On the Lipschitz continuity of the solution map in linear complementarity problems over second-order cone

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Let $\mathcal{K} \subseteq \mathbb{R}^n$ denote the second-order cone. Given an $n \times n$ real matrix M and a vector $q \in \mathbb{R}^n$, the second-order cone linear complementarity problem SOLCP(M, q) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \quad y := Mx + q \in \mathcal{K} \quad \text{and} \quad y^T x = 0.$$

We say that $M \in \mathbf{Q}$ if SOLCP(M, q) has a solution for all $q \in \mathbb{R}^n$. An $n \times n$ real matrix A is said to be a **Z**-matrix with respect to $\mathcal{K}$ iff:

$$x \in \mathcal{K}, \quad y \in \mathcal{K} \quad \text{and} \quad x^T y = 0 \implies x^T M y \leq 0.$$

Let $\Phi_M(q)$ denote the set of all solutions to SOLCP(M, q). The following results are shown in this paper:

- If $M \in \mathbf{Z} \cap \mathbf{Q}$, then Φ_M is Lipschitz continuous if and only if M is positive definite on the boundary of $\mathcal{K}$.
- If M is symmetric, then Φ_M is Lipschitz continuous if and only if M is positive definite.

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1. Introduction

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional real inner-product space. We say that a cone $\mathcal{C} \subseteq V$ is proper if $\mathrm{int}(\mathcal{C})$ is non-empty, closed, convex and pointed. The dual of the cone $\mathcal{C}$ is given by

$$\mathcal{C}^* := \{x \in V : \langle v, x \rangle \geq 0 \; \forall v \in \mathcal{C}\}.$$

Given a linear transformation $L : V \to V$, a proper cone $\mathcal{C} \subseteq V$ and a vector $q \in V$, the linear complementarity problem $\mathrm{LCP}(L, \mathcal{C}, q)$ is to find a vector $x \in V$ such that

$$x \in \mathcal{C}, \quad y := L(x) + q \in \mathcal{C}^* \quad \text{and} \quad \langle x, y \rangle = 0.$$

Complementarity problems appear in various fields. The classical linear complementarity problem is obtained by specializing $V = \mathbb{R}^n$ and $\mathcal{C} := \mathbb{R}^n_+$ for which the text of Cottle, Pang and Stone [4] is a standard reference where one can find several applications as well as theoretical and numerical results. Complementarity problem can be defined in a more general setting and is a special case of a finite dimensional variational inequality problem (VIP). A wide literature on finite dimensional VIPs and complementarity problems appears in [5].

1.1. Lipschitz continuity of solutions

Let $\Phi_{L,\mathcal{C}}(q)$ be the set of all solutions to $\mathrm{LCP}(L, \mathcal{C}, q)$. A fundamental question in linear complementarity theory is the following: When is $\Phi_{L,\mathcal{C}}$ Lipschitz continuous? To make this precise, we define the Lipschitz continuity of the set-valued map $\Phi_{L,\mathcal{C}}$.

Definition 1. Suppose $L : V \to V$ is a linear transformation such that $\mathrm{LCP}(L, \mathcal{C}, q)$ has a solution for all $q \in V$. We say that the set-valued map $\Phi_{L,\mathcal{C}}$ is Lipschitz continuous if there exists $c > 0$ such that:

$$\Phi_{L,\mathcal{C}}(q) \subseteq \Phi_{L,\mathcal{C}}(q') + c\|q - q'\|\mathbf{B} \quad \forall q, q' \in \mathbb{R}^n.$$

Here $\|.\|$ denotes the Euclidean norm and $\mathbf{B} = \{x \in V : \|x\| \leq 1\}$.

We summarize the known results from the literature:

- If L is positive definite on V (i.e., $\langle L(v), v \rangle > 0 \;\; \forall 0 \neq v \in V$) and $\mathcal{C}$ is a proper cone in V, then $\Phi_{L,\mathcal{C}}$ is Lipschitz continuous. (See Facchinei and Pang [5] or Corollary 1 in this paper.)
- Let A be an $n \times n$ real matrix such that all the principal minors are positive. Then $\Phi_{A, \mathbb{R}^n_+}$ is Lipschitz continuous. A proof of this result can be seen in [6]. Conversely, if $\Phi_{A, \mathbb{R}^n_+}$ is Lipschitz continuous then Murthy et al. [10] showed that all the principal minors of A are positive. In this setting, it is well-known that $\mathrm{LCP}(A, \mathbb{R}^n_+, q)$ has a

unique solution for all $q \in \mathbb{R}^n$ if and only if all the principal minors of A are positive. (See Cottle, Pang and Stone [4].)

