# ON THE VASCONCELOS INEQUALITY FOR THE FIBER MULTIPLICITY OF MODULES 

BALAKRISHNAN R.* AND A. V. JAYANTHAN


#### Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d>0$ with infinite residue field. Let $M$ be a finitely generated proper $R$-submodule of a free $R$-module $F$ with $\ell(F / M)<\infty$ and having rank $r$. In this article, we study the fiber multiplicity $f_{0}(M)$ of the module $M$. We prove that if $(R, \mathfrak{m})$ is a two dimensional Cohen-Macaulay local ring, then $f_{0}(M) \leq b r_{1}(M)-b r_{0}(M)+\ell(F / M)+\mu(M)-r$, where $b r_{i}(M)$ denotes the $i^{t h}$ Buchsbaum-Rim coefficient of $M$.


## 1. InTRODUCTION

Throughout the paper, we will assume that $(R, \mathfrak{m})$ is a Noetherian local ring of dimension $d>0$ with infinite residue field and $M$ is a finitely generated proper submodule of a free $R$-module $F$ with $\ell(F / M)<\infty$ and having rank $r$. Let $\mathcal{S}(F)=\underset{n \geq 0}{\bigoplus} \mathcal{S}_{n}(F)$ denote the Symmetric algebra of $F$, and $\mathcal{R}(M)=\underset{n \geq 0}{\bigoplus} \mathcal{R}_{n}(M)$ denotes the Rees algebra of $M$, which is image of the natural map from the Symmetric algebra of $M$ to the Symmetric algebra of $F$. Generalizing the notion of Hilbert-Samuel function, D. A. Buchsbaum and D. S. Rim studied the function $B F_{M}(n)=\ell\left(\mathcal{S}_{n}(F) / \mathcal{R}_{n}(M)\right)$ for $n \in \mathbb{N}$. In [3], they proved that $B F_{M}(n)$ is given by a polynomial of degree $d+r-1$ for $n \gg 0$, i.e., there exists a polynomial $B P_{M}(x) \in \mathbb{Q}[x]$ of degree $d+r-1$ such that $B F_{M}(n)=B P_{M}(n)$ for $n \gg 0$. The function $B F_{M}(n)$ is called the Buchsbaum-Rim function of $M$ with respect to $F$ and the polynomial $B P_{M}(n)$ is called the corresponding Buchsbaum-Rim polynomial. Following the notation used for the Hilbert-Samuel polynomial, one writes the Buchsbaum-Rim polynomial as

$$
B P_{M}(n)=\sum_{i=0}^{d+r-1}(-1)^{i} b r_{i}(M)\binom{n+d+r-i-2}{d+r-i-1}
$$

Key words and phrases. Buchsbaum-Rim function, Buchsbaum-Rim polynomial, Rees algebra of modules, Fiber cone of modules, Vasconcelos Inequality.

* Supported by the Council of Scientific and Industrial Research (CSIR), India.

AMS Classification 2010: 13D40, 13A30.

The coefficients $b r_{i}(M)$ for $i=0, \ldots, d+r-1$ are known as Buchsbaum-Rim coefficients. For basic properties of the Buchsbaum-Rim function and the Buchsbaum-Rim polynomial, we refer the reader to [8], [15]. In this article we study the fiber multiplicity $f_{0}(M)$ of the module $M$ and relate it with $b r_{0}(M)$ and $b r_{1}(M)$.

Let $\mathcal{F}(M):=\mathcal{R}(M) \otimes R / \mathfrak{m}$ denote the fiber cone of $M$. In Section 2, we study CohenMacaulayness of fiber cone $\mathcal{F}(M)$. The Cohen-Macaulayness of $\mathcal{F}(I)$, where $I$ is an ideal in $R$, has been of interest and has been studied widely, see for example [5], [6], [10], [13]. In [13], K. Shah studied the Hilbert function and the Cohen-Macaulayness of $\mathcal{F}(I)$.

Theorem 1.1. [13, Theorem 1] Let $(R, \mathfrak{m})$ be a local ring. Suppose $I$ is an ideal which is integral over a regular sequence $\underline{x}$ such that $I^{2}=I \underline{x}$. Then $\mathcal{F}(I)$ is Cohen-Macaulay.

We study some basic properties of the fiber cone $\mathcal{F}(M)$. We give a characterization for the Cohen-Macaulayness of $\mathcal{F}(M)$. We then prove an analogue of Theorem 1.1 in the case of modules over two dimensional Cohen-Macaulay local rings. We first recall some basics on reduction of modules.

Let $N$ be a submodule of $M$. We say that $N$ is reduction of $M$ if Rees algebra $\mathcal{R}(M)$ is integral over the $R$-subalgebra $\mathcal{R}(N)$. Equivalently, there exists $n_{0}$ such that $\mathcal{R}_{n+1}(M)=$ $N \mathcal{R}_{n}(M)$ for $n \geq n_{0}$, where the multiplication is done as $R$-submodules of $\mathcal{R}(M)$. The least integer $s$ such that $\mathcal{R}_{s+1}(M)=N \mathcal{R}_{s}(M)$ is called the reduction number of $M$ with respect to $N$, denoted as $\operatorname{red}_{N}(M)$. The reduction number of the module $M$, denoted $\operatorname{red}(M)$, is defined as $\operatorname{red}(M)=\min \left\{\operatorname{red}_{N}(M): N\right.$ is a minimal reduction of M\}. If $N$ is a submodule of $F$ generated by $d+r-1$ elements such that $\ell(F / N)<\infty$, then $N$ is said to be a parameter module. It was proved in [2] that if $\ell(F / M)<\infty$, then there exists minimal reduction generated by $d+r-1$ elements. For more details on minimal reductions, we refer the reader to [8] and [15]. In this article, we prove:

Theorem 1.2. Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring. Let $M \subset F$ be such that $\ell(F / M)<\infty$ and having rank $r$. If $\operatorname{red}(M) \leq 1$, then $\mathcal{F}(M)$ is Cohen-Macaulay.

In 4], A. Corso, C. Polini and W. Vasconcelos studied the multiplicity of the fiber cone $\mathcal{F}(I)$. One of the main result obtained in [4] is an inequality relating fiber multiplicity $f_{0}(I)$, Hilbert coefficients $e_{0}(I)$ and $e_{1}(I)$ and some other invariants of $I$ :

Theorem 1.3. [4, Theorem 2.1] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d>0$ with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal. Then we have that

$$
f_{0}(I) \leq e_{1}(I)-e_{0}(I)+\ell(R / I)+\mu(I)-d+1
$$

where $\mu(M)$ denotes the cardinality of a minimal generating set of an $R$-module $M$.

