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On inverse-positivity of sub-direct sums of matrices

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ABSTRACT

In this note, the authors consider the problem of inverse-positivity of k-subdirect sum of matrices. The main results provide a solution to an open problem posed recently.

1. Introduction

Let $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of all real numbers, the n-dimensional Euclidean space and the set of all $m\times n$ matrices over $\mathbb{R}$, respectively. We denote $\rho(A)$ as the spectral radius of $A \in \mathbb{R}^{n\times n}$, namely $\rho(A)$ is the maximum of the absolute values of the eigenvalues of A. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we say that x is *nonnegative*, i.e., $x \geqslant 0$ if and only if $x_i \geqslant 0$ for all $i = 1, 2, \ldots, n$. A matrix B is said to be *nonnegative*, denoted as $B \geqslant 0$, if all its entries are nonnegative and $A \in \mathbb{R}^{n\times n}$ is said to be *inverse-positive* if A^{-1} is nonnegative. It is known that [4], for a $A \in \mathbb{R}^{n\times n}$, A^{-1} exists and $A^{-1} \geqslant 0$ if and only if $Ax \geqslant 0 \Rightarrow x \geqslant 0$.

A matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if the off-diagonal entries of A are non-positive. Such a matrix can be written as $A = sI - B$, where $B \geqslant 0$ and $s > 0$. A is called an M-matrix if $s \geqslant \rho(B)$. If $s > \rho(B)$, then A is a nonsingular M-matrix. It is well known that an Z-matrix $A \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if and only if A is inverse-positive. One can refer to [2] and [9] for various characterizations of M-matrices.

Next, we review the notion of the sub-direct sum of matrices. The concept of the sub-direct sum was proposed by Fallat and Johnson [6]. This is a generalization of the normal sum and the direct sum of

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matrices. This concept has applications in matrix completion problems and overlapping subdomains in domain decomposition methods [7,8]. It also arises when one studies the structure of different positivity classes of matrices, for example, positive definite matrices or P-matrices. Let us recall the definition.

Definition 1.1. Let $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$ and $B = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}$, where $D \in \mathbb{R}^{(m-k)\times(m-k)}$, $E \in \mathbb{R}^{(m-k)\times k}$, $F \in \mathbb{R}^{k\times(m-k)}$, $Q \in \mathbb{R}^{k\times(n-k)}$, $S \in \mathbb{R}^{(n-k)\times k}$, $T \in \mathbb{R}^{(n-k)\times(n-k)}$ and $G, P \in \mathbb{R}^{k\times k}$. The *k-subdirect* sum of A and B is denoted as $A \oplus_k B$ and is defined as

$$A \oplus_k B = \begin{pmatrix} D & E & 0 \\ F & G+P & Q \\ 0 & S & T \end{pmatrix}. \tag{1}$$

Note that $A \oplus_k B$ is a square matrix of size $m+n-k$. When $k = 0$, the sub-direct sum reduces to the usual direct sum of matrices. The case $k = 1$ (1-subdirect sum) is treated separately from the cases $k > 1$ as their properties are qualitatively different.

The authors of [6] analyze properties of many of the positivity classes of matrices mentioned earlier under the sub-direct sum operation. In this connection, it is known that the 1-subdirect sum of two nonsingular M-matrices is again a nonsingular M-matrix while the k-subdirect sum of two nonsingular M-matrices is not a nonsingular M-matrix for $k \neq 1$. On the other hand, any nonsingular M-matrix can be written as a k-subdirect sum of two nonsingular M-matrices for any value of k. In [3], Bru et al. provide certain sufficient conditions for the the sub-direct sum of nonsingular M-matrices to be a nonsingular M-matrix. They also consider the sub-direct sum of inverses of matrices and obtain conditions for it to be nonsingular.