- Suppose $(V, \langle \cdot, \cdot \rangle, \circ)$ is a Euclidean Jordan algebra and K is the corresponding symmetric cone. If L is a linear transformation on V such that $\Phi_{L,K}$ is Lipschitz continuous, then $\det(L) > 0$ and $\langle L(v), v \rangle > 0$, where $0 \neq v \in K$ is an extreme direction. (See Theorem 4 in [1].)

- Suppose $\mathcal{S}^n$ is the space of all $n \times n$ real symmetric matrices and $\mathcal{S}^n_+$ is the cone of all positive semidefinite matrices in $\mathcal{S}^n$. Then there exists a linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ such that $\Phi_{L,\mathcal{S}^n_+}(Q)$ has a unique solution for all $Q \in \mathcal{S}^n$, but $\Phi_{L,\mathcal{S}^n_+}$ is not Lipschitz continuous. (See Example 1.3 in [8].)

  Given L and $\mathcal{C}$, in general it is hard to verify whether $\Phi_{L,\mathcal{C}}$ is Lipschitz continuous or not. In fact, an answer to the following question is not known: Does there exist a non-polyhedral symmetric cone K in a Euclidean Jordan algebra $(V, \circ, \langle \cdot, \cdot \rangle)$ such that L is not positive definite on V and $\Phi_{L,K}$ is Lipschitz continuous? Our first result of this paper answers this question. More precisely, we show the following result: Let $\mathcal{K} \subseteq \mathbb{R}^n$ be the second-order cone. There exists a family of $n \times n$ real matrices, ($\mathcal{M}$, say) such that if $M \in \mathcal{M}$, then $\Phi_{M,\mathcal{K}}$ is Lipschitz continuous, but M is not positive definite. In the second part of the paper, we consider the following question: If $\Phi_{A,\mathcal{K}}$ is Lipschitz continuous, when is A positive definite? We show that if A is symmetric, then $\Phi_{A,\mathcal{K}}$ is Lipschitz continuous if and only A is positive definite.

1.2. Second-order cone, linear complementarity problems and **Z**-transformations

In the space $\mathbb{R}^n$, the *second-order cone* (or the n-dimensional ice-cream cone or the Lorentz cone) is defined by

$$\mathcal{K}_n := \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : (x_2^2 + \cdots + x_n^2)^{1/2} \leq x_1\}.$$

The cone $\mathcal{K}_n$ is self-dual for any n and non-polyhedral when $n \geq 3$. By fixing $n \geq 3$, we will write $\mathcal{K}$ (for brevity) to denote the n-dimensional second-order cone $\mathcal{K}_n$ in the rest of the paper. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. Given a matrix $M \in M_n(\mathbb{R})$ and a vector $q \in \mathbb{R}^n$, the second-order cone linear complementarity problem $\mathrm{LCP}(M, \mathcal{K}, q)$ is to find a vector $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \quad y := Mx + q \in \mathcal{K} \quad \text{and} \quad x^T y = 0.$$

From now on we will use the notations $\mathrm{SOLCP}(M, q)$ and $\Phi_M(q)$ to denote $\mathrm{LCP}(M, \mathcal{K}, q)$ and $\Phi_{M,\mathcal{K}}(q)$ respectively. SOLCP has been of interest to several authors in recent times (see [2,12] and references therein). While the results for VIPs are applicable to SOLCP, the extra-structure available in this second-order cone framework allows us to go beyond the general study and derive specialized results.

In the theory of non-negative matrices, a real square matrix said to be a Z-matrix if all its off-diagonal entries are non-positive [3]. The definition of a Z-matrix can be reformulated as follows: $M \in M_n(I\!R)$ is a Z-matrix if and only if

$$x \in I\!R_+^n, \;\; y \in I\!R_+^n \;\; \text{and} \;\; x^T y = 0 \Longrightarrow x^T M y \le 0. \tag{1}$$

Given a proper cone $\mathcal{C}$, using the above reformulation, Gowda and Tao [9] introduced **Z**-transformations with respect to $\mathcal{C}$. The definition of **Z**-matrices with respect to the second order cone is given below:

Definition 2. We say that $M \in M_n(I\!R)$ is **Z**-matrix with respect to $\mathcal{K}$ iff:

$$x \in \mathcal{K}, \;\; y \in \mathcal{K} \;\; \text{and} \;\; x^T y = 0 \;\; \Longrightarrow x^T M y \le 0.$$

The motivation for studying **Z**-transformations with respect to a proper cone, significance and applications are well-discussed in [9]. It may be noted that if $S \in M_n(I\!R)$ and $S(\mathcal{K}) \subseteq \mathcal{K}$, then $I - S$ is a **Z**-matrix with respect to $\mathcal{K}$. Matrices that satisfy $S(\mathcal{K}) \subseteq \mathcal{K}$ are completely characterized in Loewy and Schneider [7]. When M is a **Z**-matrix with respect to $\mathcal{K}$, we have the following result from [9] which we record for later use.