Motivated by this inequality, Vasconcelos raised the question:
Question 1.4. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and $M \subset F$ with $\ell(F / M)<\infty$ and having rank $r$. Then, does the fiber multiplicity $f_{0}(M)$ satisfy

$$
f_{0}(M) \leq b r_{1}(M)-b r_{0}(M)+\ell(F / M)+\mu(M)-d-r+2 ?
$$

In Sections 3 and 4, we address the above question. In Section 3, we prove that Question 1.4 has an affirmative answer when $\operatorname{dim} R=2$. In Section 4, we establish the inequality for modules of the form $M=I \oplus \cdots \oplus I \oplus J \oplus \cdots \oplus J$, where $I$ is an $\mathfrak{m}$-primary ideal in $R$ and $J$ is a minimal reduction of $I$. For $d=1$, we provide a counter example.

Acknowledgements: We sincerely thank the referees for pointing out several errors, some of them typographical and some of them mathematical, which tremendously improved the exposition.

## 2. Fiber cone of modules

In this section, we study Cohen-Macaulay property of fiber cone of modules. We begin by recalling the definition of fiber cone $\mathcal{F}(M)$.

Definition 2.1. Let $M \subset F$ be such that $\ell(F / M)<\infty$ and having rank $r$. The fiber cone of $M$, denoted by $\mathcal{F}(M)$, is defined as

$$
\mathcal{F}(M):=\mathcal{R}(M) \otimes R / \mathfrak{m}=\bigoplus \frac{\mathcal{R}_{i}(M)}{\mathfrak{m} \mathcal{R}_{i}(M)}
$$

where $\mathcal{R}(M)$ is the Rees algebra of $M$.

The Krull dimension of $\mathcal{F}(M)$ is known as the analytic spread of $M$ and is equal to $d+r-1$, [2]. The Hilbert function of $\mathcal{F}(M)$ is given by

$$
H(\mathcal{F}(M), n)=\ell\left(\mathcal{R}_{n}(M) / \mathfrak{m} \mathcal{R}_{n}(M)\right), \text { for } n \in \mathbb{N}
$$

The corresponding Hilbert polynomial, of degree $d+r-2$ for $n \gg 0$, is written as
$H(\mathcal{F}(M), n)=f_{0}(M)\binom{n+d+r-2}{d+r-2}-f_{1}(M)\binom{n+d+r-3}{d+r-3}+\cdots+(-1)^{d+r-2} f_{d+r-2}(M)$.
The leading coefficient $f_{0}(M)$ is called the fiber multiplicity of $M$. Let $\mathcal{H}(\mathcal{F}(M), t)$ denote the Hilbert series of $\mathcal{F}(M)$, i.e.,

$$
\mathcal{H}(\mathcal{F}(M), t)=\sum_{n=0}^{\infty} H(\mathcal{F}(M), n) t^{n}
$$

We now give a characterization for the Cohen-Macaulayness of the fiber cone of a module in terms of its Hilbert series and its fiber multiplicity. See also [15, Proposition 8.40]. We skip the proof of Theorem 2.2 as it is routine.

Theorem 2.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M \subset F$ be such that $\ell(F / M)<\infty$, having rank $r$ and $N \subseteq M$ be a minimal reduction. Then the following are equivalent:
(1) Fiber cone $\mathcal{F}(M)$ is Cohen-Macaulay;
(2) $\mathcal{H}(\mathcal{F}(M), t)=\frac{1}{(1-t)^{a}} \sum_{i=0}^{b} \ell\left(\frac{\mathcal{R}_{i}(M)}{N \mathcal{R}_{i-1}(M)+\mathfrak{m} \mathcal{R}_{i}(M)}\right) t^{i}$, where $b=\operatorname{red}_{N}(M), a=\operatorname{dim} \mathcal{F}(M)$;
(3) $f_{0}(M)=\sum_{i=0}^{b} \ell\left(\frac{\mathcal{R}_{i}(M)}{N \mathcal{R}_{i-1}(M)+\mathfrak{m} \mathcal{R}_{i}(M)}\right)$.

We also note that Theorem 2.2 gives an upper bound on the reduction number of $M$ when $\mathcal{F}(M)$ is Cohen-Macaulay:

Remark 2.3. For an $R$-module $K$, let $\mu(K)$ denote the minimum number of generators of $K$. Note that if $\mathcal{F}(M)$ is Cohen-Macaulay, then

$$
f_{0}(M)=\sum_{i=0}^{b} \ell\left(\frac{\mathcal{R}_{i}(M)}{N \mathcal{R}_{i-1}(M)+\mathfrak{m} \mathcal{R}_{i}(M)}\right)=1+\sum_{i=1}^{b} \mu\left(\frac{\mathcal{R}_{i}(M)}{N \mathcal{R}_{i-1}(M)}\right) .
$$

Since $\mu\left(\frac{\mathcal{R}_{i}(M)}{N \mathcal{R}_{i-1}(M)}\right) \geq 1$ for all $1 \leq i \leq b=\operatorname{red}(M)$, we get

$$
\operatorname{red}(M) \leq f_{0}(M)-1
$$

Using the fact that $\ell(M / N+\mathfrak{m} M)=\mu(M)-\mu(N)=\mu(M)-(d+r-1)$, we obtain a better bound in the above case:

$$
\operatorname{red}(M) \leq f_{0}(M)-\mu(M)+d+r-1
$$

It is natural to expect that smaller reduction number of the module $M$ force good properties on $\mathcal{F}(M)$. If $\operatorname{red}(M)=0$, then $M$ is a parameter module and hence $\mathcal{F}(M)$ is a polynomial ring over the residue field of $R$. Next natural condition is to consider when $\operatorname{red}(M)$ is one. We extend K. Shah's, [13], result to the case of modules over two dimensional Cohen-Macaulay rings:

Theorem 2.4. Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring. Let $M \subset F$ be such that $\ell(F / M)<\infty$ and having rank $r$. If $\operatorname{red}(M) \leq 1$, then $\mathcal{F}(M)$ is Cohen-Macaulay.

