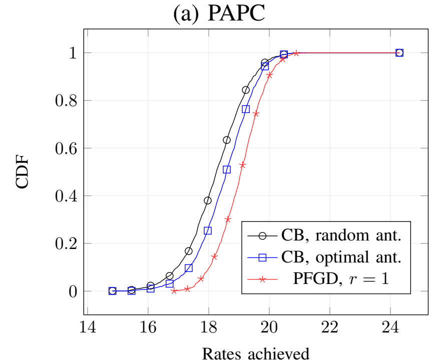
Appendix

A. Proof of Proposition 1

We can prove this by contradiction. Let $f(\mathbf{Q}) = \log|\mathbf{I}+\mathbf{H}\mathbf{Q}\mathbf{H}^H|$. We say $\mathbf{A}\succ\mathbf{B}$ if $\mathbf{A}-\mathbf{B}$ is positive definite. $f(\mathbf{Q})$ is monotonic with respect to $\mathbf{Q}$, i.e, $f(\mathbf{Q}_1) > f(\mathbf{Q}_2)$ if $\mathbf{Q}_1\succ\mathbf{Q}_2$. Suppose all the available power is not used in $\mathbf{Q}^{(J)}$. Then, we can find a $\mathbf{Q}$ that uses the full power such that $\mathbf{Q}\succ\mathbf{Q}^{(J)}$, i.e., $f(\mathbf{Q}) > f(\mathbf{Q}^{(J)})$. This is a contradiction.



(a) PAPC



(b) Joint-SPC-PGPC

Fig. 7: CDFs of rate achieved using CB (bits/channel-use) and rank-constrained capacity (bits/channel-use) using PFGD algorithm

B. Proof of Proposition 2

Let $\lambda_k \geq 0, k=1,\cdots,g$, $\lambda\geq 0$, and $\gamma_i \geq 0, i=1,\cdots,n_t$ be the lagrange multipliers for PGPC, SPC, and PAPC, respectively. Let $\mathbf{K}\succeq 0$ be the lagrange multiplier for the positive semi-definiteness constraint of the covariance matrix, $\boldsymbol{\Lambda}$ be as defined in (9), and $\boldsymbol{\Gamma}$ be a diagonal matrix with diagonal entries $\gamma_1, \gamma_2, \ldots, \gamma_{n_t}$. The lagrangian for problem (4) is

$$\begin{aligned}\mathcal{L} &= \mathbf{h}^T\mathbf{Q}\mathbf{h}^* + \text{tr}(\mathbf{K}\mathbf{Q}) - [\lambda(\text{tr}(\mathbf{Q}) - P_{tot})] - \\ &\left[\sum_{k=1}^{g}\lambda_k\left(\sum_{j\in I(k)} Q_{jj} - \tilde{P}_k\right)\right] - \left[\sum_{i=1}^{n_t}\gamma_i(Q_{ii}-\hat{P}_i)\right] \\ &= \mathbf{h}^T\mathbf{Q}\mathbf{h}^* + \text{tr}(\mathbf{K}\mathbf{Q}) - [\lambda(\text{tr}(\mathbf{Q}) - P_{tot})] - [\text{tr}(\boldsymbol{\Lambda}\mathbf{Q}) \\ &- \sum_{k=1}^{g}\lambda_k\tilde{P}_k] - [\text{tr}(\boldsymbol{\Gamma}\mathbf{Q}) - \sum_{i=1}^{n_t}\gamma_i\hat{P}_i].\end{aligned}$$