Theorem 1. *Let M be a **Z**-matrix with respect to $\mathcal{K}$. Then the following are equivalent:*

(A) *SOLCP(M, q) has a solution for all $q \in I\!R^n$.*
(B) $\det(M) > 0$ *and $M^{-1}(\mathcal{K}) \subseteq \mathcal{K}$.*
(C) *There exists $u \in \text{int}(\mathcal{K})$ such that $Mu \in \text{int}(\mathcal{K})$.*
(D) *M is positive stable.*

1.3. Main results of the paper

We summarize our main results of this paper. Let $M \in M_n(I\!R)$ be such that SOLCP(M, q) has a solution for all $q \in I\!R^n$ and $\Phi_M(q)$ be the set of all solutions to SOLCP(M, q). We show the following:

- If M is a **Z**-matrix with respect to $\mathcal{K}$, then Φ_M is Lipschitz continuous if and only if $x^T M x > 0$ for all $x \in \partial\mathcal{K} \smallsetminus \{0\}$.
- If M is a symmetric matrix, then Φ_M is Lipschitz continuous if and only if M is positive definite.

2. Preliminaries

The following notation is used throughout this paper.

1. Throughout, we use $\mathbb{R}^n$ to denote the Euclidean n-space whose elements depending on the context, are regarded as row or column vectors.
2. The boundary and the interior of a set Δ are denoted by $\partial\Delta$ and $\text{int}(\Delta)$ respectively.
3. Given an $n \times n$ matrix M, the set of all solutions to $\text{SOLCP}(M, q)$ is denoted by $\text{SOL}(M, q)$ and let $\Phi_M(q) := \text{SOL}(M, q)$.
4. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ will denote the Euclidean norm which is defined by $\|x\| := \sqrt{x^T x}$.
5. For an $n \times n$ matrix A, let $\|A\| := \inf\{c > 0 : \|Ax\| \leq c\|x\| \ \forall x \in \mathbb{R}^n\}$.
6. Let J denote the diagonal matrix $\text{diag}(1, -1, -1, \cdots, -1)$.
7. If $\text{SOLCP}(A, q)$ has a solution for all $q \in \mathbb{R}^n$, then we write $A \in \mathbf{Q}$. If A is a $\mathbf{Z}$-matrix with respect to $\mathcal{K}$, then we write $A \in \mathbf{Z}$.

Definition 3. We say that $A \in M_n(\mathbb{R})$ has GUS-property if $\text{SOLCP}(A, q)$ has a unique solution for all $q \in \mathbb{R}^n$.

Definition 4. Let $A \in \mathbf{Q}$. We say that Φ_A (defined by $\Phi_A(q) := \text{SOL}(A, q)$) is Lipschitz continuous if there exists $c > 0$ such that

$$\Phi_A(q) \subseteq \Phi_A(q') + c\|q - q'\|\mathbf{B}$$

for all q and q' in $\mathbb{R}^n$. Here $\mathbf{B} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

Definition 5. Let $A \in M_n(\mathbb{R})$. We say that A is positive definite on a set $\Delta \subseteq \mathbb{R}^n$ iff:

$$0 \neq x \in \Delta \implies x^T A x > 0.$$

The following elementary lemma will be useful in the sequel. We refer to [2] and [9] for details.