Proof. Let $N \subseteq M$ be a minimal reduction such that $\operatorname{red}(M)=\operatorname{red}_{N}(M)=1$. Now $\mu(N)=r+1$. By Theorem 2.2, it is enough to show that

$$
\mathcal{H}(\mathcal{F}(M), t)=\frac{1}{(1-t)^{r+1}}\left[1+\ell\left(\frac{M}{N+\mathfrak{m} M}\right) t\right] .
$$

Let $\left\{x_{1}, \ldots, x_{r+1}\right\}$ be a minimal generating set for $N$. Extend this to a minimal generating set $\left\{x_{1}, \ldots, x_{r+1}, y_{1}, \ldots, y_{p}\right\}$ for $M$ such that $\mu(M)=r+p+1$. Set $K=\left(y_{1}, \ldots, y_{p}\right) \subseteq M$. So $M=N+K$. Identifying the elements $x_{i}$ 's and $y_{j}$ 's with their images in $\mathcal{R}_{1}(M)$, it can be seen that $\mathcal{R}_{n}(M)$ is generated as an $R$-module by $\left\{\mathcal{R}_{n}(N), \mathcal{R}_{n-1}(N) \mathcal{R}_{1}(K), \ldots, \mathcal{R}_{1}(N) \mathcal{R}_{n-1}(K), \mathcal{R}_{n}(K)\right\}$.

Since $\operatorname{red}_{N}(M)=1, \mathcal{R}_{1}(N) \mathcal{R}_{1}(M)=\mathcal{R}_{2}(M)$. This implies that $\mathcal{R}_{i}(N) \mathcal{R}_{1}(M)=\mathcal{R}_{i+1}(M)$ for all $i \geq 1$. Now for $i \geq 2$,

$$
\begin{aligned}
\mathcal{R}_{n-i}(N) \mathcal{R}_{i}(K) & \subseteq \mathcal{R}_{n-i}(N) \mathcal{R}_{i}(M) \subseteq \mathcal{R}_{n}(M) \\
& =\mathcal{R}_{n-1}(N) \mathcal{R}_{1}(M) \\
& =\mathcal{R}_{n}(N)+\mathcal{R}_{n-1}(N) \mathcal{R}_{1}(K)
\end{aligned}
$$

Therefore

$$
\mathcal{R}_{n}(M)=\left\langle\mathcal{R}_{n}(N), \mathcal{R}_{n-1}(N) \mathcal{R}_{1}(K)\right\rangle .
$$

Set $T_{n}=\left\{x_{1}^{\alpha_{1}} \cdots x_{r+1}^{\alpha_{r+1}} \mid \alpha_{1}+\cdots+\alpha_{r+1}=n\right\}$, where products are taken in $\mathcal{R}(N)$. Let $\Delta_{1}, \ldots, \Delta_{k_{(n)}}$ and $\delta_{1}, \ldots, \delta_{k_{(n-1)}}$ denote elements of $T_{n}$ and $T_{n-1}$ respectively.
Set $S_{n}=\left\{\delta_{i} y_{j} \mid i=1, \ldots, k_{(n-1)}, j=1, \ldots, p\right\}$
Claim: $T_{n} \cup S_{n}$ is a minimal generating set for $\mathcal{R}_{n}(M)$.
It is clear that $T_{n} \cup S_{n}$ generates $\mathcal{R}_{n}(M)$. We only need to prove the minimality. So let us
assume that

$$
\sum_{i=1}^{k_{(n)}} r_{i} \Delta_{i}+\sum_{i=1}^{k_{(n-1)}} \sum_{j=1}^{p} s_{i j} \delta_{i} y_{j}=0
$$

Suppose $s_{i_{0} j_{0}} \notin \mathfrak{m}$ for some $i_{0}, j_{0}$. Rewriting the above relation, we get

$$
\begin{aligned}
\sum_{i=1}^{k_{(n)}} r_{i} \Delta_{i}+\left(\sum_{j=1}^{p} s_{i_{0} j} y_{j}\right) \delta_{i_{0}}+\sum_{i=1, i \neq i_{0}}^{k_{(n-1)}} \sum_{j=1}^{p} s_{i j} \delta_{i} y_{j} & =0 \\
\text { i.e., }\left(\sum_{j=1}^{p} s_{i_{0} j} y_{j}\right) \delta_{i_{0}}+\sum_{i=1, i \neq i_{0}}^{k_{(n-1)}} \sum_{j=1}^{p} s_{i j} \delta_{i} y_{j} & =-\sum_{i=1}^{k_{(n)}} r_{i} \Delta_{i} \in \mathcal{R}_{n}(N) .
\end{aligned}
$$

By [7, Corollary 4.5], we get $\sum_{j=1}^{p} s_{i_{0} j} y_{j} \in N$. Since $s_{i_{0} j_{0}}$ is a unit, this implies that $y_{j_{0}} \in\left(N, y_{1}, \ldots, \hat{y_{0}}, \ldots, y_{p}\right)$ contradicting the minimality of the generating set considered for $M$ above. Therefore $s_{i_{0} j_{0}} \in \mathfrak{m}$. Therefore,

$$
\sum_{i=1}^{k_{(n)}} r_{i} \Delta_{i}=-\sum_{i=1}^{k_{(n-1)}} \sum_{j=1}^{p} s_{i j} \delta_{i} y_{j} \in R_{n}(N) \cap \mathfrak{m} R_{n}(M)=\mathfrak{m} R_{n}(N)
$$

Therefore $r_{i} \in \mathfrak{m}$ for all $i$ which completes the proof of the claim.
Note that $k_{(n)}=\binom{n+r}{r}$ and $\left|T_{n} \cup S_{n}\right|=\binom{n+r}{r}+p\binom{n-1+r}{r}$. Therefore we have

$$
\begin{aligned}
\mathcal{H}(\mathcal{F}(M), t) & =\sum_{n=0}^{\infty} \mu\left(\mathcal{R}_{n}(M)\right) t^{n} \\
& =\sum_{n=0}^{\infty}\left[\binom{n+r}{r}+p\binom{n-1+r}{r}\right] t^{n} \\
& =\sum_{n=0}^{\infty}\binom{n+r}{r} t^{n}+p \sum_{n=0}^{\infty}\left[\binom{n+r}{r}-\binom{n+r-1}{r-1}\right] t^{n} \\
& =\frac{1}{(1-t)^{r+1}}+p\left[\frac{1}{(1-t)^{r+1}}-\frac{1}{(1-t)^{r}}\right] \\
& =\frac{1+p t}{(1-t)^{r+1}} .
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\ell\left(\frac{M}{N+\mathfrak{m} M}\right) & =\ell\left(\frac{M}{\mathfrak{m} M}\right)-\ell\left(\frac{N+\mathfrak{m} M}{\mathfrak{m} M}\right) \\
& =\mu(M)-\ell\left(\frac{N}{\mathfrak{m} M \cap N}\right) \\
& =\mu(M)-\ell\left(\frac{N}{\mathfrak{m} N}\right) \\
& =\mu(M)-\mu(N) \\
& =p
\end{aligned}
$$

Therefore by Theorem 2.2, $\mathcal{F}(M)$ is Cohen-Macaulay.