Recently, Abad et al. in [1] consider the same question for the class of all inverse-positive matrices, for which the set of all M-matrices is a subclass. They show that the 1-subdirect sum of inverse-positive matrices is again an inverse positive matrix whereas the k-subdirect sum does not have that property (see [1, Example 1]). Only certain special cases of inverse-positive matrices have inverse-positive k-subdirect sums. Corresponding converses are also obtained in [1], but the general case was left as an open problem. Specifically, the question of the inverse positivity of a k-subdirect sum of two inverse-positive matrices was left open. Another related question is to write an inverse-positive matrix of the particular form given by (1) as a k-subdirect sum of matrices for $k > 1$. Note that this latter assertion holds for the case of nonsingular M-matrices. That is, any nonsingular M-matrix can be written as a k-subdirect sum of M-matrices as mentioned earlier.

In this short note, we provide certain conditions under which the k-subdirect sum of inverse-positive matrices is inverse-positive. This is done in Theorem 2.4. We also present sufficient conditions for the converse to hold. In other words, if a matrix is inverse-positive, we prove that it can be written as a k-subdirect sum of inverse-positive matrices, in the presence of certain assumptions. This is presented in Theorem 2.11. We consider the case when $k \neq 1$ as for the 1-subdirect sum, these results are known [1]. We tackle the problem in its full generality by assuming that the individual summand matrices have non-trivial blocks. We deduce the corresponding results presented in [1] as immediate consequences of our results. We observe that the results presented here can also be extended to the case of Moore–Penrose inverses.

2. Main results

Let

$$A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$$

be in $\mathbb{R}^{m\times m}$ where $D \in \mathbb{R}^{k\times k}$ and nonsingular. The Schur complement of D in A, denoted by A/D, is the matrix $G - FD^{-1}E$. Note that in a similar way, we can define $A/G = D - EG^{-1}F$, if G is nonsingular. The following results about block matrices are used in the proofs of our main results.

Lemma 2.1. *Let $D \in \mathbb{R}^{(m-k)\times(m-k)}$ and $G \in \mathbb{R}^{k\times k}$ with D nonsingular and $D^{-1} \geqslant 0$. Also, let $E \in \mathbb{R}^{(m-k)\times k}$ and $F \in \mathbb{R}^{k\times(m-k)}$ where $-E \geqslant 0$ and $-F \geqslant 0$. Let $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$. Then*

(*i*) *$\det A \neq 0$ if and only if $\det(A/D) \neq 0$.*

(*ii*) *$A^{-1} \geqslant 0$ if and only if $(A/D)^{-1} \geqslant 0$.*

Proof. (*i*) We have [5], $\det A = \det D \det(A/D)$. Thus, when D is nonsingular, A is nonsingular if and only if A/D is nonsingular.

(*ii*) It can be verified by direct calculations that

$$A^{-1} = \begin{pmatrix} D^{-1} + D^{-1}E(A/D)^{-1}FD^{-1} & -D^{-1}E(A/D)^{-1} \\ -(A/D)^{-1}FD^{-1} & (A/D)^{-1} \end{pmatrix}.$$

From the expression for A^{-1}, it follows that $A^{-1} \geqslant 0$ if and only if $(A/D)^{-1} \geqslant 0$. □

The next result is similar to the result above. We skip the proof.

Lemma 2.2. *Let $P \in \mathbb{R}^{k\times k}$ and $T \in \mathbb{R}^{(n-k)\times(n-k)}$ with T nonsingular and $T^{-1} \geqslant 0$. Also, let $Q \in \mathbb{R}^{k\times(n-k)}$ and $S \in \mathbb{R}^{(n-k)\times k}$ where $-Q \geqslant 0$ and $-S \geqslant 0$. Let $B = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}$. Then*

(*i*) *$\det B \neq 0$ if and only if $\det(B/T) \neq 0$.*

(*ii*) *$B^{-1} \geqslant 0$ if and only if $(B/T)^{-1} \geqslant 0$.*

Next, we prove a determinant formula that we use in Theorem 2.4.

