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On Euclidean distance matrices

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Abstract

If A is a real symmetric matrix and P is an orthogonal projection onto a hyperplane, then we derive a formula for the Moore–Penrose inverse of PAP. As an application, we obtain a formula for the Moore–Penrose inverse of an Euclidean distance matrix (EDM) which generalizes formulae for the inverse of a EDM in the literature. To an invertible spherical EDM, we associate a Laplacian matrix (which we define as a positive semidefinite $n \times n$ matrix of rank $n-1$ and with zero row sums) and prove some properties. Known results for distance matrices of trees are derived as special cases. In particular, we obtain a formula due to Graham and Lovász for the inverse of the distance matrix of a tree. It is shown that if D is a nonsingular EDM and L is the associated Laplacian, then $D^{-1} - L$ is nonsingular and has a nonnegative inverse. Finally, infinitely divisible matrices are constructed using EDMs.

Keywords: Distance matrices; Laplacian; Trees; Infinitely divisible matrices

1. Introduction and preliminaries

A real symmetric $n \times n$ matrix D is called an Euclidean distance matrix (EDM) if there exist points $p_1, p_2, \ldots, p_n \in R^k$ such that

$$d_{ij} = (p_i - p_j)'(p_i - p_j), \quad i, j = 1, 2, \ldots, n.$$

(As usual, the transpose of a matrix A is denoted A'.) EDMs have a wide literature and applications. For details, we refer to Schoenberg [14], Gower [7], Johnson and Tarazaga [12],

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Hayden et al. [11] and the references therein. In [15], Styan and Subak-Sharpe discuss electrical network theory through EDMs. They obtain an expression for the inverse of a EDM and give physical interpretations in terms of networks.

Let

$$\mathbf{1} = (1, 1, \dots, 1)'$$

be the vector of all ones in R^n. Schoenberg [14] showed that a nonnegative symmetric matrix D with zero diagonal is a EDM if and only if

$$F := -\frac{1}{2}\left(I - \frac{\mathbf{1}\mathbf{1}'}{n}\right) D \left(I - \frac{\mathbf{1}\mathbf{1}'}{n}\right)$$

is positive semidefinite (p.s.d.). If $F = XX'$ is a decomposition of F then the rows of X give coordinates of points that generate D. In [15], it is shown that if D is an invertible EDM then

$$D^{-1} = -Y + uu',$$

where u is a nonzero vector in R^n and Y is the symmetric matrix satisfying the conditions (1) Y is p.s.d., (2) $\text{rank}(Y) = n - 1$, and (3) $\mathbf{1}'Y = 0$. It can be shown that Y is the Moore–Penrose inverse, $F^\dagger$, of F. (The definition of Moore–Penrose inverse is given later in this section.) It is easy to see that $P := I - \frac{\mathbf{1}\mathbf{1}'}{n}$ is the orthogonal projection onto the hyperplane $\{\mathbf{1}\}^\perp$.

Motivated by these results, we find a formula for the Moore–Penrose inverse of PAP where A is any symmetric matrix and P is the orthogonal projection onto the hyperplane $\{a\}^\perp$, where $a \in R^n$ is in the column space of A, satisfying $a'A^\dagger a \neq 0$. If D is a EDM, Gower [8] proved that $\mathbf{1}$ is in the column space of D and $\mathbf{1}'D^\dagger\mathbf{1} \geqslant 0$. When $\mathbf{1}'D^\dagger\mathbf{1} > 0$, Gower [8] showed that there exists a sphere of radius $\mathbf{1}'D^\dagger\mathbf{1}$ such that D is the EDM of points on the sphere. If such a sphere exists then we will say that D is a spherical EDM. We give an expression for the Moore–Penrose inverse of a spherical EDM. This generalizes the result proved by Styan and Subak-Sharpe [15]. We define the notion of a Laplacian matrix for nonsingular spherical EDMs which satisfy $\mathbf{1}'D^{-1}\mathbf{1} > 0$ and prove various properties in connection with EDMs. For nonspherical EDMs, we get an expression for the Moore–Penrose inverse by choosing the orthogonal projection onto the hyperplane $\{D\mathbf{1}\}^\perp$.