Lemma 1. *The following are true:*

(a) *Any two vectors $x \in \partial\mathcal{K}$ and $y \in \partial\mathcal{K}$ are orthogonal if and only if $y = \mu J x$ for some $\mu \geq 0$.*
(b) *If $x \notin \pm\mathcal{K}$, then there exist $a \in \partial\mathcal{K} \smallsetminus \{0\}$ and $b \in \partial\mathcal{K} \smallsetminus \{0\}$ such that $x = a - b$ and $a^T b = 0$.*
(c) *$\mathcal{K}$ is self dual. This means that $\mathcal{K} = \{x \in \mathbb{R}^n : x^T y \geq 0 \ \ \forall y \in \mathcal{K}\}$.*
(d) *$\text{int}(\mathcal{K}) = \{x : x^T y > 0 \ \ \forall y \in \mathcal{K} \smallsetminus \{0\}\}$.*

By items (a) and (d) in Lemma 1 we have the following:

Lemma 2. *Let $M \in M_n(\mathbb{R})$. Then $x \in \text{SOL}(M, q) \cap \partial\mathcal{K}$ if and only if there exists $\mu \geq 0$ such that $Mx + q = \mu J x$. Furthermore, if $x \in \text{int}(\mathcal{K}) \cap \text{SOL}(M, q)$ then $Mx + q = 0$.*

The following result follows from Theorem 4 in [1] and Theorem 4.1 in [11].

Theorem 2. *Let $A \in M_n(\mathbb{R})$. Suppose Φ_A is Lipschitz continuous. Then the following are true:*

(i) $\det(A) > 0$.
(ii) *A is positive definite on $\partial \mathcal{K}$.*

3. Lipschitz continuity of $\mathbf{Z} \cap \mathbf{Q}$ matrices

Our first objective in this paper is to show that if $M \in \mathbf{Z} \cap \mathbf{Q}$, then Φ_M is Lipschitz continuous if and only if M is positive definite on the boundary of $\mathcal{K}$. The following lemmas will be used to prove the main result.

Lemma 3. *Let $M \in M_n(\mathbb{R})$ and $\Delta \subseteq \mathbb{R}^n$ be a non-empty closed set such that*

(i) $p^T M p > 0$ *for all $p \in \Delta \smallsetminus \{0\}$.*
(ii) $p \in \Delta$ *and* $\alpha > 0 \implies \alpha p \in \Delta$.

Then there exists $c > 0$ such that

$$x \in \Phi_M(q), \ y \in \Phi_M(q') \ and \ x - y \in \Delta \implies \|x - y\| \le c\|q - q'\|.$$

Proof. Let $x \in \phi_M(q)$, $y \in \phi_M(q')$ and $x - y \in \Delta$. Suppose $x \neq y$. The vectors $w := Mx + q$ and $w' := My + q'$ belong to $\mathcal{K}$ and hence by self-duality of $\mathcal{K}$, $x^T w' \ge 0$ and $y^T w \ge 0$. Since $x^T w = 0$ and $y^T w' = 0$,

$$(x - y)^T (w - w') \le 0. \tag{2}$$

By the equation

$$(x - y)^T M (x - y) = (x - y)^T (w - w') + (x - y)^T (q - q'),$$

from (2), it follows that

$$(x - y)^T M (x - y) \le (x - y)^T (q' - q). \tag{3}$$

Let $\Delta' = \{p : p \in \Delta\} \cap \{p : \|p\| = 1\}$. Assumption (ii) implies that Δ' is non-empty compact set. Define $\alpha := \min\{p^T M p : p \in \Delta'\}$. By assumption (i), $\alpha > 0$. Write $\beta = \|x - y\|$. From (3), we now have

$$\alpha \le \frac{1}{\beta^2}(x - y)^T(q' - q).$$
$$\le \frac{\|q - q'\|}{\beta}. \tag{4}$$

If $x = y$, then

$$\|x - y\| \le \frac{1}{\alpha}\|q - q'\|. \tag{5}$$

Define $c := \frac{1}{\alpha}$. By (4) and (5),

$$\|x - y\| \le c\|q - q'\|.$$

The proof is now complete. $\quad\square$

The following result is immediate now.

Corollary 1. *Let M be a positive definite matrix. Then Φ_M is Lipschitz continuous.*

Lemma 4. *Let $A \in \mathbf{Z}$ and $z^T A z > 0$ for all non-zero $z \in \partial\mathcal{K}$. Then, $p^T A p > 0$ for all $p \notin \mathcal{K} \cup -\mathcal{K}$.*

Proof. Let $p \notin \mathcal{K} \cup -\mathcal{K}$. Item (c) in Lemma 1 implies that there exist $a \in \partial\mathcal{K} \smallsetminus \{0\}$ and $b \in \partial\mathcal{K} \smallsetminus \{0\}$ such that $a^T b = 0$ and $p = a - b$. Since $A \in \mathbf{Z}$, $a^T A b$ and $b^T A a$ are non-positive. By our assumption, $a^T A a$ and $b^T M b$ are positive. Hence

$$p^T A p = a^T A a + b^T A b - a^T A b - b^T A a > 0.$$

The proof of the lemma is complete. $\quad\square$

Theorem 3. *Let $M \in \mathbf{Z} \cap \mathbf{Q}$. Then the following are equivalent:*