The optimal $\mathbf{Q}^{(J)}$ satisfies $\frac{\partial L}{\partial \mathbf{Q}^{(J)}} = 0$, i.e., $\mathbf{h}^*\mathbf{h}^T + \mathbf{K} - \mathbf{\Lambda} - \mathbf{\Gamma} - \lambda \mathbf{I} = 0$, or $\mathbf{h}^*\mathbf{h}^T = \mathbf{W} - \mathbf{K}$, where $\mathbf{W} = \mathbf{\Lambda} + \mathbf{\Gamma} + \lambda \mathbf{I}$. This implies $\mathbf{h}^*\mathbf{h}^T\mathbf{Q}^{(J)} = \mathbf{W}\mathbf{Q}^{(J)}$, since $\mathbf{K}\mathbf{Q}^{(\mathbf{J})} = 0$ at optimal $\mathbf{Q}^{(J)}$, due to the complementary slackness condition. $\mathbf{W}$ is a diagonal matrix which implies $\mathbf{h}^*\mathbf{h}^T + \mathbf{K}$ is diagonal. Since the diagonal entries of $\mathbf{h}^*\mathbf{h}^T$ are > 0 (since each $h_i \neq 0$), and the diagonal entries of $\mathbf{K}$ are ≥ 0 (since $\mathbf{K} \succeq 0$), $\mathbf{W}$ has full rank. Therefore, rank($\mathbf{h}^*\mathbf{h}^T\mathbf{Q}^{(J)}$) = rank($\mathbf{W}\mathbf{Q}^{(J)}$) = rank($\mathbf{Q}^{(J)}$). But, rank($\mathbf{h}^*\mathbf{h}^T\mathbf{Q}^{(J)}$) $\leq$ rank($\mathbf{h}^*\mathbf{h}^T$). Therefore, we have rank($\mathbf{Q}^{(J)}$) $\leq$ rank($\mathbf{h}^*\mathbf{h}^T$) $= 1$, which means beamforming is the optimal strategy.

C. Proof of Theorem 1

First, we obtain the power allocation for each group by solving a sequence of SPC problems. Then, we derive the power allocation within a group given the group power allocation in Lemma 3. Combining these results, we get the solution in Theorem 1.

Suppose that we order groups such that

$$\frac{\sum\limits_{j\in I(1)} |h_j|^2}{\tilde{P}_1} \geq \frac{\sum\limits_{j\in I(2)} |h_j|^2}{\tilde{P}_2} \geq \cdots \geq \frac{\sum\limits_{j\in I(g)} |h_j|^2}{\tilde{P}_g}.$$

Then, $\frac{\sum_{j\in I(i)} |h_j|^2}{|\mathbf{h}|^2} \leq \frac{\tilde{P}_i}{P_{tot}}$ for some group i implies $\frac{\sum_{j\in I(s)} |h_j|^2}{|\mathbf{h}|^2} \leq \frac{\tilde{P}_s}{P_{tot}}$ for all groups $s > i$. For proving Theorem 1, first find optimal power for the first (ordered) group with only sum power constraints $\sum\limits_{j\in I(1)} P_j^{(2)}$. From the ordering of groups, if $\sum\limits_{j\in I(1)} P_j^{(2)} = \frac{\sum_{j\in I(1)} |h_j|^2}{|\mathbf{h}|^2} P_{tot} \leq \tilde{P}_1$, then none of the groups would violate PGPC, and the optimal powers under Joint SPC-PGPC $P_j^{(1)} = P_j^{(2)}$. If $\frac{\sum_{j\in I(1)} |h_j|^2}{|\mathbf{h}|^2} P_{tot} > \tilde{P}_1$, then $\sum\limits_{j\in I(1)} P_j^{(1)} = \tilde{P}_1$, and the other group powers can be found by solving a reduced problem after removing the first group and modifying the sum power constraint to $P_{tot} - \tilde{P}_1$. The justification for this step is similar to Lemma 2 using $C \subseteq G$, and $S(C) := \{\mathbf{Q} \succeq 0,\ \mathrm{tr}(\mathbf{Q}) \leq P_{tot},\ \sum\limits_{j\in I(k)} Q_{jj} \leq \tilde{P}_k,\ k \in C\}$ and $D := \{k \in G\backslash C : \sum\limits_{i\in I(k)} P_i^{(S(C))} > \tilde{P}_k\}$. Therefore, once we check PGPC for the SPC solution, we know the optimal power under Joint SPC-PGPC for at least one group. In the next step, if $\frac{\sum_{j\in I(2)} |h_j|^2}{\sum_{j\in I(i),\, i\geq 2} |h_j|^2}(P_{tot} - \tilde{P}_1) \leq \tilde{P}_2$, then none of the remaining groups will violate PGPC and we know the optimal powers. Otherwise, we remove the second group and again consider a reduced problem. This continues until $\frac{\sum_{j\in I(k+1)} |h_j|^2}{\sum_{j\in I(i),\, i\geq k+1} |h_j|^2}(P_{tot} - \sum\limits_{j=1}^{k} \tilde{P}_j) \leq \tilde{P}_{k+1}$ for some k. This k is the total number of active PGPC constraints in the optimal solution, and can be obtained as the least solution of (6).