Distance matrices of trees have a close interconnection with EDMs. A tree is a connected acyclic graph. The (i, j)-element of the distance matrix of a tree is the length of the shortest path between vertices i and j of the tree. Distance matrices of trees are special cases of nonsingular spherical EDMs. In [9], Graham and Lovász obtained a significant formula for the inverse of the distance matrix of a tree. The inverse is expressed as a sum of the Laplacian matrix of the tree and a matrix of rank one. By specializing our results to distance matrices of trees we derive the Graham and Lovász formula and also a well-known formula for the determinant of the distance matrix of a tree due to Graham and Pollack [10].

In Section 5 we show that if D is a nonsingular EDM and L is any Laplacian matrix, then $D^{-1} - L$ is a nonsingular matrix and has a nonnegative inverse.

The last section brings out some connections between EDMs and infinitely divisible matrices. In particular, we construct examples of infinitely divisible matrices based on EDMs.

We now introduce some definitions.

Definition 1.1. Let A be a real $n \times n$ matrix. Then H is called a g-inverse of A if $AHA = A$. If $HAH = H$ then we say that H is an outer inverse of A.

Definition 1.2. Let $A \in R^{n\times n}$. Then Moore–Penrose inverse of A is a matrix $A^\dagger$ satisfying the equations: $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)' = AA^\dagger$ and $(A^\dagger A)' = A^\dagger A$.

It is well known that Moore–Penrose inverse of a matrix exists and is unique. For basic properties of the Moore–Penrose inverse, see [4]. We use $R(A)$ to denote the column space of a matrix A. For $y \in R^n$, let $\mathrm{Diag}(y)$ denote the diagonal matrix with $y_1, y_2, \ldots, y_n$ along the diagonal. If X is an $n \times n$ matrix, let $\mathrm{diag}(X) = (x_{11}, \ldots, x_{nn})'$.

A symmetric $n \times n$ matrix A is called a *conditionally negative definite* (c.n.d.) matrix if for all $x \in \{\mathbf{1}\}^{\perp}$, $x'Ax \leqslant 0$. It is known that a EDM is c.n.d., see [14]. Thus an $n \times n$ EDM is negative semidefinite on an $n-1$ dimensional subspace. Hence every EDM has exactly one positive eigenvalue.

2. A formula involving the Moore–Penrose inverse

Let a be a nonzero vector in R^n. We now derive a formula for the Moore–Penrose inverse of PAP, where $P := I - \frac{aa'}{a'a}$ and A is a real symmetric matrix of order n. We first prove a preliminary result.

Lemma 2.1. *Let A be a symmetric $n \times n$ matrix and let $a \in R^n$ be a vector such that $\beta = a'A^{\dagger}a \neq 0$. If $P := I - \frac{aa'}{a'a}$ is the orthogonal projection onto the hyperplane $\{a\}^{\perp}$, then*

$$T = A^{\dagger} - \frac{(A^{\dagger}a)(a'A^{\dagger})}{\beta}$$

is an outer inverse of $K = PAP$; that is, T satisfies $TKT = T$.

Proof. We claim that $TP = PT = T$. Let $v \in R^n$. Then $v = v_1 + v_2$ where $v_1 \in \mathrm{span}(a)$ and $v_2 \in \{a\}^{\perp}$. Since P is an orthogonal projection onto $\{a\}^{\perp}$, then $Pv = v_2$. Thus, $TPv = Tv_2$. Now $Tv_1 = 0$ and hence $Tv = Tv_2$. This shows that $TP = T$. Since T is symmetric, $TP = PT$. This proves our claim. Now, we need to show that $TAT = T$. We note that,

$$\begin{aligned} TAT &= \left(A^{\dagger} - \frac{(A^{\dagger}a)(a'A^{\dagger})}{\beta}\right) A \left(A^{\dagger} - \frac{(A^{\dagger}a)(a'A^{\dagger})}{\beta}\right) \\ &= \left(A^{\dagger}A - \frac{A^{\dagger}aa'A^{\dagger}A}{\beta}\right)\left(A^{\dagger} - \frac{(A^{\dagger}a)(a'A^{\dagger})}{\beta}\right) \\ &= A^{\dagger} - \frac{(A^{\dagger}a)(a'A^{\dagger})}{\beta} - \frac{(A^{\dagger}a)(a'A^{\dagger})}{\beta} + \frac{(A^{\dagger}a)(a'A^{\dagger})}{\beta} = T. \end{aligned}$$

