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On the solution of dual integral equations

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Abstract

A quick method of solution of dual integral equations involving a kernel comprised of trigonometric functions is explained. Certain solvability criteria are obtained in terms of forcing functions for the unique solution of the dual integral equations.

Keywords: Singular integral equation; Logarithmic singularity; Trigonometric kernels; Surface tension; Water wave scattering

1. Introduction

Dual integral equations are often encountered in different branches of mathematical physics and they generally arise while solving a boundary value problem with mixed boundary conditions (see Sneddon [1] and Debnath Lokenath [2]). Chakrabarti et al. [3,4] studied linear water wave scattering by a vertical barrier by reducing the corresponding boundary value problem to dual integral equations with a trigonometric kernel. The behavior of one of the integrals of these dual integral equations at the point where the boundary condition changes plays a crucial role in determining their solution.

The dual integral equations

$$\begin{aligned} &\frac{2}{\pi}\int_0^\infty A(\xi)\left[\sum_{k=0}^{n} c_{k+1}\frac{\partial^{2k+1}}{\partial y^{2k+1}} + c_0\right]\sin\xi y\,\mathrm{d}\xi = f(y), \quad y \in L \\ &\frac{2}{\pi}\int_0^\infty \xi A(\xi)\left[\sum_{k=0}^{n} c_{k+1}\frac{\partial^{2k+1}}{\partial y^{2k+1}} + c_0\right]\sin\xi y\,\mathrm{d}\xi = g(y), \quad y \in (0,\infty)\setminus L, \end{aligned} \tag{1.1}$$

where c_i, $i = 0, 1, 2, \dots, 2n+1$, are real or complex constants and $L = (0, a)$ or (a, ∞), $a > 0$, are uniquely solvable when the functions g and f are suitably differentiable. They arise in connection with the radiation or scattering of surface water waves propagating in deep water with surface tension, associated with partial vertical wave-makers or barriers taken into account (see [5]). In this context, L represents the complement of the vertical wave-maker or barrier position along the positive axis.

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In the present note, we attempt to solve the above dual integral equations (1.1) completely by converting them into logarithmic singular integral equations. With the aid of the bounded solutions of these singular integral equations, a unique solution for the dual integral equations is obtained under certain restrictions or conditions on the forcing functions f and g.

2. The method of solution

We start with solving the dual integral equations (1.1) with $L = (0, a)$. They can be equivalently written as a set of differential equations

$$T\left[\frac{2}{\pi}\int_0^\infty A(\xi)\sin\xi y\,\mathrm{d}\xi\right] = f(y), \quad 0 < y < a$$
$$T\left[\frac{2}{\pi}\int_0^\infty \xi A(\xi)\sin\xi y\,\mathrm{d}\xi\right] = g(y), \quad a < y < \infty,$$

where $T = \sum_{k=0}^{n} c_{k+1}\frac{\partial^{2k+1}}{\partial y^{2k+1}} + c_0$. After solving the above $(2n+1)$th-order ordinary differential equations, they transform into a new set of dual integral equations

$$\frac{2}{\pi}\int_0^\infty A(\xi)\sin\xi y\,\mathrm{d}\xi = \sum_{k=0}^{2n} D_k \mathrm{e}^{\lambda_k y} + T^{-1}[f(y)] \equiv h_1(y), \quad 0 < y < a \tag{2.1}$$

$$\frac{2}{\pi}\int_0^\infty \xi A(\xi)\sin\xi y\,\mathrm{d}\xi = \sum_{k=0}^{2n} E_k \mathrm{e}^{\lambda_k y} + T^{-1}[g(y)] \equiv h_2(y), \quad a < y < \infty, \tag{2.2}$$

where $T^{-1}[f(y)], T^{-1}[g(y)]$ are the particular integrals with respect to the differential operator T and $\lambda_k, k = 0, 1, 2, \ldots, n$, are the roots of the polynomial equation $\sum_{k=0}^{n} c_{k+1}x^{2k+1} + c_0 = 0$.

Accommodating zero along the y axis, we must have from Eq. (2.1) that

$$h_1^{(2i)}(y) = 0, \quad i = 0, 1, 2, \ldots, n, \text{ i.e., } \sum_{k=0}^{2n} D_k\lambda_k^{2i} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}\left[T^{-1}(f(y))\right]\Bigg|_{y=0} = 0, i = 0, 1, 2, \ldots, n, \tag{2.3}$$

where we follow the notation that the superscript in parentheses for a function, such as $h_1(y)$, denotes the order of differentiation.