One of the key ideas used in the above proof is the analytical independence of the generators of $N,[7$, Corollary 4.5]. This result is proved in dimension 2 and as far as we know, an analogue of this result in higher dimensions is not known. Once this result is generalized to higher dimensions, the above proof goes through for higher dimensions as well.

At this stage, we would also like to compare the case of ideals with that of modules. In the case of ideals, the above result, in much more generality, has a much simpler proof due to the existence of the associated graded ring and the beautiful and one of the most basic results on regular sequences, namely Valabrega-Valla theorem. In the case of modules, both associated graded ring as well as a Valabrega-Valla type theorem are missing.

Following is an interesting observation on the Cohen-Macaulayness of $F(M)$ in a special case.

Proposition 2.5. Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring and $M=I \oplus J$, where $I$ is an $\mathfrak{m}$-primary ideal $J$ be a minimal reduction of $I$. If $\operatorname{red}(M) \leq 1$, then $I=J$.

Proof. Since $\operatorname{red}(M) \leq 1$, by Theorem 2.4, $\mathcal{F}(M)$ is Cohen-Macaulay.
Now by Theorem 2.2, $f_{0}(M)=\mu(M)-2=\mu(I)+\mu(J)-2=\mu(I)$. The Hilbert function
of $\mathcal{F}(M)$ is given by

$$
\begin{aligned}
H(\mathcal{F}(M), n) & =\ell\left(\frac{\mathcal{R}_{n}(M)}{\mathfrak{m} \mathcal{R}_{n}(M)}\right) \\
& =\sum_{i=0}^{n} \ell\left(\frac{J^{i} I^{n-i}}{\mathfrak{m} J^{i} I^{n-i}}\right) \\
& =\ell\left(\frac{I^{n}}{\mathfrak{m} I^{n}}\right)+\sum_{i=1}^{n-1} \ell\left(\frac{I^{n}}{\mathfrak{m} I^{n}}\right)+\ell\left(\frac{J^{n}}{\mathfrak{m} J^{n}}\right) \\
& =n H(\mathcal{F}(I), n)+H(\mathcal{F}(J), n) .
\end{aligned}
$$

Therefore $f_{0}(M)=2 f_{0}(I)$. Since $\operatorname{red}(I) \leq 1, \mathcal{F}(I)$ is Cohen-Macaulay and hence $f_{0}(I)=$ $\mu(I)-1$. Hence $\mu(I)=2$ and hence a parameter ideal. Therefore $I=J$.

## 3. VASCONCELOS INEQUALITY FOR $d=2$

In this section, we prove the Vasconcelos inequality for modules over two dimensional Cohen-Macaulay rings. We adopt the technique used to prove the inequality for $f_{0}(I)$ in [4]. The idea involves using the knowledge of the Hilbert polynomial of the Sally module $S_{N}(M)$, where $N$ is a minimal reduction of $M$. The notion of Sally module $S_{N}(M)$ was introduced in [1], extending the corresponding notion of ideals, [16]. We first recall the definition of $S_{N}(M)$ from [1].

Definition 3.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M \subset F$ be an $R$-module and $N \subseteq M$ be an $R$-submodule. Then the Sally module of $M$ with respect to $N$ is defined as $S_{N}(M):=$ $\underset{n \geq 1}{\oplus} \frac{\mathcal{R}_{n+1}(M)}{M \mathcal{R}_{n}(N)}$.

Note that if $N$ is a reduction of $M$, then $S_{N}(M)$ is a finitely generated $\mathcal{R}(N)$-module and $S_{N}(M)=0$ if and only if $\operatorname{red}_{N}(M)=1$. If $r=1$, then this definition coincides with the definition of Sally module of an ideal $I$ with respect to a minimal reduction $J$. It is to be noted that in this case, $\operatorname{dim}_{\mathcal{R}(J)} S_{J}(I)=d$ if $S_{J}(I) \neq 0$, 15]. Note that the proof of this result given by Corso et al., [4, Theorem 2.1], can not be adapted to the case of modules. Another approach to the dimension is through the Hilbert function. In the case of rank one, the Hilbert polynomial of the Sally module can be computed and then using Northcott inequality along with Huneke-Ooishi theorem one can conclude that the dimension of the Sally module is $d$. This approach also fails in the case of modules since an analogue
of Northcott inequality (for $d \geq 3$ ) and Huneke-Ooishi theorem (even for $d \geq 2$ ) are not known.

Suppose $M \subset F$ is of rank $r$ with $\ell(F / M)<\infty$ and $N$ is a minimal reduction of $M$. We first show that if $S_{N}(M) \neq 0$, then $\operatorname{dim}_{\mathcal{R}(N)} S_{N}(M) \leq d+r-1$. Set $T=\oplus T_{n}=\oplus \frac{S_{n}(F)}{M \mathcal{R}_{n-1}(N)}$ and $S_{N}(M)=\oplus S_{n}=\oplus \frac{R_{n+1}(M)}{M \mathcal{R}_{n}(M)}$. Let $B F_{M}(n)$ and $B F_{N}(n)$ denote the Buchsbaum-Rim functions of $M$ and $N$ respectively, i.e., $B F_{M}(n)=\ell\left(\frac{S_{n}(F)}{\mathcal{R}_{n}(M)}\right)$ and $B F_{N}(n)=\ell\left(\frac{S_{n}(F)}{\mathcal{R}_{n}(N)}\right)$. Observe that

$$
B F_{M}(n) \leq \ell\left(T_{n}\right) \leq B F_{N}(n)
$$

Since Buchsbaum-Rim polynomials of $M$ and $N$ are of degree $d+r-1$ with same leading coefficients, it follows that the Hilbert polynomial of $T$ is of degree $d+r-1$ with the same leading coefficient as that of Buchsbaum-Rim polynomial of $M$. Note that

$$
\ell\left(S_{n-1}\right)=\ell\left(T_{n}\right)-B F_{M}(n)
$$

Therefore it follows that the Hilbert polynomial of Sally module is of degree at most $d+r-2$ and hence $\operatorname{dim} S_{N}(M) \leq d+r-1$. If $d=2$, then by [1, Theorem 3.2], it follows that, for $n \gg 0$

$$
\begin{align*}
\ell\left(S_{n-1}\right)= & {\left[b r_{1}(M)-b r_{0}(M)+\ell(F / M)\right]\binom{n+r-1}{r}-b r_{2}(M)\binom{n+r-2}{r-1} } \\
& +\cdots+(-1)^{r} b r_{r+1}(M) \tag{1}
\end{align*}
$$