This completes the proof. □

Theorem 2.1. *Let A be a symmetric $n \times n$ matrix and let $a \in R(A)$ be a vector such that $\beta := a'A^{\dagger}a \neq 0$. If $P := I - \frac{aa'}{a'a}$ is the orthogonal projection onto the hyperplane $\{a\}^{\perp}$, then*

$$T := A^{\dagger} - \frac{(A^{\dagger}a)(a'A^{\dagger})}{\beta}$$

is the Moore–Penrose inverse of $K = PAP$.

Proof. We claim that $PAA^{\dagger}$ is symmetric. Since $a \in R(A)$, there exists $w \in R^n$ such that $Aw = a$. Now we have

$$PAA^{\dagger} = \left(I - \frac{aa'}{a'a}\right)AA^{\dagger} = AA^{\dagger} - \frac{(Aw)(w'AAA^{\dagger})}{a'a} = AA^{\dagger} - \frac{(Aw)(w'A)}{a'a}.$$

Thus $PAA^\dagger$ is symmetric. We claim that $B = PA(A^\dagger a)(a'A^\dagger) = 0$. Note that $A^\dagger a = A^\dagger Aw$. Therefore $B = PA(A^\dagger Aw)(w'AA^\dagger)$. Since $AA^\dagger Aw = Aw = a$ and $Pa = 0$ it follows that $B = 0$. We now prove that $KT = TK$. We see that

$$\begin{aligned} KT &= PAPT \\ &= PAT \quad (\text{as } PT = TP = T) \qquad (1)\\ &= PAA^\dagger - \frac{B}{\beta} \\ &= PAA^\dagger. \qquad (2) \end{aligned}$$

Since $PAA^\dagger$ is symmetric, as already noted, we get $KT = TK$. By Lemma 2.1, T is an outer inverse of K. It remains to show that T is a g-inverse of K. Now,

$$KTK = KTPAP = KTAP = PAA^\dagger AP = PAP = K.$$

This completes the proof. □

3. Spherical Euclidean distance matrices

Let D be a EDM. Gower [8] proved that $DD^\dagger\mathbf{1} = \mathbf{1}$, $\mathbf{1}'D^\dagger\mathbf{1} \geqslant 0$ and D is spherical if and only if $\mathbf{1}'D^\dagger\mathbf{1} > 0$. Hence $\mathbf{1} \in R(D)$. Let $\beta := \mathbf{1}'D^\dagger\mathbf{1}$. Using Theorem 2.1 we get the following result:

Theorem 3.1. *Let D be a spherical EDM and let $P := I - \frac{1}{n}\mathbf{1}\mathbf{1}'$. If $G := -\frac{1}{2}PDP$, then setting $u = D^\dagger\mathbf{1}$,*

$$D^\dagger = -\tfrac{1}{2}G^\dagger + \frac{1}{\beta}uu'. \qquad (3)$$

Continuing with the notation of Theorem 3.1, Schoenberg [14] proved that G is p.s.d. and

$$D = \operatorname{diag}(G)\mathbf{1}\mathbf{1}' + \mathbf{1}\mathbf{1}'\operatorname{diag}(G) - 2G. \qquad (4)$$

From (3) we deduce,

$$-\tfrac{1}{2}G^\dagger DG^\dagger = G^\dagger. \qquad (5)$$

We now consider the case when D is spherical and nonsingular. In this case, let us define $L^\dagger := -\frac{1}{2}PDP$. Clearly $\mathbf{1}'L^\dagger = 0$ and $L^\dagger$ is p.s.d. Since D is nonsingular and P is of rank $n-1$, we see that $\operatorname{rank}(L^\dagger) = n-1$. Therefore $\mathbf{1}'L = 0$, L is p.s.d. and $\operatorname{rank}(L) = n-1$. These observations motivate the next definition.