Lemma 2.3. *Let $C = \begin{pmatrix} D & E & 0 \\ F & Y & Q \\ 0 & S & T \end{pmatrix}$ where D and T are nonsingular. Then $\det C = \det D \det T \det(Y - FD^{-1}E - QT^{-1}S)$.*

Proof. Let $C = \begin{pmatrix} X & \tilde{Q} \\ \tilde{S} & T \end{pmatrix}$, where $X = \begin{pmatrix} D & E \\ F & Y \end{pmatrix}$, $\tilde{Q} = \begin{pmatrix} 0 \\ Q \end{pmatrix}$ and $\tilde{S} = \begin{pmatrix} 0 & S \end{pmatrix}$. We have $\det C = \det T \det(C/T)$, where $C/T = X - \tilde{Q}T^{-1}\tilde{S}$

Now, $X - \tilde{Q}T^{-1}\tilde{S} = \begin{pmatrix} D & E \\ F & Y \end{pmatrix} - \begin{pmatrix} 0 \\ Q \end{pmatrix} T^{-1} \begin{pmatrix} 0 & S \end{pmatrix} = \begin{pmatrix} D & E \\ F & Y - QT^{-1}S \end{pmatrix}$. Again, we have $\det(X - \tilde{Q}T^{-1}\tilde{S}) = \det D \det((X - \tilde{Q}T^{-1}\tilde{S})/D) = \det D \det(Y - QT^{-1}S - FD^{-1}E)$. □

We state the first main result of this note below. This result presents a sufficient condition for the inverse-positivity of k-subdirect sum of inverse-positive matrices.

Theorem 2.4. *Let* $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$ *and* $B = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}$ *be inverse-positive matrices of orders m and n respectively, where* $D \in \mathbb{R}^{(m-k)\times(m-k)}$, $E \in \mathbb{R}^{(m-k)\times k}$, $F \in \mathbb{R}^{k\times(m-k)}$, $Q \in \mathbb{R}^{k\times(n-k)}$, $S \in \mathbb{R}^{(n-k)\times k}$, $T \in \mathbb{R}^{(n-k)\times(n-k)}$ *and* $G, P \in \mathbb{R}^{k\times k}$ *with* $D^{-1} \geqslant 0, T^{-1} \geqslant 0, -E \geqslant 0, -F \geqslant 0, -Q \geqslant 0, -S \geqslant 0$, $(A/D)^{-1} \geqslant 0$ *and* $(B/T)^{-1} \geqslant 0$. *If, in addition,* $(G+P-FD^{-1}E-QT^{-1}S)^{-1}$ *exists and is nonnegative, then* $(A \oplus_k B)^{-1} \geqslant 0$.

Proof. Let $C = A \oplus_k B$. From Lemma 2.3, we have the formula: $\det C = \det D \det T \det(G+P-FD^{-1}E-QT^{-1}S)$. Hence, from the assumptions of the theorem, it follows that C is nonsingular.

Let $C(x_1, x_2, x_3) \in \mathbb{R}_+^{m-k} \times \mathbb{R}_+^{k} \times \mathbb{R}_+^{n-k}$, where $\mathbb{R}_+^j$ is the nonnegative orthant in $\mathbb{R}^j$. We show that $(x_1, x_2, x_3) \in \mathbb{R}_+^{m-k} \times \mathbb{R}_+^{k} \times \mathbb{R}_+^{n-k}$. Then $D(x_1 + D^{-1}Ex_2) = Dx_1 + Ex_2 \geqslant 0$. This implies that $x_1 + D^{-1}Ex_2 \geqslant 0$ as $D^{-1} \geqslant 0$. Similarly, $T^{-1}Sx_2 + x_3 \geqslant 0$ since $T(T^{-1}Sx_2 + x_3) = Sx_2 + Tx_3 \geqslant 0$ and $T^{-1} \geqslant 0$.

Again, consider $Dx_1 + Ex_2 = u_1 \geqslant 0$. Then, $Dx_1 = -Ex_2 + u_1$ and hence $x_1 = -D^{-1}Ex_2 + D^{-1}u_1$. Therefore, $Fx_1 = -FD^{-1}Ex_2 + FD^{-1}u_1$. In a similar way, we get $Qx_3 = -QT^{-1}Sx_2 + QT^{-1}u_3$, where $u_3 \geqslant 0$. Substituting for Fx_1 and Qx_3 in $Fx_1 + (G+P)x_2 + Qx_3$, we get $0 \leqslant Fx_1 + (G+P)x_2 + Qx_3 = (G+P-FD^{-1}E-QT^{-1}S)x_2 + FD^{-1}u_1 + QT^{-1}u_3$.