Once we know k, the first k groups use full group power and we get $P_j^{(1)}$ for all j belonging to the first k groups using the following Lemma.

Lemma 3. *(Allocation of power within a group) For the Joint-SPC-PGPC problem, let power P_k' be assigned to group k in the optimal solution. Then, the optimal power $P_l^{(1)}$ for each antenna $l \in I(k)$ is given by*

$$P_l^{(1)} = \frac{|h_l|^2}{\sum\limits_{j\in I(k)} |h_j|^2} P_k'. \tag{20}$$

Proof. Let f_k^l be the fraction of the group power P_k' assigned to antenna l, i.e., $P_l = P_k' f_k^l$, where $l \in I(k)$. Let S_1 be the feasible set of covariance matrices for the Joint SPC-PGPC problem. Using Lemma 1, we have

$$\begin{aligned} \arg\max_{\mathbf{Q}\in S_1} \mathbf{h}^T\mathbf{Q}\mathbf{h}^* &= \arg\max_{\mathbf{q}\in \mathbb{Q}^1} \left(\sum_{l=1}^{n_t} |h_l|\sqrt{P_l}\right)^2 \\ &= \arg\max_{\mathbf{q}\in \mathbb{Q}^1} \left(\sum_{l=1}^{n_t} |h_l|\sqrt{P_l}\right). \end{aligned}$$

where

$$\mathbb{Q}^1 := \left\{\mathbf{q} : \mathbf{q} = \left[\frac{\sqrt{P_1}h_1^*}{|h_1|}, \cdots, \frac{\sqrt{P_{n_t}}h_{n_t}^*}{|h_{n_t}|}\right], \mathbf{q}\mathbf{q}^H \in S_1\right\}.$$

Now, we have

$$\begin{aligned} \max_{\mathbf{q}\in\mathbb{Q}^1} \sum_{k=1}^{g} \sum_{l\in I(k)} |h_l|\sqrt{P_l} &= \max_{\{f_k^l\}} \sum_{k=1}^{g} \sum_{l\in I(k)} |h_l|\sqrt{P_k' f_k^l} \\ &= \sum_{k=1}^{g} \max_{\{f_k^l\}} \left(\sum_{l\in I(k)} |h_l|\sqrt{P_k' f_k^l}\right). \end{aligned}$$

Each $\sum\limits_{l\in I(k)} |h_l|\sqrt{P_k' f_k^l}$ is maximized subject to $\sum\limits_{l\in I(k)} f_k^l = 1$ by $f_k^l = \frac{|h_l|^2}{\sum_{j\in I(k)} |h_j|^2}$. Thus, we have (20). ☐

For the groups $k+1$ to g, the optimal power is found using only SPC $P_{tot} - \sum\limits_{j=1}^{k} \tilde{P}_j$.

D. Proof of Theorem 2

From (12), since we do not have PAPC constraints, we have

$$\mathbf{Q}^{(1)} = (\mathbf{\Lambda} + \lambda \mathbf{I})^{-1} - (\mathbf{H}^H \mathbf{H})^{-1}. \tag{21}$$

The left hand side of (11) is invertible. Therefore, we have

$$\lambda + \lambda_i > 0, \ \forall i = 1, 2, \cdots, g, \tag{22}$$

Now, we find $\lambda + \lambda_i$, $i = 1, 2, \cdots, g$, to complete the solution. We consider two cases.