Definition 3.1. Let L be a symmetric $n \times n$ matrix. Then we say that L is a Laplacian matrix if L is p.s.d. with rank $n-1$ and has row sums zero.

We remark that a conventional Laplacian has nonpositive, integer off-diagonal entries but we do not require this property here. We now prove the following result:

Theorem 3.2. *Let D be a spherical, nonsingular EDM. Then there exists a unique Laplacian matrix L satisfying*

$$D^{-1} = -\frac{1}{2}L + \frac{uu'}{\beta}, \tag{6}$$

where $u = D^{-1}\mathbf{1}$, $\beta = \mathbf{1}'D^{-1}\mathbf{1}$.

Proof. We only need to show that the Laplacian associated with D is unique, as (6) follows from Theorem 3.1 and the definition of the Laplacian. Let $z := D^{-1}\mathbf{1}$. Suppose, $D^{-1} = -\frac{1}{2}L + uu' = -\frac{1}{2}M + vv'$. Then, $z = uu'\mathbf{1} = vv'\mathbf{1}$. Since $\operatorname{rank}(uu') = 1$, and $z \in R(uu')$, then $z = \beta u$ for a nonzero β. Thus, $uu' = k^2 zz'$ for some nonzero k. Similarly, $vv' = c^2 zz'$. Since $uu'\mathbf{1} = vv'\mathbf{1}$, then $c^2 = k^2$. Therefore, $uu' = vv'$ and hence $L = M$. □

The above result was obtained, in a different form, in Styan and Subak-Sharpe [15].

For a matrix A, we denote by $A(i, j)$ the submatrix obtained by deleting row i and column j of A. The matrix $A(i, i)$ is denoted $A(i)$. We now deduce some simple properties of the Laplacian.

Proposition 3.1. *Let D be a spherical EDM and let L be the corresponding Laplacian. Then*

1. $-\frac{1}{2}LDL = L$.
2. *For* $i \neq j \in \{1, 2, \ldots, n\}$, $d_{ij} = \frac{\det(L(i,j))}{\det(L(i))}$.

Proof. The first equation follows from (5). We now prove the second equation. We first claim that if L is a Laplacian matrix, then for any two g-inverses of L, say, $S = (s_{ij})$ and $T = (t_{ij})$,

$$s_{ii} + s_{jj} - s_{ij} - s_{ji} = t_{ii} + t_{jj} - t_{ij} - t_{ji} \quad \text{for all } i \neq j.$$

Let x be the column vector with $x_i = 1$ and $x_j = -1$ and with its remaining coordinates zero. Clearly, $\mathbf{1}'x = 0$. Thus x belongs to the orthogonal complement of the null space of L, which is the same as $R(L)$ and hence there exists a vector $y \in R^n$ such that $Ly = x$. It is easy to see that $x'Sx = y'Ly = s_{ii} + s_{jj} - s_{ij} - s_{ji}$ and $x'Tx = y'Ly = t_{ii} + t_{jj} - t_{ij} - t_{ji}$. This proves our claim. Let H be the $n \times n$ matrix with $H(i) = L(i)^{-1}$ and each entry in row and column i equal to zero. The matrix H depends on i but we suppress this in the notation. Note that for all j, $h_{ii} = h_{ij} = h_{ji} = 0$ and for $j \neq i$, $h_{jj} = \frac{\det(L(i,j))}{\det(L(i))}$. It is easily verified that H is a g-inverse of L. Thus, $h_{ii} + h_{jj} - h_{ij} - h_{ji} = \frac{\det(L(i,j))}{\det(L(i))}$. Now the result follows from the first equation and the above claim. □

We now find an expression for the determinant of a spherical, nonsingular EDM. If A is a symmetric matrix with $\mathbf{1}'A = 0$, then all the cofactors of A are equal. We call this common cofactor value of A as the *common cofactor*.