(i) Φ_M *is Lipschitz continuous.*
(ii) $x^T M x > 0$ *for all $\partial\mathcal{K} \smallsetminus \{0\}$.*

Proof. (i)$\Longrightarrow$ (ii) is a known result, see Theorem 2. We now prove the converse (ii) $\Rightarrow$ (i). By Theorem 4 in [2], M has GUS-property. Let $u, v \in \mathbb{R}^n$, $\{x\} = \Phi_M(u)$ and $\{y\} = \Phi_M(v)$. We now show that there exists $C > 0$ such that

$$\|x - y\| \le C\|u - v\| \quad \forall u, v \in \mathbb{R}^n.$$

If $x = y$, then we have

$$\|x - y\| \le \|u - v\|. \tag{6}$$

Suppose $x \ne y$. Consider the following possible cases:

(c1) $x - y \in \pm\partial\mathcal{K}$.
(c2) $x - y \notin \pm\mathcal{K}$.
(c3) $x - y \in \pm\mathrm{int}(\mathcal{K})$.

Assume that $x - y \in \pm \partial \mathcal{K}$. Define

$$\Omega := \{p : 0 \neq p \in \pm \partial \mathcal{K}\}.$$

By assumption (ii), $p^T M p > 0$ for all $p \in \Omega$. Now $x = \Phi_M(u)$, $y = \Phi_M(v)$ and $x - y \in \Omega$. If $\alpha > 0$ and $p \in \Omega$, then $\alpha p \in \Omega$. By Lemma 3 we can find $C_1 > 0$ such that

$$\|x - y\| \leq C_1 \|u - v\|. \tag{7}$$

Assume (c2). Define

$$\nabla := \{p : p \notin \pm \mathcal{K}\}.$$

In view of Lemma 4,

$$p^T M p > 0 \quad \forall p \in \nabla.$$

By Lemma 3, there exists $C_2 > 0$ such that

$$\|x - y\| \leq C_2 \|u - v\|. \tag{8}$$

Consider case (c3). Without loss of generality, let $x - y \in \text{int}(\mathcal{K})$. Then $x \in \text{int}(\mathcal{K})$ and by Lemma 2,

$$Mx + u = 0. \tag{9}$$

If $My + v = 0$, then by (9),

$$x = -M^{-1}u \quad \text{and} \quad y = -M^{-1}v.$$

So,

$$\|x - y\| = \|M^{-1}(u - v)\| \leq \|M^{-1}\|\|u - v\|. \tag{10}$$

If $y \in \text{int}(\mathcal{K})$, then $My + v = 0$ and hence (10) holds. Now, assume that $y \in \partial \mathcal{K}$. By Lemma 1, there exists $\mu \geq 0$ such that $My + v = \mu J y$. In view of (10), it suffices to assume that $\mu > 0$ and $y \neq 0$.

By an easy verification, we deduce that if $A \in M_n(\mathbb{R})$ is non-singular, then

$$x \in \text{SOL}(A, q) \Longleftrightarrow Ax + q \in \text{SOL}(A^{-1}, -A^{-1}q). \tag{11}$$

In view of (9) and (11) we have

$$0 \in \text{SOL}(M^{-1}, -M^{-1}u) \quad \text{and} \quad \mu J y \in \text{SOL}(M^{-1}, -M^{-1}v). \tag{12}$$

Since $M \in \mathbf{Z} \cap \mathbf{Q}$, by item (B) in Theorem 1,

$$M^{-1}(\mathcal{K}) \subseteq \mathcal{K}. \tag{13}$$

Define

$$\Lambda := (\mathcal{K} \cup -\mathcal{K}) \smallsetminus \{0\}.$$

As $\mathcal{K}$ is self dual, equation (13) implies that $x^T M^{-1} x \geq 0$ for all $x \in \Lambda$. If $x^T M^{-1} x = 0$ for some $x \in \mathcal{K}$, then $x \in \mathrm{SOL}(M^{-1}, 0)$ and hence $x = 0$. Therefore,

$$x^T M^{-1} x > 0 \quad \forall x \in \Lambda.$$

Since $\mu J y \in \Lambda$, by Lemma 3 and (12), there exists $\lambda > 0$ such that

$$\|\mu J y\| \leq \lambda \|M^{-1}(u - v)\| \leq \theta \|u - v\|,$$

where $\theta = \lambda \|M^{-1}\|$. By (9), we see that

$$\begin{aligned}
\|M(y - x)\| &= \|(My + v) - (Mx + u) + (u - v)\| \\
&\leq \|M(y - x) + v - u\| + \|v - u\| \\
&= \|My + v\| + \|v - u\| \\
&= \|\mu J y\| + \|v - u\| \\
&\leq (\theta + 1)\|u - v\|.
\end{aligned}$$