Now, $u_1 \geqslant 0$ and $D^{-1} \geqslant 0$ imply that $D^{-1}u_1 \geqslant 0$ and so $-FD^{-1}u_1 \geqslant 0$ as $-F \geqslant 0$. Using similar arguments, we get $-QT^{-1}u_3 \geqslant 0$. Thus, we get $(G+P)x_2 \geqslant 0$. Since $(G+P)^{-1} \geqslant 0$, we get $x_2 \geqslant 0$.

Again, $x_2 \geqslant 0$, $-E \geqslant 0$ and $D^{-1} \geqslant 0$ imply that $-D^{-1}Ex_2 \geqslant 0$. This in turn implies that $x_1 \geqslant 0$ as $x_1 + D^{-1}Ex_2 \geqslant 0$. Using similar arguments, we get $x_3 \geqslant 0$. Hence the theorem. □

Next, we show that certain results of [1] can be obtained as corollaries of our result.

Corollary 2.5. *[1, Proposition 8] Let* $A = \begin{pmatrix} D & 0 \\ F & G \end{pmatrix}$ *and* $B = \begin{pmatrix} P & 0 \\ S & T \end{pmatrix}$ *be inverse-positive matrices with* $D^{-1} \geqslant 0, G^{-1} \geqslant 0, P^{-1} \geqslant 0, T^{-1} \geqslant 0, -F \geqslant 0$ *and* $-S \geqslant 0$. *If the matrix* $G + P$ *is inverse-positive, then* $C = A \oplus_k B$ *is inverse-positive.*

Proof. The proof follows from Theorem 2.4 by taking $E = 0$ and $Q = 0$. □

Corollary 2.6. *[1, Proposition 10] Let* $A = \begin{pmatrix} D & 0 \\ F & G \end{pmatrix}$ *and* $B = \begin{pmatrix} P & Q \\ 0 & T \end{pmatrix}$ *be inverse-positive matrices with* $D^{-1} \geqslant 0, G^{-1} \geqslant 0, P^{-1} \geqslant 0, T^{-1} \geqslant 0, -F \geqslant 0$ *and* $-Q \geqslant 0$. *In addition, if* $G + P$ *is inverse-positive, then* $A \oplus_k B$ *is inverse-positive.*

Proof. Taking $E = 0$ and $S = 0$ in Theorem 2.4, we get the result. □

The following examples illustrate Theorem 2.4.

Example 2.7. Consider

$$A = \left(\begin{array}{c|c} D & E \\ \hline F & G \end{array}\right) = \left(\begin{array}{c|cc} 1.2587 & -0.5874 & -0.1259 \\ \hline -0.1259 & 1.2587 & -0.5874 \\ -0.5874 & -0.1259 & 1.2587 \end{array}\right)$$

and

$$B=\left(\begin{array}{c:c} P & Q \\ \hdashline S & T \end{array}\right)=\left(\begin{array}{cc:c} 8.9616 & -6.8279 & -0.5121 \\ -0.5121 & 8.9616 & -6.8279 \\ \hdashline -6.8279 & -0.5121 & 8.9616 \end{array}\right).$$

Then $A^{-1} \geqslant 0$ and $B^{-1} \geqslant 0$. Also $(A/D)^{-1} = \begin{pmatrix} 1.0001 & 0.5001 \\ 0.3334 & 1.0001 \end{pmatrix}$, $(B/T)^{-1} = \begin{pmatrix} 0.2500 & 0.2000 \\ 0.1667 & 0.2500 \end{pmatrix}$ and $(G + P - QT^{-1}S - FD^{-1}E)^{-1} = \begin{pmatrix} 0.1959 & 0.1495 \\ 0.1226 & 0.1959 \end{pmatrix}$ are nonnegative. Thus the conditions in Theorem 2.4 are satisfied. Also, the inverse of 2-subdirect sum of A and B is

$$(A \oplus_2 B)^{-1} = \begin{pmatrix} 0.8465 & 0.1037 & 0.0894 & 0.0740 \\ 0.0894 & 0.1959 & 0.1495 & 0.1251 \\ 0.1037 & 0.1226 & 0.1959 & 0.1562 \\ 0.0740 & 0.1562 & 0.1251 & 0.2158 \end{pmatrix} \geqslant 0.$$