Theorem 3.3. *Let D be a nonsingular spherical EDM and let L be the corresponding Laplacian. If* γ *is the common cofactor of L and* $\alpha = \mathbf{1}'D^{-1}\mathbf{1}$ *then*

$$\det(D^{-1}) = \left(-\frac{1}{2}\right)^{n-1} \frac{\gamma}{\alpha}. \tag{7}$$

Proof. The proof follows from (6) and the multilinearity of the determinant. □

We now consider nonspherical EDMs. In this case $\mathbf{1}'D^{\dagger}\mathbf{1} = 0$. Hence Theorem 3.2 is not true. However by choosing the orthogonal projection $P_1 := I - \frac{uu'}{u'u}$, where $u = D\mathbf{1}$, we get the following result:

Theorem 3.4. *Let D be a nonspherical EDM and let $\beta := \mathbf{1}'D\mathbf{1}$. If $K := -\frac{1}{2}P_1 D P_1$ where $P_1 := I - \frac{uu'}{u'u}$, and $u = D\mathbf{1}$, then*

$$D^{\dagger} = -\frac{1}{2}K^{\dagger} + \frac{1}{\beta}\mathbf{1}\mathbf{1}'. \tag{8}$$

Proof. Let $K_1 := P_1 D P_1$. By Theorem 2.1, we have

$$K_1^{\dagger} = D^{\dagger} - \frac{1}{\beta}\mathbf{1}\mathbf{1}'.$$

Now,

$$K^{\dagger} = \left(-\frac{1}{2}K_1\right)^{\dagger} = -2{K_1}^{\dagger} = -2\left(D^{\dagger} - \frac{1}{\beta}\mathbf{1}\mathbf{1}'\right).$$

Therefore,

$$D^{\dagger} = -\frac{1}{2}K^{\dagger} + \frac{1}{\beta}\mathbf{1}\mathbf{1}'$$

and the proof is complete. □

Continuing with the notation of Theorem 3.4, let $u = D\mathbf{1}$ and $U = \mathrm{Diag}(u)$. From (8) we have

$$UD^{-1}U = -\frac{1}{2}UK^{\dagger}U + \frac{1}{\beta}U\mathbf{1}\mathbf{1}'U. \tag{9}$$

Put $H := UK^{\dagger}U$. Then

$$UD^{-1}U = -\frac{1}{2}H + \frac{1}{\beta}U\mathbf{1}\mathbf{1}'U. \tag{10}$$

Since $\mathbf{1}'H = 0$, then all the cofactors of H are equal. Let γ_1 be the common cofactor of H. Using (10) and the multilinearity of the determinant we see that

$$\det(D^{-1}) = \frac{1}{\Pi_{i=1}^{n} u_i^2}\left(-\frac{1}{2}\right)^{n-1}\gamma_1\beta.$$

Thus, using the notation introduced above, we have the following theorem:

Theorem 3.5. *Let D be a nonsingular nonspherical EDM. Then*

$$\det(D^{-1}) = \frac{1}{\Pi_{i=1}^{n} u_i^2}\left(-\frac{1}{2}\right)^{n-1}\beta\gamma_1,$$

where $\beta := \mathbf{1}'D\mathbf{1}$, $u = D\mathbf{1}$, γ_1 is the common cofactor of $H := UK^{\dagger}U$ and $U := Diag(u)$.

4. Distance matrices of trees

Let $T = (V, \mathscr{E})$ denote a tree with the set of vertices V and the set of edges $\mathscr{E}$. We assume that $V = \{1, 2, \ldots, n\}$, and the edges are unordered pairs (i, k), $i \neq k$. To each edge (i, k) we assign a number $w_{ik} = 1$ if $i \neq k$ and (i, k) is an edge of T, $i, k \in V$. If $i \neq k$ and (i, k) is not an edge of T then we define $w_{ik} = 0$. The *Laplacian* is then the matrix

$$L = \begin{pmatrix} \sum_k w_{1k} & -w_{12} & -w_{13} & \dots & -w_{1n} \\ -w_{21} & \sum_k w_{2k} & -w_{23} & \dots & -w_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ -w_{n1} & -w_{n2} & -w_{n3} & \dots & \sum_k w_{nk} \end{pmatrix}.$$

Clearly the row sums of L are zero. It is well known that L is p.s.d. with rank $n - 1$. The distance matrix E of T is an $n \times n$ matrix with the (i, j)-entry equal to the distance (i.e. the length of the shortest path) between vertices i and j. In this section, we obtain the results due to Graham and Lovász [9] and Graham and Pollack [10] as special cases of our earlier results.