Then we get

$$\|y - x\| = \|M^{-1} M(y - x)\| \leq \|M^{-1}\| \|M(y - x)\| \leq \|M^{-1}\|(\theta + 1)\|u - v\|. \tag{14}$$

Define

$$C := \max\{1, C_1, C_2, \|M^{-1}\|, \|M^{-1}\|(\theta + 1)\}.$$

By (6), (7), (8), (10) and (14), we deduce that

$$\|x - y)\| \leq C\|u - v\| \quad \forall u, v \in \mathbb{R}^n.$$

Therefore Φ_M is Lipschitz continuous. The proof of the theorem is complete. $\quad \square$

An easy consequence of the above theorem is the following:

Corollary 2. *Let* $M \in \mathbf{Z}$ *and* $e = (1, 0, \ldots, 0)$. *If* $Me \in \mathrm{int}(\mathcal{K})$, *then* Φ_M *is Lipschitz continuous.*

Proof. Since e and Me belong to int$(\mathcal{K})$, Theorem 1 implies $M \in \mathbf{Q}$. We claim that M is positive definite on $\partial\mathcal{K}$. Let $x \in \partial\mathcal{K} \smallsetminus \{0\}$. Without loss of generality assume that $x = (1/2, x_2, \ldots, x_n)$. Then $y := Jx = (1/2, -x_2, \ldots, -x_n)$. Define

$$A := \begin{bmatrix} x^T M x & x^T M y \\ y^T M x & y^T M y \end{bmatrix}.$$

Since $M \in \mathbf{Z}$, the off-diagonal entries of A are non-positive. Let $e' = (1, 1)$. Since $Me \in$ int$(\mathcal{K})$ and $x + y = e$, by item (d) in Lemma 1 it follows that $Ae' = (x^T Me, y^T Me) \in \mathbb{R}^2_{++}$. By a well-known result on P-matrices (see [3], Theorem 2.3 of chapter 6), it follows that all the principal minors of A are positive. Hence $x^T M x > 0$. By Theorem 3, Φ_M is Lipschitz continuous. The proof is complete. $\square$

3.1. Examples

We now give an example of a matrix $M \in M_n(\mathbb{R})$ that is not positive definite but Φ_M is Lipschitz continuous.

Example 1. For $\alpha > 0$, define

$$M_\alpha := \begin{bmatrix} 1 - \alpha & -\alpha & 0 \\ \alpha & 1 + \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By an easy verification, we see that M_α can be written as $I - S_\alpha$, where S_α is nilpotent and $S_\alpha(\mathcal{K}) \subseteq \mathcal{K}$. Since $\det(M_\alpha) > 0$ and $M_\alpha^{-1} = I + S_\alpha$, from Theorem 1, it follows that $M_\alpha \in \mathbf{Z} \cap \mathbf{Q}$. We claim that

$$x^T M_\alpha x > 0 \ \ \forall x \in \partial\mathcal{K} \smallsetminus \{0\} \iff 0 < \alpha < 2.$$

Suppose $x^T M_\alpha x > 0 \ \ \forall x \in \partial\mathcal{K} \smallsetminus \{0\}$. By choosing $p = (1, 0, 1)$, we find that $p^T M_\alpha p = 2 - \alpha$. So, $0 < \alpha < 2$. For any $y = (y_1, y_2, y_3) \in \partial\mathcal{K} \smallsetminus \{0\}$, we have

$$\begin{aligned} y^T M_\alpha y &= (1 - \alpha)y_1^2 + (1 + \alpha)y_2^2 + y_3^2 \\ &= y_1^2 + y_2^2 + y_3^2 + \alpha(y_2^2 - y_1^2) \\ &> y_1^2 + y_2^2 + y_3^2 + 2(y_2^2 - y_1^2) \\ &= 2y_1^2 + 2(y_2^2 - y_1^2) = 2y_2^2 \geq 0. \end{aligned}$$

If $1 < \alpha < 2$, then M_α is not positive definite but Φ_M is Lipschitz continuous.