It is not known whether the equality $b r_{0}(M)-b r_{1}(M)=\ell(F / M)$ implies $\operatorname{red}_{N}(M)=1$ and hence we can not possibly conclude from the above equation that $\operatorname{dim}_{\mathcal{R}(N)} S_{N}(M)=r$. Therefore, we ask:

Question 3.2. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring. Let $M \subset F$ be an $R$-module of rank $r$ with $\ell(F / M)<\infty$ and $N \subseteq M$ be a minimal reduction of $M$. If $S_{N}(M)$ is non-zero, then is $\operatorname{dim}_{\mathcal{R}(N)} S_{N}(M)=d+r-1$ ?

Keeping in mind the case $r=1$, we would like to ask another question, an affirmative answer to which will give an affirmative answer to Question 3.2,

Question 3.3. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring. Let $M \subset F$ be an $R$-module of rank $r$ with $\ell(F / M)<\infty$ and $N$ be a minimal reduction of $M$. Is the multiplicity of the Sally module, $e_{0}\left(S_{N}(M)\right)=\ell(F / M)+b r_{1}(M)-b r_{0}(M)$ ?

We now prove the Vasconcelos inequality for modules over 2-dimensional Cohen-Macaulay local rings.

Theorem 3.4. Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring. Let $M \subset F$ be such that $\ell(F / M)<\infty$ and having rank $r$. Then

$$
f_{0}(M) \leq b r_{1}(M)-b r_{0}(M)+\ell(F / M)+\mu(M)-r .
$$

If $\operatorname{red}(M) \leq 1$, then the equality holds.

Proof. Let $N$ be a minimal reduction of $M$. Let us choose $f_{1}, \ldots, f_{k}$ from $M$ such that $M=\left(N, f_{1}, \ldots, f_{k}\right)$, where $k=\mu(M)-\mu(N)=\mu(M)-r-1$. Let $S_{N}(M)$ be the Sally module of $M$ with respect to $N$. For $i=1, \ldots, k$, let $g_{i}$ denote image of $f_{i}$ in $\mathcal{R}_{1}(M)$. Consider the following $\mathcal{R}(N)$-module homomorphisms

$$
i: \mathcal{R}(N) \rightarrow \mathcal{R}(M) \text { and } \phi: \mathcal{R}(N)^{k} \rightarrow \mathcal{R}(M)
$$

where $i$ is the natural inclusion map and $\phi$ is defined by $\phi\left(e_{i}\right)=g_{i}$ for $i=1, \ldots, k$, where $\left\{e_{1}, \ldots, e_{k}\right\}$ is the standard basis for $\mathcal{R}(N)^{k}$. Now consider the following graded exact sequence of $\mathcal{R}(N)$-modules

$$
\mathcal{R}(N) \oplus \mathcal{R}(N)^{k}[-1] \xrightarrow{\psi} \mathcal{R}(M) \longrightarrow S_{N}(M)[-1] \longrightarrow 0
$$

where $\psi$ is induced by $i$ and $\phi$. Tensor the above sequence with $-\otimes R / \mathfrak{m}$ to get the following graded exact sequence with corresponding induced maps

$$
\mathcal{F}(N) \oplus \mathcal{F}(N)^{k}[-1] \xrightarrow{\psi} \mathcal{F}(M) \longrightarrow S_{N}(M)[-1] \otimes \frac{R}{\mathfrak{m}} \longrightarrow 0
$$

Taking lengths of the graded parts we get, for $n \in \mathbb{N}$,

$$
\begin{aligned}
\ell\left([\mathcal{F}(M)]_{n}\right) & \leq \ell\left([\mathcal{F}(N)]_{n}+\left[\mathcal{F}(N)^{k}\right]_{n-1}\right)+\ell\left(\left[\frac{S_{N}(M)}{\mathfrak{m} S_{N}(M)}\right]_{n-1}\right) \\
& \leq \ell\left([\mathcal{F}(N)]_{n}\right)+\ell\left(\left[\mathcal{F}(N)^{k}\right]_{n-1}\right)+\ell\left(\left[S_{N}(M)\right]_{n-1}\right)
\end{aligned}
$$

Note that for $n \gg 0$,

$$
\begin{aligned}
\ell\left([\mathcal{F}(M)]_{n}\right)= & \sum_{i=0}^{r}(-1)^{i} f_{i}(M)\binom{n+r-i}{r-i}, \\
\ell\left([\mathcal{F}(N)]_{n}\right)= & \binom{n+r}{r}, \\
\ell\left(\left[\mathcal{F}(N)^{k}\right]_{n-1}\right)= & k\binom{n+r-1}{r}, \\
\ell\left(\left[S_{N}(M)\right]_{n-1}\right)= & {\left[b r_{1}(M)-b r_{0}(M)+\ell(F / M)\right]\binom{n+r-1}{r} } \\
& +\sum_{i=1}^{r}(-1)^{i} b r_{i+1}(M)\binom{n+r-1-i}{r-i},
\end{aligned}
$$

where the last equality follows from (1). It follows, by comparing the leading coefficients, that

$$
\begin{aligned}
f_{0}(M) & \leq 1+k+b r_{1}(M)-b r_{0}(M)+\ell(F / M) \\
& =b r_{1}(M)-b r_{0}(M)+\ell(F / M)+\mu(M)-r .
\end{aligned}
$$

Now assume that $\operatorname{red}(M)=1$. It follows from Theorem [2.4 that $\mathcal{F}(M)$ is Cohen-Macaulay and hence $f_{0}(M)=1+\mu(M)-\mu(N)=\mu(M)-r$. Since $\operatorname{red}(M)=1$, by [1, Theorem 3.3], we get $b r_{0}(M)-b r_{1}(M)=\ell(F / M)$. Therefore the result follows.

As a consequence, we obtain a bound on the reduction number, similar to that of [11, Corollary 1.5]. It may be noted that in [11], the bound is derived without the CohenMacaulay assumption on the fiber cone.

