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ON k-INTERSECTION EDGE COLOURINGS

Rahul Muthu, N. Narayanan

and

C.R. Subramanian

The Institute of Mathematical Sciences
Taramani, Chennai-600113, India

e-mail: {rahulm,narayan,crs}@imsc.res.in

Abstract

We propose the following problem. For some $k \geq 1$, a graph G is to be properly edge coloured such that any two adjacent vertices share at most k colours. We call this the *k-intersection edge colouring*. The minimum number of colours sufficient to guarantee such a colouring is the k-intersection chromatic index and is denoted $\chi'_k(G)$. Let f_k be defined by

$$f_k(\Delta) = \max_{G:\Delta(G)=\Delta} \{\chi'_k(G)\}.$$

We show that $f_k(\Delta) = \Theta(\frac{\Delta^2}{k})$. We also discuss some open problems.

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1. Introduction

All the graphs we consider are simple and finite. A colouring of the edges of a graph G is proper, if no two incident edges are coloured the same. The minimum number of colours sufficient to guarantee such a colouring is known as the chromatic index of G and its standard notation is $\chi'(G)$. Throughout, $\Delta = \Delta(G)$ stands for the maximum degree of the graph G.

We introduce the notion of *k-intersection edge colouring* as a proper edge colouring of a graph, in which no more than k colours seen by any vertex u are also seen by any fixed neighbour of u. When we say 'colours seen by a vertex', we mean the set of colours received by the edges incident to that particular vertex. If $\mathcal{E}(u)$ denotes the edges incident to the vertex u, then we need to have $|\mathcal{C}(\mathcal{E}(u)) \cap \mathcal{C}(\mathcal{E}(v))| \leq k$ for every edge $e = (u, v)$, where $\mathcal{C}(\mathcal{E}(x))$ is the set of colours seen by the vertex x in the colouring $\mathcal{C}$. We are interested in bounding the *k-intersection chromatic index* $\chi'_k(G)$, which is the minimum number of colours sufficient to guarantee a k-intersection edge colouring of a graph G.

In a proper edge colouring, the number of common colours between a pair of adjacent vertices can be as high as $\Delta(G)$. At the other extreme, for a distance-2 edge colouring (see [1, 5]), the number of allowed common colours is exactly 1 (a proper edge colouring of a graph G is a distance-2 colouring if any two edges separated by a single edge receive distinct colours). Also, notice that when G is a Δ-regular graph and $k = \Delta - 1$, the colouring is a adjacent vertex distinguishing (AVD) edge colouring [3] (an edge colouring is AVD if no pair of adjacent vertices see the same set of colours on its edges). It is easy to verify that maximum value of distance-2 chromatic index is of the order $\Theta(\Delta^2)$, while $\Delta + 1$ is an upper bound for the chromatic index $\chi'(G)$. The notion of k-intersection edge colouring simultaneously generalises all the three above notions by allowing the maximum number of common colours to be bounded by some k between 1 and Δ, inclusive of both. Our study is motivated by a desire to know about what happens to the chromatic index when the maximum number of common colours is bounded by k.

It is well-known from Vizing's work [8] that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for any graph G. Vizing's proof is also constructive leading to a deterministic polynomial time algorithm to obtain a proper edge colouring of any graph.

For $k, \Delta \geq 1$, define

$$f_k(\Delta) = \max_{G:\Delta(G)=\Delta} \{\chi'_k(G)\}.$$

In this short paper, we obtain tight estimates on $f_k(\Delta)$. Our proofs are based on probabilistic arguments and we make use of Vizing's result as well as probability tools like Lovász Local Lemma (Symmetric and General forms stated in the next section) and the Chernoff bound. See [2, 4, 6] for further details. Specifically, we prove the following.

Theorem 1. *For the function $f_k(\Delta)$ defined above, we have $f_k(\Delta) = \Theta(\frac{\Delta^2}{k})$.*

We give two lemmas which will imply the above theorem. $K_{\Delta+1}$ denotes the complete graph on $\Delta + 1$ vertices.

Lemma 2. *For any graph G with $\Delta = \Delta(G)$,*

1. $\chi'_k(G) \leq \lceil \frac{2\Delta}{k} \rceil (\Delta + 1)$, *if* $20 \log \Delta \leq k \leq \Delta$.
2. $\chi'_k(G) \leq \frac{22\Delta^2}{k}$, *if* $1 \leq k \leq \Delta$.