Example 2. Let

$$M := \begin{bmatrix} 1 & 4\sqrt{2} & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

If $x = (-2\sqrt{2}, 1, 0)^T$, then $x^T M x = 0$ and hence M is not positive definite. By an easy verification, we find that

$$6J - (JM + M^T J) = \begin{bmatrix} 4 & -4\sqrt{2} & 0 \\ -4\sqrt{2} & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is positive semidefinite. It can be shown that $A \in M_n(\mathbb{R})$ is a **Z**-matrix with respect to $\mathcal{K}$ if and only if the matrix $\gamma J - (AJ + JA^T)$ is positive semidefinite for some $\gamma \in \mathbb{R}$ (see Example 4 in [9]). Hence $M \in \mathbf{Z}$. Since $Me \in \text{int}(\mathcal{K})$, by Corollary 2, Φ_M is Lipschitz continuous.

3.2. Symmetric matrices

We now show that if $M \in M_n(\mathbb{R})$ is a symmetric **Q**-matrix, then Φ_M is Lipschitz continuous if and only if M is positive definite. The following lemma will be used in the proof of main result.

Lemma 5. *Let $Q \in M_n(\mathbb{R})$ be non-singular. Define $\Omega := Q\mathcal{K}$ and $\Omega^* := Q^{-T}\mathcal{K}$. If u and v are any two non-zero orthogonal vectors in $\mathbb{R}^n$ such that $u, v \notin \partial\Omega \cup -\partial\Omega$ and $u, v \notin \partial\Omega^* \cup -\partial\Omega^*$, then at least one vectors in the set $\{u, -u, v, -v\}$ does not belong to $\Omega \cup \Omega^*$.*

Proof. Assume the contrary. Then there exist two non-zero vectors u and v in $\mathbb{R}^n$ satisfying the hypothesis of the lemma such that

$$\{u, -u, v, -v\} \subseteq \Omega \cup \Omega^*. \tag{15}$$

This means that either $u \in \text{int}(\Omega)$ or $u \in \text{int}(\Omega^*)$. Without loss of generality, let $u \in \text{int}(\Omega)$. Then, $u = Qx$, where $x \in \text{int}(\mathcal{K})$. If $v \in \text{int}(\Omega^*)$, then $v = Q^{-T}y$ for some $y \in \text{int}(\mathcal{K})$ and hence $u^T v = x^T y$. Since x and y belong to $\text{int}(\mathcal{K})$, $x^T y > 0$; so $u^T v > 0$. But u and v are orthogonal. This contradiction implies that $v \notin \text{int}(\Omega^*)$. By a similar argument, $-v \notin \text{int}(\Omega^*)$. Thus, $v \notin \text{int}(\Omega^*) \cup \partial\Omega \cup \partial\Omega^*$. By (15), $v \in \text{int}(\Omega)$. As $\Omega \cap -\Omega = \{0\}$, $-v \notin \text{int}(\Omega)$. Therefore, $-v \notin \Omega \cup \Omega^*$. This is a contradiction. Hence the lemma must be true. $\square$

Theorem 4. *Let A be an $n \times n$ symmetric matrix. Suppose $A \in \mathbf{Q}$. Then the following are equivalent:*

(i) *A is positive definite.*

(ii) Φ_A *is Lipschitz continuous.*

Proof. (i) $\Longrightarrow$ (ii) follows from Corollary 1. We now show that (ii)$\Longrightarrow$(i). By Theorem 4 in [1] and Theorem 4.1 in [11] it follows that $\det(A) > 0$ and $x^T A x > 0$ for all $x \in \partial\mathcal{K} \smallsetminus \{0\}$. Furthermore by (11) in Theorem 3 it follows that $\Phi_{A^{-1}}$ is Lipschitz continuous and $A^{-1} \in \mathbf{Q}$. Hence $x^T A^{-1} x > 0$ for all $x \in \partial\mathcal{K} \smallsetminus \{0\}$. In addition, by Theorem 4 in [1], we see that $\det(A) > 0$.