Now, consider [6, Example 4.2]. The matrices under consideration are nonsingular M-matrices. We see that these matrices fail to satisfy one of the conditions in Theorem 2.4 and hence the 2-subdirect sum is not a nonsingular M-matrix.

Example 2.8. Let $A = \left(\begin{array}{c:c} D & E \\ \hdashline F & G \end{array}\right) = \left(\begin{array}{c:cc} 2 & -1 & -1 \\ \hdashline -1 & 5 & 0 \\ -1 & -9 & 5 \end{array}\right)$ and $B = \left(\begin{array}{cc:c} 5 & -9 & -1 \\ 0 & 5 & -1 \\ \hdashline -1 & -1 & 2 \end{array}\right)$. Both A and B are nonsingular M-matrices. Here, we see that $D^{-1} = T^{-1} \geqslant 0$, $-E = -S \geqslant 0$ and $-F = -Q \geqslant 0$. Also, we have $(A/D)^{-1} = \frac{1}{31}\begin{pmatrix} 9 & 1 \\ 19 & 9 \end{pmatrix} \geqslant 0$ and $(B/T)^{-1} = \frac{1}{31}\begin{pmatrix} 9 & 19 \\ 1 & 9 \end{pmatrix} \geqslant 0$. But, $(G + P - FD^{-1}E - QT^{-1}S)^{-1} = \frac{-1}{19}\begin{pmatrix} 9 & 10 \\ 10 & 9 \end{pmatrix} \ngeqslant 0$ and hence $(A \oplus_2 B)^{-1} \ngeqslant 0$. Thus $A \oplus_2 B$ is not a nonsingular M-matrix.

We recall the definition of a regular splitting of a matrix and a characterization of nonnegativity of the inverse of a matrix using such splittings [2]. We use this result to demonstrate a converse of Theorem 2.4.

Definition 2.9. Let $A \in \mathbb{R}^{n \times n}$. The decomposition $A = M - N$ is said to be a regular splitting of A if M is nonsingular, $M^{-1} \geqslant 0$ and $N \geqslant 0$.

Theorem 2.10. *[2, Theorem 5.6, Chapter 7] Let $A = M - N$ be a regular splitting of $A \in \mathbb{R}^{n \times n}$. Then, $A^{-1} \geqslant 0$ if and only if $\rho(M^{-1}N) < 1$.*

We now prove the second main result of this note. This presents conditions under which a given inverse-positive matrix C can be written as a k-subdirect sum of two inverse-positive matrices A and B, for $k > 1$.

Theorem 2.11. *Let $D \in \mathbb{R}^{n_1 \times n_1}$, $E \in \mathbb{R}^{n_1 \times k}$, $F \in \mathbb{R}^{k \times n_1}$, $Y \in \mathbb{R}^{k \times k}$, $Q \in \mathbb{R}^{k \times n_2}$, $S \in \mathbb{R}^{n_2 \times k}$ and $T \in \mathbb{R}^{n_2 \times n_2}$ with D, T and Y nonsingular. Let $D^{-1} \geqslant 0$, $Y^{-1} \geqslant 0$, $T^{-1} \geqslant 0$, $-E \geqslant 0$, $-F \geqslant 0$, $-Q \geqslant 0$ and $-S \geqslant 0$. Further, assume that $\rho(Y^{-1}QT^{-1}S) < 1 - \rho(Y^{-1}FD^{-1}E)$. Let*

$$C = \begin{pmatrix} D & E & 0 \\ F & Y & Q \\ 0 & S & T \end{pmatrix}$$

be an inverse positive matrix (of size $n = n_1 + n_2 + k$). Then there exist real numbers a, b such that $0 < a, b < 1$ and for $G = aY$, $P = bY$, $G + P = Y$, $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix} \in \mathbb{R}^{(n_1+k)\times(n_1+k)}$, $B = \begin{pmatrix} P & Q \\ S & T \end{pmatrix} \in \mathbb{R}^{(k+n_2)\times(k+n_2)}$ are inverse-positive with $C = A \oplus_k B$.