(i) $\lambda_i > 0$ for group i: If $\lambda_i > 0$ for group i then, $\sum_{j \in I(i)} Q_{jj} = \tilde{P}_i$ since $\lambda_i (\sum_{j \in I(i)} Q_{jj} - \tilde{P}_i) = 0$.

$$\sum_{j \in I(i)} \frac{1}{\lambda + \lambda_i} - a_i = \tilde{P}_i \implies \frac{1}{\lambda + \lambda_i} = \frac{\tilde{P}_i + a_i}{n_i} \tag{23}$$

Thus, we have the expression in the first case in (15). Since $\lambda_i > 0$, $\frac{1}{\lambda + \lambda_i} < \frac{1}{\lambda}$. Therefore, note that we have

$$\frac{\tilde{P}_i + a_i}{n_i} < \frac{1}{\lambda}, \ \forall i \text{ such that } \lambda_i > 0. \tag{24}$$

(ii) $\lambda_i = 0$ for group i: $\lambda_i = 0$ for atleast one i implies $\lambda > 0$ (from (22)). The group power constraint for group i implies

$$\frac{\tilde{P}_i + a_i}{n_i} \geq \frac{1}{\lambda}, \forall i \text{ such that } \lambda_i = 0. \tag{25}$$

Because of our ordering of groups, (24) and (25), all groups for which $\lambda_i > 0$ will come first followed by groups with $\lambda_i = 0$. Let k be the number of groups where $\lambda_i > 0$. From Proposition 1, $\mathbf{Q^{(1)}}$ uses full available sum power. Therefore,

$$P_{tot} = \text{tr}(\mathbf{Q}) = \sum_{i=1}^{k} \tilde{P}_i + \sum_{i>k} \frac{n_i}{\lambda} - \sum_{i>k} a_i$$
$$\implies \frac{1}{\lambda} = \frac{P_{tot} + \sum_{i>k} a_i - \sum_{i=1}^{k} \tilde{P}_i}{\sum_{i>k} n_i}. \tag{26}$$

Thus, we have the expression in the second case in (15). Now, we will find k. Because of our ordering k is the solution of following inequalities :

$$\frac{\tilde{P}_k + a_k}{n_k} < \frac{P_{tot} + \sum_{j>k} a_j - \sum_{j=1}^{k} \tilde{P}_j}{\sum_{j>k} n_j} \tag{27}$$

$$\frac{\tilde{P}_{k+1} + a_{k+1}}{n_{k+1}} \geq \frac{P_{tot} + \sum_{j>k} a_j - \sum_{j=1}^{k} \tilde{P}_j}{\sum_{j>k} n_j}. \tag{28}$$

Because of the ordering of groups $\frac{\tilde{P}_i + a_i}{n_i}$ is non-decreasing in i. k is the least i for which (28) is satisfied. Since (28) is satisfied for $k = g - 1$, there exists atleast one solution for k in $\{0, 1, \cdots, g-1\}$. Finally, from k, (23), and (26), we can find $\mathbf{Q^{(1)}}$. $\mathbf{Q^{(1)}}$ is full rank when $(\mathbf{\Lambda} + \lambda \mathbf{I})^{-1} - (\mathbf{H}^H \mathbf{H})^{-1} \succ 0$.

E. Proof of Theorem 4

To find $\Pi_e(\mathbf{V})$, we find the matrix $\mathbf{U}$ which is nearest to $\mathbf{V}$ in the constraint set, i.e, solve

$$\begin{aligned} \min_{\mathbf{U}} \quad & \sum_{l=1}^{n_t} \|\mathbf{u}_l - \mathbf{v}_l\|^2 \\ \text{s.t.} \quad & \sum_{l=1}^{n_t} \|\mathbf{u}_l\|^2 \leq P_{tot}, \sum_{l \in I(i)} \|\mathbf{u}_l\|^2 \leq \tilde{P}_i, \ \forall i. \end{aligned} \tag{29}$$