Suppose A is not positive definite. Since A is symmetric and $x^T A x > 0$ for a non-zero vector $x \in I\!\!R^n$, A has at least one positive eigenvalue. Let k be the number of positive eigenvalues of A. Put $s = n - k$. Since $\det(A) > 0$, $s \geq 2$. There exists an $n \times n$ non-singular matrix Q such that

$$B := QAQ^T = \begin{bmatrix} I_k & 0 \\ 0 & -I_s \end{bmatrix}.$$

Here I_s and I_k denote identity matrices of order s and k respectively. Let $\Delta := Q\mathcal{K}$. Since Q is non-singular, we have the following:

$$\partial(Q\mathcal{K}) = Q(\partial\mathcal{K}) \quad \text{and} \quad \text{int}(Q\mathcal{K}) = Q(\text{int}(\mathcal{K})). \tag{16}$$

The set Δ is a closed convex cone with dual

$$\Delta^* = Q^{-T}\mathcal{K}.$$

We now show that if $x \in \partial\Delta^* \smallsetminus \{0\}$, then $x^T B x > 0$. If $x \in \partial\Delta^* \smallsetminus \{0\}$, by (16), there exists $0 \neq y \in \partial\mathcal{K}$ such that $x = Q^{-T}y$. Hence,

$$x^T B x = x^T QAQ^T x = y^T Q^{-1} QAQ^T Q^{-T} y = y^T A y > 0.$$

Since $B^{-1} = B$, by a similar argument, we get

$$x^T B^{-1} x > 0 \ \ \forall x \in \partial\Delta \smallsetminus \{0\}.$$

Hence if $x = (x_1, \dots, x_n) \in \partial\Delta \cup \partial\Delta^* \smallsetminus \{0\}$, then

$$\sum_1^k x_i^2 > \sum_{k+1}^n x_i^2. \tag{17}$$

Since $s \geq 2$, the right-hand side of (17) is non-zero if $x_{n-1} \neq 0$ or $x_n \neq 0$. We now choose two vectors in $I\!\!R^n$, namely, $u = (0, \dots, 0, 1)$ and $v = (0, \dots, 1, 0)$. From (17), we see that none of the vectors in the set $\{u, -u, v, -v\}$ belong to $\partial\Delta \cup \partial\Delta^*$ and hence u and v satisfy the hypothesis of Lemma 5.

Without loss of generality, let u be the vector for which there exist $x \in \partial\Delta \smallsetminus \{0\}$ and $y \in \partial\Delta^* \smallsetminus \{0\}$ such that $u = x - y$ and $x^T y = 0$. Since $u_1 = u_2 = \ldots = u_{n-1} = 0$, $x_i - y_i = 0$ for all $1 \leq i \leq n - 1$. By using the orthogonality of x and y, we find that

$$x_n y_n = -\sum_1^{n-1} y_i^2. \tag{18}$$

By applying (17) to x and y and using (18) and Cauchy–Schwarz inequality, we get

$$(\sum_1^k y_i^2)^2 = (\sum_1^k x_i^2)(\sum_1^k y_i^2) > (\sum_{k+1}^n x_i^2)(\sum_{k+1}^n y_i^2)$$

$$\geq (\sum_{k+1}^n x_i y_i)^2 \tag{19}$$

$$= (\sum_1^k y_i^2)^2,$$

which is not possible. Hence A must be positive definite. The proof of the theorem is complete. □

The following result is implicit from the proof of the above theorem.

Theorem 5. *Let A be an $n \times n$ symmetric matrix. Then A is positive definite if and only if $\det(A) > 0$ and A and A^{-1} are positive definite on $\partial\mathcal{K}$.*

We conclude the paper with an example of a symmetric matrix A such that $\det(A) > 0$ and A is positive definite on $\partial\mathcal{K}$, but A^{-1} is not positive definite on $\partial\mathcal{K}$.

Example 3. Let $A := \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. If $x \in \partial\mathcal{K}$, then $x^T A x = x_1^2 > 0$ and hence A is positive definite on $\partial\mathcal{K}$. But if $y = (1, 0, 1) \in \partial\mathcal{K}$, we see that $y^T A^{-1} y = -0.5$. Hence A^{-1} is not positive definite on $\partial\mathcal{K}$.

4. Conclusion

For **Z**-matrices with respect to $\mathcal{K}$, we have obtained a sufficient condition for the Lipschitz continuity of the solution map of a second-order cone linear complementarity problem. When the given matrix is symmetric, we have shown that the corresponding solution map is Lipschitz continuous if and only if it is positive definite. The proof of both these results depends upon the fact that all the boundary points and the extreme vectors of the second-order cone coincide. But this is not true in general for a symmetric

cone in a Euclidean Jordan algebra. Hence we do not know whether our results can be extended to other symmetric cones. This may be a topic for further research.

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