Corollary 3.5. Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring and $M \subset F$ be such that $\ell(F / M)<\infty$ and having rank r. Assume that the fiber cone $\mathcal{F}(M)$ is CohenMacaulay. Then

$$
\operatorname{red}(M) \leq b r_{1}(M)-b r_{0}(M)+\ell(F / M)+1
$$

Proof. By Remark 2.3 and Theorem 3.4,

$$
\operatorname{red}(M) \leq f_{0}(M)-\mu(M)+r+1 \leq b r_{1}(M)-b r_{0}(M)+\ell(F / M)+1
$$

In [1], we obtained a Northcott type inequality for the Buchsbaum-Rim coefficients and also proved that $\operatorname{red}(M) \leq 1$ ensures the equality. Now we prove a partial converse, i.e., the equality in the Northcott inequality yields the reduction number to be at most one under the assumption that the fiber cone is Cohen-Macaulay.

Corollary 3.6. Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring and $M \subset F$ be such that $\ell(F / M)<\infty$ and having rank $r$. Then the following are equivalent:
(1) $\mathcal{F}(M)$ is Cohen-Macaulay and $b r_{0}(M)-b r_{1}(M)=\ell(F / M)$,
(2) $\operatorname{red}(M) \leq 1$.

Proof. (1) $\Rightarrow(2)$ : Follows from Corollary 3.5,
$(2) \Rightarrow(1)$ : Follows from [1, Theorem 3.3].

## 4. Direct sum of ideals

In this section, we study the Vasconcelos inequality for modules which are direct sums of an $\mathfrak{m}$-primary ideal. We begin by producing an example to show that the inequality does not hold for modules over 1-dimensional rings. Then we proceed to prove the result for $d \geq 2$.

Example 4.1. [9, Example 6.2] Let $k$ be a field and $R=k \llbracket t^{7}, t^{15}, t^{17}, t^{33} \rrbracket, I=\left(t^{7}, t^{17}, t^{33}\right)$ and $J=\left(t^{7}\right)$. Then $R$ is a one dimensional Noetherian local domain and $I$ is an $\mathfrak{m}$-primary ideal with minimal reduction $J$. Since $\ell\left(I^{2} / J I\right)=1$ and $I^{3}=J I^{2}$, by [12], $H_{I}(n)=P_{I}(n)$ for all $n>1$, where $H_{I}(n)$ and $P_{I}(n)$ denote the Hilbert function and Hilbert polynomial of $I$ respectively. It can be easily computed that $P_{I}(n)=7 n-5$. The fiber cone $\mathcal{F}(I)$ is Cohen-Macaulay [9, Theorem 3.4] and its multiplicity is $f_{0}(I)=4$. Let $M=I \oplus I$. The Buchsbaum-Rim polynomial corresponding to $M \subseteq F=R^{2}$ is given by

$$
B P_{M}(n)=(n+1)(7 n-5)=14\binom{n+1}{2}-5\binom{n}{1}-5 .
$$

Therefore

$$
b r_{0}(M)=14, b r_{1}(M)=5, \ell(F / M)=6, \mu(M)=6, f_{0}(M)=f_{0}(I)=4
$$

Hence we have $f_{0}(M)=4>b r_{1}(M)-b r_{0}(M)+\ell(F / M)+\mu(M)-(d+r-2)=2$.

Remark 4.2. Let $M=I \oplus \cdots \oplus I \subseteq F=R^{r}$. Let $S(F) \cong R\left[t_{1}, \ldots, t_{r}\right]$ and $\mathcal{R}(M) \cong$ $R\left[I t_{1}, \ldots, I t_{r}\right]$, where $t_{1}, \ldots, t_{r}$ are indeterminates over $R$. The homogeneous $R$-submodule $\mathcal{R}_{n}(M)$ of $\mathcal{R}(M)$ is given by

$$
\mathcal{R}_{n}(M) \cong \sum_{i_{1}+\cdots+i_{r}=n} I^{n} t_{1}^{i_{1}} \cdots t_{r}^{i_{r}} .
$$

Therefore, for $n \geq 0$, we have

$$
\begin{aligned}
\mu\left(\mathcal{R}_{n}(M)\right) & =\ell\left(\frac{\mathcal{R}_{n}(M)}{\mathfrak{m} \mathcal{R}_{n}(M)}\right)=\binom{n+r-1}{r-1} \mu\left(I^{n}\right) \quad \text { and } \\
B F_{M}(n) & =\ell\left(\frac{\mathcal{S}_{n}(F)}{\mathcal{R}_{n}(M)}\right)=\binom{n+r-1}{r-1} \ell\left(R / I^{n}\right) .
\end{aligned}
$$

Hence for $n \gg 0$,

$$
\mu\left(\mathcal{R}_{n}(M)\right)=\binom{n+r-1}{r-1} \sum_{i=0}^{d}(-1)^{i} f_{i}(I)\binom{n+d-1-i}{d-1-i} .
$$

Therefore

$$
f_{0}(M)=\binom{d+r-2}{r-1} f_{0}(I)
$$

Similarly we have

$$
\begin{aligned}
b r_{0}(M) & =\binom{d+r-1}{r-1} e_{0}(I) \\
b r_{1}(M) & =(d-1)\binom{d+r-2}{r-2} e_{0}(I)+\binom{d+r-2}{r-1} e_{1}(I) \\
\ell(F / M) & =r \ell(R / I), \text { and } \mu(M)=r \mu(I)
\end{aligned}
$$

We conclude the article by presenting a class of modules for which the Vasconcelos inequality holds true, namely, modules which are direct sum of two $\mathfrak{m}$-primary ideals $I$ and $J$, where one of them, say $J$, is a reduction of $I$. In this case, it can be seen that the fiber multiplicity $f_{0}(M)$ and the Buchsbaum-Rim coefficients $b r_{0}(M)$ and $b r_{1}(M)$ depend only on $I$ and $r$, not on the number of copies of $J$ involved in the direct sum. Recall that if $I$ is an $\mathfrak{m}$-primary ideal in a Cohen-Macaulay local ring $R$, then $e_{0}(I)=\mu(I)+\ell(R / I)-d+\ell(\mathfrak{m} I / \mathfrak{m} J)$, 6].