Proof. For any nonzero real number r, take $G = rY$. Then the determinant of the matrix $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$ is a polynomial in r of degree at most k. Let Z_A be the set of all real zeros of this polynomial. Similarly, let Z_B be the set of all real zeros of the polynomial obtained from the determinant of $B = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}$ by taking $P = sY$, for $s \neq 0$. Note that Z_A and Z_B may be empty due to the fact that the polynomials above may not have real zeros.

Choose $a \in (\mathbb{R} \setminus Z_A) \cap (0, 1)$ such that $1 - a \in (\mathbb{R} \setminus Z_B) \cap (0, 1)$, $a > \rho(Y^{-1}FD^{-1}E)$ and $1 - a > \rho(Y^{-1}QT^{-1}S)$. This is possible since both Z_A and Z_B are finite sets and $\rho(Y^{-1}FD^{-1}E) < 1$. Set $b = 1 - a$. By setting $G = aY$ and $P = bY$, we obtain the invertibility of A and B. Also, we have $G + P = Y$ and hence $C = A \oplus_k B$.

Consider the Schur complement of D in A, $A/D = G - FD^{-1}E$. Then $A/D = M_1 - N_1$ is a regular splitting of A/D where $M_1 = G = aY$ and $N_1 = FD^{-1}E$ with M_1 nonsingular, $M_1^{-1} \geqslant 0$ and $N_1 \geqslant 0$. Hence, by Theorem 2.10, $(A/D)^{-1} \geqslant 0$ as $\rho(M^{-1}N) = \rho\left(\frac{Y^{-1}}{a}FD^{-1}E\right) < 1$. Again, applying Lemma 2.1, it follows that $A^{-1} \geqslant 0$. In a similar way, we get $B^{-1} \geqslant 0$. □

The following results in [1] can be obtained as consequences of Theorem 2.11. Note that these are the converses of Corollary 2.5 and Corollary 2.6, respectively.

Corollary 2.12. *[1, Proposition 9] Let $C = \begin{pmatrix} D & 0 & 0 \\ F & Y & 0 \\ 0 & S & T \end{pmatrix}$ be an inverse-positive matrix with $D^{-1} \geqslant 0$, $Y^{-1} \geqslant 0$, $T^{-1} \geqslant 0$, $-F \geqslant 0$ and $-S \geqslant 0$. Then $C = A \oplus_k B$ for some inverse-positive matrices A and B.*

Proof. Proof follows by taking $E = 0$ and $Q = 0$ in Theorem 2.11. □

Corollary 2.13. *[1, Proposition 11] Let $C = \begin{pmatrix} D & 0 & 0 \\ F & Y & Q \\ 0 & 0 & T \end{pmatrix}$ be an inverse-positive matrix with $D^{-1} \geqslant 0$, $Y^{-1} \geqslant 0$, $T^{-1} \geqslant 0$, $-F \geqslant 0$ and $-Q \geqslant 0$. Then $C = A \oplus_k B$ for some inverse-positive matrices A and B.*

Proof. Set $E = 0$ and $S = 0$ in Theorem 2.11. □

We illustrate Theorem 2.11 as follows.

Example 2.14. Consider the matrix $C = \left(\begin{array}{c:c:c} D & E & 0 \\ \hdashline F & Y & Q \\ \hdashline 0 & S & T \end{array}\right) = \left(\begin{array}{cc:cc:cc} -2 & 1 & 0 & 0 & 0 & 0 \\ 7 & -3 & 0 & -1 & 0 & 0 \\ \hdashline -1 & 0 & -3 & 8 & -1 & 0 \\ 0 & 0 & 7 & -2 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 & 7 & -3 \end{array}\right)$.