Several relations between L and E are known. By induction it can be easily shown that

$$-\frac{LEL}{2} = L. \tag{11}$$

We now obtain the Graham and Lovász formula.

Theorem 4.1. *Let T be a tree on n vertices with Laplacian L and distance matrix E. Let δ_i denote the degree of the vertex i, $i = 1, 2, \dots, n$ and let $\delta = (\delta_1, \dots, \delta_n)$ and $\lambda := (2 - \delta_1, \dots, 2 - \delta_n)$. Then*

$$E^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\lambda\lambda'.$$

Proof. From (11) we get the following equation, noting that $P = LL^{\dagger} = I - \frac{\mathbf{1}\mathbf{1}'}{n}$:

$$-\tfrac{1}{2}PEP = L^{\dagger}. \tag{12}$$

By Theorem 3.2,

$$E^{-1} = -\frac{1}{2}L + \frac{(E^{-1}\mathbf{1})(\mathbf{1}'E^{-1})}{\mathbf{1}'E^{-1}\mathbf{1}}. \tag{13}$$

It is easy to verify, by induction on n, that

$$E\lambda = (n-1)\mathbf{1} \quad \text{and} \quad \mathbf{1}'\lambda = 2, \tag{14}$$

and hence

$$\lambda' E\lambda = 2(n-1). \tag{15}$$

It follows from (13)–(15) that

$$E^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\lambda\lambda'$$

and the proof is complete. □

We now obtain the Graham and Pollack [10] formula for the determinant of the distance matrix of a tree. By the matrix–tree theorem, any cofactor of L is the number of spanning trees and hence the common cofactor, say, $K = 1$. Thus by Theorem 3.3 we have:

Theorem 4.2. *Let T be a tree on n vertices and let E be the distance matrix of T. Then* $\det(E) = (-1)^{n-1}(n-1)2^{n-2}$.

5. Inverse EDMs perturbed by Laplacians

Let D be the distance matrix of a tree. If L is the Laplacian matrix of an arbitrary connected graph, then it was shown in [1] that $(D^{-1} - L)^{-1}$ is a nonnegative matrix. Now let D be a EDM. Motivated by the result in [1], we may ask the following question: If L is a Laplacian, is $(D^{\dagger} - L)^{\dagger}$ necessarily a nonnegative matrix? From numerical experiments, we found that for small $\epsilon > 0$, $(D^{\dagger} - \epsilon L)^{\dagger}$ is not a nonnegative matrix when D is singular. However when D is a nonsingular EDM, we observed that $D^{-1} - L$ has a nonnegative inverse. We give a proof of this in the next theorem. First, we prove the following lemma which is true even for singular EDMs.

We recall that a matrix S is called a signature matrix if it is a diagonal matrix with diagonal entries as 1 or -1. A matrix $A \in R^{n\times n}$ is called a N-matrix if all the principal minors are negative.

Lemma 5.1. *Let D be a EDM and let L be a Laplacian matrix. Then the following are true*:

1. $D^{\dagger} - L$ *is a nonsingular matrix.*
2. $D^{\dagger} - L$ *is c.n.d.*
3. $\det(L - D^{\dagger}) < 0$.

Proof. Suppose that there exists a nonzero vector $x \in R^n$ such that $(D^{\dagger} - L)x = 0$. Then, $\mathbf{1}'D^{\dagger}x = 0$. Put $y = D^{\dagger}x$. Then $y \in \{\mathbf{1}\}^{\perp}$. Now, $y'Dy \leqslant 0$ since D is c.n.d. and therefore $x'D^{\dagger}x \leqslant 0$. Hence $x'Lx \leqslant 0$. Because L is p.s.d., $x'Lx = 0$ and therefore, $Lx = 0$. Thus, $x = \beta\mathbf{1}$ for some nonzero β. We now have $D^{\dagger}\mathbf{1} = 0$. This contradicts the result $DD^{\dagger}\mathbf{1} = \mathbf{1}$.