The Lagrangian for the above problem is:

$$L(\mathbf{U}, \lambda, \{\lambda_i\}) = \sum_{l=1}^{n_t} \|\mathbf{u}_l - \mathbf{v}_l\|^2 + \lambda \left(\sum_{l=1}^{n_t} \|\mathbf{u}_l\|^2 - P_{tot} \right) + \sum_{i=1}^{g} \lambda_i \left(\sum_{l \in I(i)} \|\mathbf{u}_l\|^2 - \tilde{P}_i \right).$$

At optimal $\mathbf{U}$, $\frac{\partial L}{\partial \mathbf{u}_l^{(*)}} = 0$, i.e., $\mathbf{u}_l^{(*)}(1+\lambda+\lambda_i) = \mathbf{v}_l$, $l \in I(i)$. Since $(1 + \lambda + \lambda_i) \neq 0$

$$\mathbf{u}_l^{(*)} = \frac{\mathbf{v}_l}{(1 + \lambda + \lambda_i)}, \ l \in I(i). \tag{30}$$

Now, we find $(1 + \lambda + \lambda_i)$, $i = 1, 2, \cdots, g$, to complete the solution. We consider two cases.

(i) $\lambda_i > 0$ for group i: From KKT conditions $\lambda_i (\sum_{l \in I(i)} \|\mathbf{u}_l^{(*)}\|^2 - \tilde{P}_i) = 0$, which implies

$$\frac{1}{(1 + \lambda + \lambda_i)^2} = \frac{\tilde{P}_i}{\sum_{l \in I(i)} \|\mathbf{v}_l\|^2}. \tag{31}$$

Also, since $\lambda_i > 0$

$$\frac{\tilde{P}_i}{\sum_{l \in I(i)} \|\mathbf{v}_l\|^2} < \frac{1}{(1 + \lambda)^2}, \ \forall i \text{ s.t. } \lambda_i > 0. \tag{32}$$

(ii) $\lambda_i = 0$ for group i: Here $\mathbf{u}_l^{(*)} = \frac{\mathbf{v}_l}{(1+\lambda)}$.

From the per-group power constraints, we have

$$\frac{\tilde{P}_i}{\sum_{l \in I(i)} \|\mathbf{v}_l\|^2} \geq \frac{1}{(1 + \lambda)^2}, \ \forall i \text{ s.t. } \lambda_i = 0. \tag{33}$$

Let k number of groups satisfy $\lambda_i > 0$. The rest of the groups satisfy $\lambda_i = 0$. Given the arrangement of groups as in the ordered set R, groups corresponding to first k elements in R will satisfy $\lambda_i > 0$. Furthermore, since $P_{tot} \leq \sum_{i=1}^{g} \tilde{P}_i$, the full available sum power P_{tot} is used in the optimal $\mathbf{U}$. Therefore, we have

$$\sum_{i=R(k+1)}^{R(g)} \left(\sum_{l \in I(i)} \frac{\|\mathbf{v}_l\|^2}{(1+\lambda)^2} \right) = P_{tot} - \sum_{i=R(1)}^{R(k)} \tilde{P}_i$$

$$\implies \frac{1}{(1+\lambda)^2} = \frac{P_{tot} - \sum_{i=R(1)}^{R(k)} \tilde{P}_i}{\sum_{i=R(k+1)}^{R(g)} \left(\sum_{l \in I(i)} \|\mathbf{v}_l\|^2 \right)}. \quad (34)$$

Because of our ordering in set R, k is the solution of the inequalities (17) and

$$\frac{\tilde{P}_{R(k)}}{\sum_{l \in I(R(k))} \|\mathbf{v}_l\|^2} < \frac{P_{tot} - \sum_{i=R(1)}^{R(k)} \tilde{P}_i}{\sum_{i=R(k+1)}^{R(g)} \left(\sum_{l \in I(i)} \|\mathbf{v}_l\|^2 \right)}. \quad (35)$$

Because of the ordering of groups $\frac{\tilde{P}_{R(i)}}{\sum_{l \in I(R(i))} \|\mathbf{v}_l\|^2}$ is non-decreasing in i. k is the least i for which (17) is satisfied. Since (17) is satisfied for $k = g-1$, there exists atleast one solution for k in $\{0, 1, \cdots, g-1\}$.

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