We have, $C^{-1} = \frac{1}{36}\begin{pmatrix} 129 & 43 & 7 & 3 & 21 & 7 \\ 294 & 86 & 14 & 6 & 42 & 14 \\ 6 & 2 & 2 & 6 & 6 & 2 \\ 21 & 7 & 7 & 3 & 21 & 7 \\ 21 & 7 & 7 & 3 & 129 & 43 \\ 42 & 14 & 14 & 6 & 294 & 86 \end{pmatrix} \geqslant 0$. Also, $S = E \leqslant 0$, $Q = F \leqslant 0$ and $T = D$. Again, $D^{-1} = T^{-1} = \begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix} \geqslant 0$, $Y^{-1} = \frac{1}{50}\begin{pmatrix} 2 & 8 \\ 7 & 3 \end{pmatrix} \geqslant 0$ and $\rho(Y^{-1}FD^{-1}E) = \rho(Y^{-1}QT^{-1}S) = \frac{7}{50}$. Thus, $\rho(Y^{-1}FD^{-1}E) < 1 - \rho(Y^{-1}QT^{-1}S)$. Hence, the conditions of Theorem 2.11 are satisfied.

Now, we have $Z_A = \{0\}$ and $Z_B = \{0, \frac{43}{50}\}$. Choose $a = b = \frac{1}{2}$. Clearly, $a > \rho(Y^{-1}FD^{-1}E)$ and $b > \rho(Y^{-1}QT^{-1}S)$. Now, take $G = \frac{1}{2}Y$, $P = \frac{1}{2}Y$, $A = \left(\begin{array}{c:c} D & E \\ \hdashline F & G \end{array}\right) = \left(\begin{array}{cc:cc} -2 & 1 & 0 & 0 \\ 7 & -3 & 0 & -1 \\ \hdashline -1 & 0 & \frac{-3}{2} & 4 \\ 0 & 0 & \frac{7}{2} & -1 \end{array}\right)$ and $B = \left(\begin{array}{c:c} P & Q \\ \hdashline S & T \end{array}\right) = \left(\begin{array}{cc:cc} \frac{-3}{2} & 4 & -1 & 0 \\ \frac{7}{2} & -1 & 0 & 0 \\ \hdashline 0 & 0 & -2 & 1 \\ 0 & -1 & 7 & -3 \end{array}\right)$. Then $A^{-1} = \frac{1}{18}\begin{pmatrix} 75 & 25 & 7 & 3 \\ 168 & 50 & 14 & 6 \\ 6 & 2 & 2 & 6 \\ 21 & 7 & 7 & 3 \end{pmatrix} \geqslant 0$, $B^{-1} = \frac{1}{18}\begin{pmatrix} 2 & 6 & 6 & 2 \\ 7 & 3 & 21 & 7 \\ 7 & 3 & 75 & 25 \\ 14 & 6 & 168 & 50 \end{pmatrix} \geqslant 0$ and $C = A \oplus_2 B$.

Remark 2.15. The conditions provided in the above Theorem 2.11 are sufficient but not necessary, as shown by the following example.

Example 2.16. Let $C = \left(\begin{array}{c:c:c} D & E & 0 \\ \hdashline F & Y & Q \\ \hdashline 0 & S & T \end{array}\right) = \left(\begin{array}{c:cc:c} 0 & 0 & 6 & 0 \\ \hdashline 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c:cc} 0 & 0 & 6 \\ \hdashline 0 & 1 & 0 \\ 3 & 0 & -1 \end{array}\right) \oplus_2 \left(\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right) = A \oplus_2 B.$

Here C is an inverse-positive matrix of the form as in the right-hand side of Eq. (1). Also, $C^{-1} = \frac{1}{6}\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $A^{-1} = \frac{1}{18}\begin{pmatrix} 1 & 0 & 6 \\ 0 & 18 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ and $B^{-1} = I$ are nonnegative. However, C does not satisfy any of the conditions of Theorem 2.11.

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