We now prove (2). Let $x \in \{\mathbf{1}\}^{\perp}$. We claim that $D^{\dagger}$ is c.n.d. We note that $D^{\dagger}$ has exactly one positive eigenvalue. Since $DD^{\dagger}\mathbf{1} = \mathbf{1}$, from Theorem 4.1 in [5], $D^{\dagger}$ is c.n.d. Now $-L$ is negative semidefinite. Hence $D^{\dagger} - L$ is a c.n.d. matrix.

Let $A := D^{\dagger} - L$. Then A is a nonsingular c.n.d. matrix. If A is negative definite then $\mathbf{1}'A\mathbf{1} < 0$. This is a contradiction. Thus, A has exactly one positive eigenvalue and hence $\det(L - D^{\dagger}) < 0$. $\square$

We now obtain the following identity:

Lemma 5.2. *Let D be a nonsingular spherical EDM and L be the corresponding Laplacian matrix. If D is nonsingular and $\gamma \geqslant 0$ then*

$$(D^{-1} - \gamma L)^{-1} = \left(\frac{\gamma}{1+2\gamma}\right)\left(\frac{2}{\mathbf{1}'D^{-1}\mathbf{1}}\right)\mathbf{1}\mathbf{1}' + \frac{1}{1+2\gamma}D. \tag{16}$$

Proof. From (6), $L = -2(D^{-1} - \frac{1}{k}uu')$, where $u := D^{-1}\mathbf{1}$ and $k := \mathbf{1}'D^{-1}\mathbf{1}$. Now the proof follows by direct verification. $\square$

From the above result we note that $D^{-1} - L$ has a nonnegative inverse if L is the corresponding Laplacian matrix of D. This fact holds even when L is any Laplacian as shown in the following result:

Theorem 5.1. *Let D be a nonsingular EDM. If L is a Laplacian matrix, then* $(D^{-1} - L)^{-1} > 0$.

Proof. As before, let $D^{-1}(i)$ denote the principal submatrix obtained by deleting the ith row and the ith column. We claim that $-D^{-1}(i)$ is p.s.d. By the interlacing theorem, $-D^{-1}(i)$ can have at most one nonnegative eigenvalue. But $d_{ii} = 0$ and hence $\det(D^{-1}(i)) = 0$. Thus, 0 is an eigenvalue of $D^{-1}(i)$. Therefore, $-D^{-1}(i)$ is p.s.d. This implies that $\alpha L(i) - D^{-1}(i)$ is positive definite for any $\alpha > 0$. Now, the inertia of $\alpha L - D^{-1}$ is the same as $-D^{-1}$ and hence $\det(\alpha L - D^{-1}) < 0$. Thus, $\alpha L - D^{-1}$ is an N-matrix. By Lemma 5 in Parthasarathy and Ravindran [13], there exists a signature matrix S_α such that $S_\alpha(\alpha L - D^{-1})^{-1}S_\alpha < 0$. For $\alpha = 0$ we note that $S_\alpha = I$ or $-I$. By continuity, $S_\alpha = I$ or $-I$ for all α. Hence $(D^{-1} - L)^{-1} > 0$. This completes the proof. □

6. A note on infinitely divisible matrices

In this section, we construct infinitely divisible matrices from distance matrices. We now define an infinitely divisible matrix. Let $A = (a_{ij})$ be a nonnegative symmetric matrix and $r \geqslant 0$. Recall that the rth Hadamard power of A is defined by $A^{\circ r} := (a_{ij}^r)$.

Definition 6.1. Let A be a p.s.d. matrix. We say that A is infinitely divisible if $A^{\circ r}$ is p.s.d. for all $r \geqslant 0$.

Infinitely divisible matrices are studied in detail by Horn [6]. Interesting examples can be found in Bhatia [3]. We now construct infinitely divisible matrices from distance matrices. First we state a result due to Löwner, see, for example, [3]. We say that a matrix A is conditionally positive definite (c.p.d.) if $-A$ is c.n.d.