The dual integral equations (2.1) and (2.2) can be differentiated up to $2i,\ i = 0, 1, 2, \ldots, n$, times and the resulting sets of dual integral equations are given by

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty (-1)^i\xi^{2i}A(\xi)\sin\xi y\,\mathrm{d}\xi &= \sum_{k=0}^{2n} D_k\lambda_k^{2i}\mathrm{e}^{\lambda_k y} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}T^{-1}[f(y)]\\ &\equiv h_1^{(2i)}(y), \quad 0 < y < a\end{aligned} \tag{2.4}$$

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty (-1)^i\xi^{2i+1}A(\xi)\sin\xi y\,\mathrm{d}\xi &= \sum_{k=0}^{2n} E_k\lambda_k^{2i}\mathrm{e}^{\lambda_k y} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}T^{-1}[g(y)]\\ &\equiv h_2^{(2i)}(y), \quad a < y < \infty, \text{ for } i = 0, 1, 2, \ldots, n.\end{aligned} \tag{2.5}$$

We define

$$P_{2i+1}(y) = \frac{2}{\pi}(-1)^i\int_0^\infty \xi^{2i+1}A(\xi)\sin\xi y\,\mathrm{d}\xi, \quad 0 < y < a,\ i = 0, 1, 2, \ldots, n \tag{2.6}$$

and then we clearly see that $P_{2i+1}(y) = (-1)^i P_1^{(2i)}(y),\ i = 1, 2, \ldots, n$.

From the Eqs. (2.5) and (2.6), by the Fourier sine transform, we derive that

$$\xi^{2i+1}A(\xi) = \int_0^\infty Q_{2i+1}(y)\sin\xi y\,\mathrm{d}y, \tag{2.7}$$

where

$$Q_{2i+1}(y) = \begin{cases} \text{(i)} \quad P_1^{(2i)}(y), \quad 0 < y < a \\ \text{(ii)} \quad (-1)^i h_2^{(2i)}(y), \quad a < y < \infty, \end{cases} \quad i = 0, 1, 2, \ldots, n.$$

By equating $A(\xi)$ recursively in the relation (2.7) and using integration by parts, we must have the following conditions:

$$h_2^{(2i)}(a) = 0, \quad i = 0, 1, 2, \ldots, n-1,$$

i.e.,

$$\sum_{k=0}^{2n} E_k \lambda_k^{2i} e^{\lambda_k a} + \frac{d^{2i}}{dy^{2i}} \left[T^{-1}[g(y)] \right] \Big|_{y=a} = 0, \quad i = 0, 1, 2, \ldots, n-1 \tag{2.8}$$

and

$$P^{(2i+1)}(a) = (-1)^i h_2^{(2i+1)}(a) = 0, \quad i = 0, 1, 2, \ldots, n-1. \tag{2.9}$$

Note that the conditions (2.9) are not suitable for representing them in terms of the unknown constants. For this purpose, we derive a set of logarithmic singular integral equations and utilize their bounded solution. The restriction of the bounded solution will be clear a little later.

Substituting $\xi^{2i} A(\xi), i = 0, 1, 2, \ldots, n$, from Eq. (2.7) into the relation (2.4), we derive a set of singular integral equations

$$\int_0^a P_1^{(2i)}(u) \log \left| \frac{u+t}{u-t} \right| du = -\int_a^\infty h_2^{(2i)}(u) \log \left| \frac{u+t}{u-t} \right| du + \sum_{k=0}^{2n} D_k \lambda_k^{2i} e^{\lambda_k t} + \frac{d^{2i}}{dt^{2i}} \left[T^{-1}[f(t)] \right] \equiv R_i(t), \quad i = 0, 1, 2, \ldots, n, \tag{2.10}$$

where we have utilized the following relation (see [6], Eq. 3.741(1)):

$$\int_0^\infty \frac{\sin \xi y \, \sin \xi t}{\xi} d\xi = -\frac{1}{2} \log \left| \frac{y-t}{y+t} \right|, \quad \text{for } y, t \in (0, \infty).$$