Lemma 3. $\chi'_k(K_{\Delta+1}) \geq \frac{\Delta^2}{2k}$.

Note that Statement 2 of Lemma 2 itself implies the upper bound of Theorem 1. But Statement 1 improves the constant from 22 to 2 and its proof is based on a different colouring argument. For both statements, we have not tried to optimize the constants involved. Here, the logarithms are the natural ones with respect to the base e.

2. Upper Bound

Proof (Lemma 2: Part 1). We first handle the case $k \geq 20 \log \Delta$ by making use of the symmetric form of Lovász Local Lemma (stated below).

Lemma 4 (Symmetric form of Lovász Local Lemma)**.** *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be events in a probability space. Suppose that each event $\mathcal{A}_i$ is mutually independent of all but at most d other events $\mathcal{A}_j$. Further assume that $\mathbf{Pr}(\mathcal{A}_i) \leq p$ for $1 \leq i \leq n$. If*

$$ep(d + 1) \leq 1$$

then $\mathbf{Pr}(\bigwedge_{i=1}^{n} \bar{\mathcal{A}}_i) > 0$.

First, we use Vizing's theorem to obtain a proper $(\Delta+1)$-edge colouring $\mathcal{C}'$. We then design the following random experiment. For each edge coloured a, we assign a new colour chosen independently and uniformly randomly from the set $\{a_1, a_2, \ldots, a_\eta\}$ where η is to be fixed later. Let the resulting colouring be $\mathcal{C}$. Let $e_{u,i}$ denote the edge coloured i incident (if any) to the vertex u in the original colouring $\mathcal{C}'$. Given two edges $e_{u,i}, e_{v,i}$ coloured i in $\mathcal{C}'$, the probability that they receive the same colour in $\mathcal{C}$ is $1/\eta$. Let $\mathcal{C}'_u$ denote the set of colours used on edges incident to vertex u in the initial colouring $\mathcal{C}'$. For any edge $e = \{u, v\} \in E(G)$, let $s_e = |\mathcal{C}'_u \cap \mathcal{C}'_v| - 1$. Let ζ_e stand for the number of common colours i (other than the color of e) such that edges $e_{u,i}$ and $e_{v,i}$ get the same colour in the new colouring $\mathcal{C}$. A "bad" event is that for some edge $e = \{u, v\}$, the vertices u and v have at least $k+1$ colours in common in the new colouring $\mathcal{C}$, or equivalently, $\zeta_e \geq k$. Absence of every such event implies that we have a desired colouring. We have $Exp(\zeta_e) = \dfrac{s_e}{\eta}$.

Let $B(n, p)$ denote the sum of n independent and identically distributed indicator variables each having expectation p. By the well-known Chernoff bound (see [2, 7]), we have

$$\mathbf{Pr}(B(n,p) \geq (1+\epsilon)np) \leq \left(\frac{e^{\epsilon}}{(1+\epsilon)^{1+\epsilon}}\right)^{np} \text{ for any } \epsilon > 0.$$

Note that $\zeta_e = B(s_e, 1/\eta)$. Also, ζ_e is stochastically dominated by $B(\Delta, 1/\eta)$. We now set $\eta = \lceil \frac{2\Delta}{k} \rceil$. For the sake of simplicity, we ignore the ceilings (without affecting the correctness of the arguments) and treat $\eta = \frac{2\Delta}{k}$. Hence, we have

$$\mathbf{Pr}(\zeta_e \geq k) \leq \mathbf{Pr}(B(\Delta, 1/\eta) \geq k) \leq e^{\frac{-(0.38)k}{2}}.$$

Using $k \geq 20 \log \Delta$, we get

$$\mathbf{Pr}(\zeta_e \geq k) \leq e^{-(3.8)\log \Delta} \leq \Delta^{-3.8}.$$

Thus, we obtain an upper bound on the probability of having more than k colours in common between the endpoints of any fixed edge. Since two such events are dependent only if they share an edge, each event is mutually independent of all but at most $2\Delta(\Delta-1)$ other events. Now, to apply the symmetric form of Lovász Local Lemma, we need to verify that

$$e \times \Delta^{-3.8} \times (2\Delta(\Delta-1)+1) \leq 1$$

which is true if

$$\frac{2e\Delta^2}{\Delta^{3.8}} \leq 1 \iff \frac{2e}{\Delta^{1.8}} \leq 1 \iff (2e)^{5/9} \leq \Delta$$

which is true for every $\Delta \geq 3$.