Theorem 4.3. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with $d \geq 2$. Let $I$ be an $\mathfrak{m}$-primary ideal and $J$ be a reduction. Let $\mathbf{I}=I \oplus \cdots \oplus I$ (u-times), $\mathbf{J}=J \oplus \cdots \oplus J(v$ times) and $M=\mathbf{I} \oplus \mathbf{J} \subset F=R^{r}$, where $r=u+v$. Then

$$
f_{0}(M) \leq b r_{1}(M)-b r_{0}(M)+\ell(F / M)+\mu(M)-(d+r-2)
$$

Proof. Since the assertion is proved for $d=2$ in the previous section, we may assume that $d \geq 3$. Let us assume that result holds for $M^{\prime}=I \oplus \cdots \oplus I(r$ times $)$. Since $J$ is a reduction of $I, J I^{s}=I^{s+1}$ for some $s \in \mathbb{N}$ and $e_{0}(I)=e_{0}(J)$. Let

$$
\Delta=\sum_{i=s}^{n}\binom{i+u-1}{u-1}\binom{n-i+v-1}{v-1} \text { and } \delta=\sum_{i=0}^{s-1}\binom{i+u-1}{u-1}\binom{n-i+v-1}{v-1} .
$$

Note that $\Delta=\binom{n+r-1}{r-1}-\delta$. The Buchsbaum-Rim function is given by

$$
\begin{aligned}
B F(n)= & \Delta \ell\left(R / I^{n}\right)+\sum_{i=0}^{s-1}\left[\binom{i+u-1}{u-1}\binom{n-i+v-1}{v-1} \ell\left(R / I^{i} J^{n-i}\right)\right] \\
= & {\left[\binom{n+r-1}{r-1}-\delta\right] \ell\left(R / I^{n}\right) } \\
& +\sum_{i=0}^{s-1}\left[\binom{i+u-1}{u-1}\binom{n-i+v-1}{v-1} \ell\left(R / I^{i} J^{n-i}\right)\right] \\
= & \binom{n+r-1}{r-1} \ell\left(R / I^{n}\right)+\sum_{i=0}^{s-1}\left[\binom{i+u-1}{u-1}\binom{n-i+v-1}{v-1} \ell\left(I^{n} / I^{i} J^{n-i}\right)\right] .
\end{aligned}
$$

Note for each $i=0, \ldots, s-1$,

$$
\ell\left(\frac{R}{I^{n}}\right) \leq \ell\left(\frac{R}{I^{i} J^{n-i}}\right) \leq \ell\left(\frac{R}{J^{n}}\right)
$$

Hence for each fixed $i$ and $n \gg 0$, the function $\ell\left(R / I^{i} J^{n-i}\right)$ is given by polynomial of degree $d$ with leading coefficient $e_{0}(I)$. This implies $\ell\left(I^{n} / I^{i} J^{n-i}\right)=\ell\left(R / I^{i} J^{n-i}\right)-\ell\left(R / I^{n}\right)$ is given by polynomial of degree at most $d-1$ for large $n$.
By considering $n \gg 0$, the Buchsbaum-Rim polynomial of $M$ is given by

$$
B P(n)=\binom{n+r-1}{r-1} P_{I}(n)+O\left(n^{d+r-3}\right) .
$$

Note that the first term in the above expression is the Buchsbaum-Rim polynomial of $M^{\prime}=$ $I \oplus \cdots \oplus I(r$-times $)$. Therefore $b r_{0}(M)=b r_{0}\left(M^{\prime}\right)$ and $b r_{1}(M)=b r_{1}\left(M^{\prime}\right)$. Similarly we show
that $f_{0}(M)=f_{0}\left(M^{\prime}\right)$. Let $s=\operatorname{red}_{J}(I)$. Then $J I^{i}=I^{i+1}$ for $i \geq s$.

$$
\begin{aligned}
\mu\left(\mathcal{R}_{n}(M)\right)= & {\left[\binom{n+r-1}{r-1}-\sum_{i=0}^{s-1}\left[\binom{i+u-1}{u-1}\binom{n-i+v-1}{v-1}\right] \mu\left(I^{n}\right)\right.} \\
& +\sum_{i=0}^{s-1}\left[\binom{i+u-1}{u-1}\binom{n-i+v-1}{v-1} \ell\left(\frac{I^{i} J^{n-i}}{\mathfrak{m} J^{i} J^{n-i}}\right)\right] \\
= & \binom{n+r-1}{r-1} \mu\left(I^{n}\right) \\
& +\sum_{i=0}^{s-1}\binom{i+u-1}{u-1}\binom{n-i+v-1}{v-1}\left[\ell\left(\frac{I^{i} J^{n-i}}{\mathfrak{m} I^{i} J^{n-i}}\right)-\ell\left(\frac{I^{n}}{\mathfrak{m} I^{n}}\right)\right] \\
= & \binom{d+r-2}{r-1} f_{0}(I)\binom{n+d+r-2}{d+r-2}+O\left(n^{d+r-3}\right) .
\end{aligned}
$$

Hence $f_{0}(M)=\binom{d+r-2}{r-1} f_{0}(I)=f_{0}\left(M^{\prime}\right)$. Also note that

$$
\ell(F / M)=u \ell(R / I)+v \ell(R / J)=\ell\left(F / M^{\prime}\right)-v \ell(R / I)+v \ell(R / J)
$$

and $\mu(M)=u \mu(I)+v \mu(J)=\mu\left(M^{\prime}\right)-v \mu(I)+v \mu(J)$.

Therefore

$$
\begin{aligned}
b r_{1}(M)- & b r_{0}(M)+\ell(F / M)+\mu(M)-(d+r-2)-f_{0}(M) \\
= & b r_{1}\left(M^{\prime}\right)-b r_{0}\left(M^{\prime}\right)+\ell\left(F / M^{\prime}\right)+\mu\left(M^{\prime}\right)-(d+r-2)-f_{0}\left(M^{\prime}\right) \\
& +v[\ell(R / J)+\mu(J)-(\ell(R / I)+\mu(I))] \\
\geq & 0 .
\end{aligned}
$$