The bounded solutions of the logarithmic singular integral equations (see [7]), described by the relation (2.10), are given by

$$P_1^{(2i)}(u) = \frac{2}{\pi} u \sqrt{a^2 - u^2} \int_0^a \frac{R_i^{(1)}(t)}{\sqrt{a^2 - t^2}(u^2 - t^2)} dt, \quad 0 < u < a, \tag{2.11}$$

provided that

$$\int_0^a \frac{R_i^{(1)}(t)}{\sqrt{a^2 - t^2}} dt = 0, \quad i = 0, 1, 2, \ldots, n. \tag{2.12}$$

The bounded property of the solution is that $P_1^{(2i)}(u) = 0, i = 0, 1, 2, \ldots, n$, at the end points 0 and a. Now, we multiply the relation (2.11) by u and integrate between 0 and a. After using the integral

$$\int_0^a \frac{t^2 \sqrt{b^2 - t^2}}{t^2 - x^2} dt = -\frac{\pi}{2} x^2$$

which is evaluated by the standard contour integration technique, we obtain that

$$a\, P_1^{(2i-1)}(a) = -\int_0^a \frac{t^2 R_i^{(1)}(t)}{\sqrt{a^2 - t^2}} dt, \quad i = 1, 2, \ldots, n. \tag{2.13}$$

From the relations (2.9) and (2.13), the conditions expressed in terms of the unknown constants are obtained as

$$\int_0^a \frac{t^2 R_i^{(1)}(t)}{\sqrt{a^2-t^2}}\mathrm{d}t + a\, h_2^{(2i-1)}(a) = 0, \quad i = 1, 2, \ldots, n \tag{2.14}$$

The relations (2.3), (2.8), (2.12) and (2.14) together will determine the $4n+1$ unknown parameters that appear in Eqs. (2.1) and (2.2) completely. Thus, we conclude here that the dual integral equations (1.1) have a unique solution provided a set of solvability criteria are satisfied by the forcing functions f and g. We remark here that in Eq. (2.5), accommodating infinity along the y axis, one can equate certain constants, whose coefficient is $\mathrm{e}^{\lambda_k y}$ with the root λ_k having a positive real part, to zero. These extra conditions are naturally utilized in the physical problems of practical interest.

For the case of L being (a, ∞) in the relation (1.1), the equivalent form of the dual integral equations

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty (-1)^i \xi^{2i+1} A(\xi)\sin\xi y\,\mathrm{d}\xi &= \sum_{k=0}^{2n} D_k \lambda_k^{2i}\mathrm{e}^{\lambda_k y} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}T^{-1}[f(y)] \\ &\equiv h_1^{(2i)}(y), \quad 0 < y < a \end{aligned}\tag{2.15}$$

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty (-1)^i \xi^{2i} A(\xi)\sin\xi y\,\mathrm{d}\xi &= \sum_{k=0}^{2n} E_k \lambda_k^{2i}\mathrm{e}^{\lambda_k y} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}T^{-1}[g(y)] \\ &\equiv h_2^{(2i)}(y), \quad a < y < \infty, \text{ for } i = 1, 2, \ldots, n, \end{aligned}\tag{2.16}$$

can be similarly solved. By defining $P_{2i+1}(y), i = 0, 1, 2, \ldots, n$, as in (2.6) for $a < y < \infty$, we obtain from the relation (2.15) that

$$\xi^{2i+1}A(\xi) = \int_0^\infty Q_{2i+1}(y)\sin\xi y\,\mathrm{d}y, \tag{2.17}$$

with

$$Q_{2i+1}(y) = \begin{cases}(-1)^i h_1^{(2i)}(y), & 0 < y < a \\ P_1^{(2i)}(y), & a < y < \infty,\end{cases} \qquad i = 0, 1, 2, \ldots, n.$$