Hence, the result follows for the case $k \geq 20 \log \Delta$ by fixing $\eta = \lceil \frac{2\Delta}{k} \rceil$. Since even a number strictly less than but sufficiently close to 2 will suffice, a more tight analysis shows that we can do away with the additive sub-linear term. We skip the details. ■

Now we look at the case when $k < 20 \log \Delta$ where the above arguments fail because the uniform upper bound on the probabilities $\mathbf{Pr}(\zeta_e \geq k)$ is not sufficiently small to apply the symmetric version of Local Lemma.

***Proof* (*Lemma* 2: *Part* 2).** We assume that $\Delta \geq 6$, since otherwise k is at most 5 and even a distance-2 colouring suffices, since the number of colours used would be at most $2\Delta(\Delta - 1) + 1 < 22\Delta^2/k$ for the range $k \leq \Delta \leq 5$. For proving Part 2, we use the most general form of Lovász Local Lemma as given below.

Lemma 5 (The Lovàsz Local Lemma (general form))**.** *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be events in a probability space Ω such that each event A_i is mutually independent of all events in $\mathcal{A} - (\{A_i\} \cup \mathcal{D}_i)$, for some $\mathcal{D}_i \subseteq \mathcal{A}$. Also suppose that there exist $x_1, \ldots, x_n \in (0, 1)$ such that*

$$Pr(A_i) \leq x_i \prod_{A_j \in \mathcal{D}_i} (1 - x_j), \quad 1 \leq i \leq n.$$

Then $Pr(\overline{A_1} \wedge \cdots \wedge \overline{A_n}) > 0$.

Consider the following random experiment. Colour each edge uniformly and independently at random with one of $c = 22\Delta^2/k$ colours where α is to be determined later. We define the following two types of *bad* events.

- Type I A pair of incident edges e, f receive the same colour. Denote it by $E_{e,f}$.
- Type II For an edge $e = (u, v)$, and sets $S_1 \subseteq \mathcal{E}(u) \setminus \{e\}$, $S_2 \subseteq \mathcal{E}(v) \setminus \{e\}$, with $|S_1| = |S_2| = k$, these edges are properly coloured with a set of k colours. We denote it by E_{e,S_1,S_2}.

Suppose that the random colouring $\mathcal{C}$ be such that none of the above events hold. Then $\mathcal{C}$ is proper and also is such that no two adjacent vertices share more than k colours in common. We show that with positive probability this happens. In order to apply the Local Lemma, observe the following.

Lemma 6. 1. *For each event $\mathcal{E}_{e,f}$ of Type* I, *$Pr(E_{e,f}) = \frac{1}{c}$.*
2. *For each event of Type* II, *$Pr(E_{e,s_1,s_2}) = \frac{\binom{c}{k} k!^2}{c^{2k}} \leq \frac{k!}{c^k}$.*

We obtain an upper bound on the number of events of any type whose outcome depends on a given edge. We then multiply this by the number of edges whose colouring affects a given event $\mathcal{E}$ to get an upper bound on the number of other events of each type on which $\mathcal{E}$ could possibly depend.

Lemma 7. *For an edge e, the following holds true:*

- *at most $2\Delta - 2$ events of Type* I *depend on e,*
- *at most $2\Delta\binom{\Delta-1}{k}\binom{\Delta-2}{k-1} < \frac{2\Delta^{2k}}{k!(k-1)!}$ events of Type* II *depend on e.*

Proof. Since the edges are all coloured independently of each other, the outcome of any event depends exclusively on the colouring of the set of edges on which it is defined. It follows that the colour received by a given edge influences exactly those events whose definition are based on that edge.

The event of type I is based on a pair of incident edges. Thus, to compute the number of events of type I, whose outcome is dependent on an edge e, we have to compute the number of pairs of incident edges to which e belongs. The number of such pairs can be easily be seen to be at most 2Δ. Similarly, an event of type II is defined on a set of $2k$ edges, incident on a fixed edge, k at each endpoint. The number of such sets of $2k$ edges containing e can is at most $2\Delta\binom{\Delta-1}{k}\binom{\Delta-2}{k-1}$. This is explained as follows. For an edge $f = (w, x)$, we need to identify an edge e incident to either w or to x. There are at most 2Δ possibilities for this. We then need to choose $k-1$ edges for the set (containing f) from among the remaining at most $\Delta - 2$ edges and choose another k edges from the other endpoint of e. ■

The outcome of events of Types I and II depends respectively on exactly 2 and $2k$ participating edges.