It remains to prove that Vasconcelos inequality holds for $M^{\prime}=I \oplus \cdots \oplus I(r$ - times $)$. Set $\Lambda=b r_{1}\left(M^{\prime}\right)-b r_{0}\left(M^{\prime}\right)+\ell\left(F / M^{\prime}\right)+\mu\left(M^{\prime}\right)-(d+r-2)-f_{0}\left(M^{\prime}\right)$ and $e_{i}=e_{i}(I)$ for $i=0,1, \ldots, d$. Then from Remark 4.2, it follows that

$$
\begin{align*}
\Lambda= & (d-1)\binom{d+r-2}{r-2} e_{0}+\binom{d+r-2}{r-1} e_{1}-\binom{d+r-1}{r-1} e_{0}+r[\ell(R / I)+\mu(I)] \\
& -(d+r-2)-\binom{d+r-2}{r-1} f_{0}(I)  \tag{2}\\
\geq & (d-1)\binom{d+r-2}{r-2} e_{0}+\binom{d+r-2}{r-1} e_{1}-\binom{d+r-1}{r-1} e_{0}+r[\ell(R / I)+\mu(I)] \\
& -(d+r-2)-\binom{d+r-2}{r-1}\left[e_{1}-e_{0}+\ell(R / I)+\mu(I)-(d-1)\right] \\
= & (d-1)\binom{d+r-2}{r-2} e_{0}+\binom{d+r-2}{r-1} e_{1}-\left[\binom{d+r-2}{r-1}+\binom{d+r-2}{r-2}\right] e_{0} \\
& +r[\ell(R / I)+\mu(I)]-(d+r-2)-\binom{d+r-2}{r-1}\left[e_{1}-e_{0}+\ell(R / I)+\mu(I)-(d-1)\right] \\
= & {\left[(r-1) \frac{(d-2)}{d} e_{0}-\ell(R / I)-\mu(I)+d-1\right]\binom{d+r-2}{r-1} } \\
& +r[\ell(R / I)+\mu(I)]-(d+r-2) .
\end{align*}
$$

If $r=2$, then the last expression reduces to

$$
\Lambda \geq(d-2)\left[e_{0}-\ell(R / I)-\mu(I)+d\right] \geq 0
$$

If $r=3$, then we have

$$
\begin{aligned}
\Lambda & \geq\left[\frac{2(d-2)}{d} e_{0}-\ell(R / I)-\mu(I)+d-1\right]\binom{d+1}{2}+3[\ell(R / I)+\mu(I)]-(d+1) \\
& =\left[\frac{d-4}{d} e_{0}+\ell\left(\frac{\mathfrak{m} I}{\mathfrak{m} J}\right)-1\right]\binom{d+1}{2}+3\left[e_{0}+d-\ell(\mathfrak{m} I / \mathfrak{m} J)\right]-(d+1),
\end{aligned}
$$

where the last equality holds since $e_{0}=\ell(R / I)+\mu(I)-d+\ell(\mathfrak{m} I / \mathfrak{m} J)$. Now splitting the proof into the cases $d=3$ and $d \geq 4$ together with $\ell(\mathfrak{m} I / \mathfrak{m} J)=0$ or $\ell(\mathfrak{m} I / \mathfrak{m} J)>0$, one can easily obtain the inequality $\Lambda \geq 0$.

If $r \geq 4$, then $\frac{(r-1)(d-2)}{d} \geq 1$ and the equality holds if and only if $r=4$ and $d=3$. It is straightforward to verify the inequality $\Lambda \geq 0$ if $d=3$ and $r=4$. If $d>3$ or $r>4$, then one obtains the required inequality by splitting the proof into the cases $\ell(\mathfrak{m} I / \mathfrak{m} J)=0$ and $\ell(\mathfrak{m} I / \mathfrak{m} J) \geq 1$.

Remark 4.4. Suppose $I$ is a parameter ideal. Then $e_{0}(I)=\ell(R / I), e_{1}(I)=0, \mu(I)=d$ and $f_{0}(I)=1$. Therefore, it can be seen from the expression (2) that
(1) if $r=2$, then Vasconcelos inequality becomes an equality,
(2) if $r=3$, the equality holds if and only if $e_{0}(I)=1$ which happens if and only if $R$ is a regular local ring and $I$ is the unique maximal ideal;
(3) if $r \geq 4$, the equality never holds.

## References

[1] Balakrishnan R. and A. V. Jayanthan, A Northcott type inequality for Buchsbaum-Rim coefficients. J. Commut. Algebra, 8 (2016), no.4, 493-512.
[2] J. Brennan, B. Ulrich, W.V. Vasconcelos, The Buchsbaum-Rim polynomial of a module, J. Algebra 241 (2001) 379-392.
[3] D. A. Buchsbaum and D.S. Rim, A generalized Koszul complex II. Depth and multiplicity,Trans. Amer. Math. Soc. 111 (1964), 197-224.
[4] A. Corso, C. Polini, and W. V. Vasconcelos, Multiplicity of the special fiber of blowups. Math. Proc. Cambridge Philos. Soc., 140(2)(2006), 207-219.
[5] C. D'Cruz, K. N. Raghavan, and J. K. Verma, Cohen-Macaulay fiber cones. In Commutative algebra, algebraic geometry, and computational methods (Hanoi, 1996). Springer, Singapore, 1999, 233-246.
[6] S. Goto, Cohen-Macaulayness and negativity of $A$-invariants in Rees algebras associated to m-primary ideals of minimal multiplicity. J. Pure Appl. Algebra, 152(1-3)(2000), 93-107. Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998).
[7] F. Hayasaka, E. Hyry, A family of graded modules associated to a module, Comm. Algebra 36 (11) 2008, 4201-4217.
[8] C. Huneke and I. Swanson, Integral closure of ideals, rings, and modules, London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006.
[9] A. V. Jayanthan, T. J. Puthenpurakal, and J. K. Verma, On fiber cones of $\mathfrak{m}$-primary ideals. Canad. J. Math., 59(1)(2007), 109-126.
[10] A. V. Jayanthan and J. K. Verma, Hilbert coefficients and depth of fiber cones. J. Pure Appl. Algebra, 201(1-3)(2005), 97-115.
[11] M. E. Rossi, A bound on the reduction number of a primary ideal. Proc. Amer. Math. Soc., 128(5)(2000), 1325-1332.
[12] J. D. Sally, Hilbert coefficients and reduction number 2. J. Algebraic Geom., 1(2)(1992), 325-333.
[13] K. Shah, On the Cohen-Macaulayness of the fiber cone of an ideal. J. Algebra, 143(1)(1991), 156-172.
[14] A. Simis, B. Ulrich, W. Vasconcelos, Rees algebras of modules, Proc. Lond. Math. Soc. (3)87 (2003) 610-646.
[15] W. Vasconcelos, Integral Closure:Rees algebra, Multiplicities, Algorithms. Springer Monographs in Mathematics, 2005.
[16] W. V. Vasconcelos, Hilbert functions, analytic spread, and Koszul homology. Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992), 401-422, Contemp. Math., 159, Amer. Math. Soc., Providence, RI, 1994.

E-mail address: r.balkrishnan@gmail.com

E-mail address: jayanav@iitm.ac.in

Department of Mathematics, Indian Institute of Technology Madras, Chennai, INDIA 600036.