Then, the conditions of the solvability criteria are specified as

$$\text{(i) } h_2^{(2i)}(0) = 0, \quad \text{i.e.,} \quad \sum_{k=0}^{2n} E_k\lambda_k^{2i} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}\left[T^{-1}(f(y))\right]\Big|_{y=0} = 0, \quad i = 0, 1, 2, \ldots, n.$$

$$\text{(ii) } h_1^{(2i)}(a) = 0, \quad \text{i.e.,} \quad \sum_{k=0}^{2n} D_k\lambda_k^{2i}\mathrm{e}^{\lambda_k a} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}\left[T^{-1}[g(y)]\right]\Big|_{y=a} = 0, \quad i = 0, 1, 2, \ldots, n-1.$$

$$\text{(iii) } P^{(2i+1)}(a) = (-1)^i h_1^{(2i+1)}(a) = 0, \quad i = 0, 1, 2, \ldots, n-1, \tag{2.18}$$

where in this case

$$P^{(2i+1)}(y) = \frac{2}{\pi}(-1)^i\int_0^\infty \xi^{2i+1}A(\xi)\sin\xi y\,\mathrm{d}\xi, \quad a < y < \infty,\ i = 0, 1, 2, \ldots, n.$$

$$\text{(iv) } \int_a^\infty \frac{t S_i^{(1)}(t)}{\sqrt{a^2-t^2}}\mathrm{d}t = 0, \quad i = 0, 1, 2, \ldots, n,$$

with

$$S_i(t) = -\int_a^\infty h_1^{(2i)}(u)\log\left|\frac{u+t}{u-t}\right|\mathrm{d}u + \sum_{k=0}^{2n} E_k\lambda_k^{2i}\mathrm{e}^{\lambda_k t} + \frac{\mathrm{d}^{2i}}{\mathrm{d}t^{2i}}\left[T^{-1}[g(t)]\right].$$

Condition (iii) is not in a suitable form to be expressed in terms of the unknown constants and it cannot be modified as in the previous case. Using certain integral relations, an equivalent relation between $h_1^{(2i+1)}(y)$ and $h_2^{(2i+1)}(y)$ is derived as in the following:

Multiplying the Eq. (2.16) by $\frac{1}{\sqrt{y^2-a^2}}$ and integrating between a and ∞, after using the integral relation (see [6] pp. 419)

$$\frac{2}{\pi}\int_a^\infty \frac{\sin \xi y}{\sqrt{y^2-a^2}}\,\mathrm{d}y = J_0(a\xi),$$

where J_0 is a Bessel function, gives

$$\int_0^\infty \xi^{2i} A(\xi) J_0(a\xi)\mathrm{d}\xi = (-1)^i \int_a^\infty \frac{h_2^{(2i)}(y)}{\sqrt{y^2-a^2}}\,\mathrm{d}y. \tag{2.19}$$

Making the substitution from the relation (2.17) into the relation (2.19) and use of the integral (see [6] pp. 744)

$$\int_0^\infty J_0(a\xi)\frac{\sin \xi y}{\xi}\,\mathrm{d}\xi = \frac{\pi}{2}$$

gives

$$-P_1^{(2i-1)}(a) + (-1)^i \int_0^a h_1^{(2i)}(y)\mathrm{d}y = (-1)^i \frac{\pi}{2}\int_a^\infty \frac{h_2^{(2i)}(y)}{\sqrt{y^2-a^2}}\,\mathrm{d}y, \quad i = 1, 2, \ldots, n. \tag{2.20}$$

From the relations (2.18) and (2.20), we determine the required condition as

$$h_1^{(2i-1)}(a) + \int_a^\infty \frac{h_2^{(2i)}(y)}{\sqrt{y^2-a^2}}\,\mathrm{d}y = 0, \quad i = 1, 2, \ldots, n.$$

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References

[1] I.N. Sneddon, The use of Integral Transforms, McGraw-Hill, New York, 1972.
[2] D. Lokenath, Integral Transforms and Their Applications, CRC Press, Boca Raton USA, 1995.
[3] A. Chakrabarti, B.N. Mandal, Solution of some dual integral equations, ZAMM 78 (1998) 141–144.
[4] A. Chakrabarti, S. Banerjea, T. Sahoo, A unified approach to problems of scattering of surface water waves by vertical barriers, J. Austral. Math. Soc. Ser. B 39 (1997) 93–103.
[5] P.F. Rhodes-Robinson, The effect of surface tension on the progressive waves due to incomplete vertical wave-makers in water of infinite depth, Proc. R. Soc. Lond. A 435 (1991) 293–319.
[6] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic press, London, 1980.
[7] A. Chakrabarti, S.R. Manam, Solution of the logarithmic singular integral equation, Appl. Math. Lett. 16 (2003) 369–373.