Theorem 6.1. *If A is a symmetric matrix with positive entries then A is infinitely divisible if and only if its Hadamard logarithm* $\log^\circ A = (\log(a_{ij}))$ *is c.p.d.*

Now from the previous section, we see that if D is any nonsingular EDM and L is a Laplacian matrix then $D^{-1} - L$ is a c.n.d. matrix and its inverse is positive. Hence from Theorem 4.1 in [5], it follows that the inverse of $D^{-1} - L$ is a c.n.d. matrix. Now put $A = (D^{-1} - L)^{-1}$. By Theorem 4.4.4. in [2], $\log^\circ A$ is c.n.d. and hence $\left(\log\left(\frac{1}{a_{ij}}\right)\right)$ is a c.p.d. matrix. Thus, by Theorem 6.1, we have the following result:

Theorem 6.2. *Let D be a nonsingular EDM and L be a Laplacian matrix. If $A := (D^{-1} - L)^{-1}$, then $A^{\circ -1} = \left(\frac{1}{a_{ij}}\right)$ is an infinitely divisible matrix.*

By a similar argument as before one can prove the following:

Theorem 6.3. *Let D be an EDM. For $k > 0$, let $T := D + kJ$. Then $T^{\circ -1} := \left(\frac{1}{t_{ij}}\right)$ is an infinitely divisible matrix.*

Example. Let $D = (d_{ij})$ be the distance matrix of a path on n-vertices, where $n > 2$. Then $d_{ij} = |i - j|$, $1 \leqslant i, j \leqslant n$. By Theorem 6.3 it follows that $(|i - j| + k)$ is an infinitely divisible matrix for all $k > 0$.

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References

[1] R.B. Bapat, S.J. Kirkland, M. Neumann, On distance matrices and Laplacians, Linear Algebra Appl. 401 (2005) 193–209.
[2] R.B. Bapat, T.E.S. Raghavan, Nonnegative matrices and its applications, in: Encyclopedia of Mathematics and its Applications, vol. 64, Cambridge University Press, Cambridge, 1997.
[3] R. Bhatia, Infinitely divisible matrices, Amer. Math. Monthly 113 (2006) 221–235.
[4] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, Wiley-Interscience, 1974.
[5] J.P. Crouzeix, J.A. Ferland, Criteria for quasi-convexity and pseudo-convexity: relationships and comparisons, Math. Program. 23 (1982) 193–205.
[6] R.A. Horn, The theory of infinitely divisible matrices and kernels, Trans. Amer. Math. Soc. 136 (1969) 269–286.
[7] J.C. Gower, Euclidean distance geometry, Math. Sci. 7 (1982) 1–14.
[8] J.C. Gower, Properties of Euclidean and non-Euclidean distance matrices, Linear Algebra Appl. 67 (1985) 81–97.
[9] R.L. Graham, L. Lovász, Distance matrix polynomials of trees, Adv. Math. 29 (1) (1978) 60–88.
[10] R.L. Graham, H.O. Pollack, On the addressing problem for loop switching, Bell. System Tech. J. 50 (1971) 2495–2519.
[11] T.L. Hayden, R. Reams, J. Wells, Methods for constructing distance matrices and the inverse eigenvalue problem, Linear Algebra Appl. 295 (1999) 97–112.
[12] C.R. Johnson, P. Tarazaga, Connections between the real positive semidefinite and distance matrix completion problems, Linear Algebra Appl. 223 (1995) 375–391.
[13] T. Parthasarathy, G. Ravindran, N-matrices, Linear Algebra Appl. 139 (1990) 89–102.
[14] I.J. Schoenberg, Metric spaces and positive definite functions, Trans. Amer. Math. Soc. 44 (1938) 522–536.
[15] G.E. Styan, G.E. Subak-Sharpe, Inequalities and equalities with the Campbell–Youla generalized inverse of the indefinite admittance matrix of resistive networks, Linear Algebra Appl. 250 (1997) 349–370.