In order to apply the the Local Lemma we now need to choose reals which will ensure that the inequalities are satisfied. We will choose $x = \frac{2}{c}$ and $y = \frac{k!2^k}{c^k}$ as the reals associated with events of Type I, II respectively.

To verify that the colouring $\mathcal{C}$ satisfies the required properties, it is sufficient to show that the following inequalities hold:

$$\frac{1}{c} \leq \frac{2}{c}\left(1 - \frac{2}{c}\right)^{2\times 2\Delta}\left(1 - \frac{k!2^k}{c^k}\right)^{2\times\frac{2\Delta^{2k}}{k!(k-1)!}} \tag{1}$$

and

$$\frac{k!}{c^k} \leq \frac{k!2^k}{c^k}\left(1 - \frac{2}{c}\right)^{2k\times 2\Delta}\left(1 - \frac{k!2^k}{c^k}\right)^{2k\times\frac{2\Delta^{2k}}{k!(k-1)!}}. \tag{2}$$

Simplifying and taking roots, we see that both, (1) and (2), follow from

$$1 \leq 2\left(1 - \frac{2}{c}\right)^{4\Delta}\left(1 - \frac{k!2^k}{c^k}\right)^{\frac{4\Delta^{2k}}{k!(k-1)!}}. \tag{3}$$

Substituting $c = 22\Delta^2/k$, (3) is equivalent to

$$1 \le 2\left(1 - \frac{k}{11\Delta^2}\right)^{4\Delta}\left(1 - k!\left(\frac{k}{11\Delta^2}\right)^k\right)^{\frac{4\Delta^{2k}}{k!(k-1)!}}, \tag{4}$$

and further to

$$1 \le 2\beta_1^{\frac{4k}{11\Delta}}\beta_2^{\frac{4}{(k-1)!}\left(\frac{k}{11}\right)^k}, \tag{5}$$

where

$$\beta_1 = \left(1 - \frac{k}{11\Delta^2}\right)^{\frac{11\Delta^2}{k}} \text{ and } \beta_2 = \left(1 - \frac{k!k^k}{(11)^k\Delta^{2k}}\right)^{\frac{(11)^k\Delta^{2k}}{(k!k^k)}}.$$

By using the assumptions $\Delta \ge 6$ and $k \le \Delta$, one can verify that $\beta_1, \beta_2 \ge 1/4$ and also that the sum of the exponents of β_1 and β_2 in (5) is at most 1/2. This establishes inequality (5). It follows that there exist a k-intersection edge colouring of G using $22\Delta^2/k$ colours. ∎

3. A Lower Bound

To show that the bound is tight, we show that complete graphs require at least $\frac{\Delta^2}{2k}$ colours. We use the standard notation $K_{\Delta+1}$ to denote the complete graph on $\Delta + 1$ vertices. Without loss of generality, assume that $V(K_{\Delta+1}) = \{1, 2, \ldots, \Delta + 1\}$.

Consider any k-intersection edge colouring C of $K_{\Delta+1}$. Starting from vertex 1, we scan the vertices in increasing order. There are Δ colours used on $\mathcal{E}(1)$. Since $|\mathcal{C}(\mathcal{E}(1)) \cap \mathcal{C}(\mathcal{E}(2))| \le k$, at least $\Delta - k$ colours should have been used on $\mathcal{E}(2)$ which are not used for $\mathcal{E}(1)$. Similarly, $\mathcal{E}(3)$ should use at least $\Delta - 2k$ colours which are not used on $\mathcal{E}(1) \cup \mathcal{E}(2)$. Continuing in this way, we see that a minimum of $\Delta + \Delta - k + \cdots + \Delta - \lfloor \frac{\Delta}{k} \rfloor \times k$ colours are required to colour $K_{\Delta+1}$. Using s to denote $\lfloor \Delta/k \rfloor$, the summation above can be re-written as

$$\Delta(s+1) - k(1 + 2 + \cdots + s) = (s+1)(\Delta - ks/2) \ge (s+1)(\Delta/2) \ge \frac{\Delta^2}{2k}.$$

This establishes the lower bound. ∎

4. Remarks and Open problems

It would be interesting to know, if the lower bound is tight for other classes of graphs like bicliques (complete bipartite graphs). An interesting problem is to design an efficient algorithm which produce a k-intersection edge colouring matching the upper bounds derived here.

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