Non-trivial classical backgrounds with vanishing quantum corrections

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Abstract

Vacuum polarization and particle production effects in classical electromagnetic and gravitational backgrounds can be studied by the effective lagrangian method. Background field configurations for which the effective lagrangian is zero are special in the sense that the lowest order quantum corrections vanishes for such configurations. We propose here the conjecture that there will be neither particle production nor vacuum polarization in classical field configurations for which all the scalar invariants are zero. We verify this conjecture, by explicitly evaluating the effective lagrangian, for *non-trivial* electromagnetic and gravitational backgrounds with vanishing scalar invariants. The implications of this result are discussed.

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I. INTRODUCTION

The effective lagrangian approach is probably the most unambiguous approach available at present to study the evolution of quantum fields in classical backgrounds [1–4]. In this approach, an effective lagrangian is obtained for the classical background field by integrating out the degrees of freedom corresponding to the quantum field. The effective lagrangian thus obtained, in general, has a real and an imaginary part. The real part is interpreted as the 'vacuum-to-vacuum' transition amplitude, *i.e.* the amplitude for the quantum field to remain in the initial vacuum state at late times (vacuum polarization) and the existence of a nonzero imaginary part is attributed to the instability of the vacuum (particle production).

Several non-perturbative features of the theory can be understood if the effective lagrangian can be evaluated *exactly* for an *arbitrary* background field configuration. But, the evaluation of the effective lagrangian for an arbitrary classical background proves to be an uphill task. Therefore, there has been numerous attempts in literature [5–13] to evaluate the effective lagrangian explicitly for a *given* electromagnetic or gravitational background.

Symmetry considerations suggest that it should be possible to express the effective lagrangian, at least formally, in terms of invariant scalars describing the classical background (gauge invariant quantities involving the field tensor $F_{\mu\nu}$ and its derivatives in the case of electromagnetism and coordinate invariant scalars involving the Riemann curvature tensor $R_{\mu\nu\lambda\rho}$ and its derivatives in the case of gravity). The existence of an imaginary part to the effective lagrangian—and other features—should be related to the actual values of some of these scalars. Consider, for example, the simple case a quantized complex scalar field interacting with an electromagnetic background that is constant both in space and time. Schwinger, in his classic paper [1], had evaluated the effective lagrangian for such a background by integrating out the degrees of freedom corresponding to the quantum field. (Schwinger had in fact considered the quantum field to be a spinor field, but his result also holds good for the complex scalar field we consider in this paper.) He showed that the resulting expression for the effective lagrangian depends only on the two gauge invariant quantities $\mathcal{G} = F^{\mu\nu}F_{\mu\nu}$ and $\mathcal{F} = \epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$. Further, the effective lagrangian had an imaginary part only if $\mathcal{G} < 0$, thereby implying that constant magnetic fields cannot produce particles while constant electric fields can. Schwinger's result, of course, had been obtained only for constant $F_{\mu\nu}$'s and it is not easy to evaluate the effective lagrangian for a more general case (see [14] for an attempt in this direction). Also, for an arbitrary electromagnetic background, there is no a priori reason as to why the effective lagrangian cannot depend on invariant quantities involving the derivatives of $F_{\mu\nu}$'s, for instance, say, $\partial_\lambda F^{\mu\nu}\partial^\lambda F_{\mu\nu}$.

The situation is still worse in the case of gravitational backgrounds. The gravitational analogue of Schwinger's electromagnetic example would be the case of a constant gravitational field, *i.e.* a spacetime whose $R_{\mu\nu\lambda\rho}$'s are constants. It would certainly be a worthwhile effort to evaluate the effective lagrangian for such a background. Though, considerable amount work has been done in this direction in literature (see, for instance, references [15–18]), we are yet to have a covariant criterion for particle production by constant gravitational fields (analogous to the criterion $\mathcal{G} < 0$ Schwinger had obtained for the constant electromagnetic background). Also, since the gravitational interaction is not renormalizable, it is not easy at all to regularize the effective lagrangian (see, for *e.g.*, sections 6.11 and 6.12 of reference [2] in this context).

In this paper, we investigate a related but more restricted question. We ask: Can one find non-trivial background field configurations for which the (regularized) effective lagrangian vanishes identically? That is, we are interested in finding classical field configurations in which neither vacuum polarization nor particle production takes place. Such configurations certainly enjoy some special status because these are the ones for which lowest order semiclassical corrections vanish. What kind of classical field configurations will have this feature?

The effective lagrangian for the *constant* electromagnetic background reduces to zero when the gauge invariant quantities $\mathcal{F}$ and $\mathcal{G}$ are set to zero. Apart form this case, at least one more non-trivial electromagnetic field configuration is already known in literature for which the effective lagrangian proves to be zero. Schwinger, in his pioneering paper [1] also calculates the effective lagrangian for a plane electromagnetic wave background (for which gauge invariant quantities $\mathcal{F}$ and $\mathcal{G}$ are zero) and shows that it vanishes identically. These results suggest the following conjecture. *The effective lagrangian will be zero if all the scalar invariants describing the background vanish identically.* In this paper, we present examples of non-trivial electromagnetic and gravitational backgrounds with vanishing scalar invariants to support our conjecture. We evaluate the effective lagrangian explicitly using Schwinger's proper time formalism for the case of a quantized complex scalar field and show that it identically vanishes in these backgrounds.

This paper is organized as follows. In section II we present an example from electromagnetism and in section III we present an example from gravity. We explicitly evaluate the effective lagrangian and show that it vanishes identically in these backgrounds. Finally, in section IV, we discuss the wider implications of our analysis.

II. EFFECTIVE LAGRANGIAN FOR THE ELECTROMAGNETIC EXAMPLE

A. Preliminaries

The system we shall consider in this section consists of a complex scalar field Φ interacting with an electromagnetic field represented by the vector potential A^μ. It is described by the action

$$\mathcal{S}[A^\mu, \Phi] = \int d^4x\, \mathcal{L}(A^\mu, \Phi) = \int d^4x \Big\{ (\partial_\mu \Phi + iqA_\mu \Phi)\,(\partial^\mu \Phi^* - iqA^\mu \Phi^*) \\ - m^2 \Phi\Phi^* - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \Big\}, \tag{1}$$

where q and m are the charge and the mass associated with a single quantum of the complex scalar field, the asterisk denotes complex conjugation and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

The electromagnetic field is assumed to behave classically, hence A_μ is just a c-number while the complex scalar field is assumed to be a quantum field so that Φ is an operator valued distribution. In such a situation, we can obtain an effective lagrangian for the classical electromagnetic background by integrating out the degrees of freedom corresponding to the quantum field as follows

$$\exp\, i \int d^4x\, \mathcal{L}_{eff}(A^\mu) \equiv \int \mathcal{D}\Phi \int \mathcal{D}\Phi^*\, \exp\, i\, \mathcal{S}[A^\mu, \Phi], \tag{3}$$

where we have set $\hbar = c = 1$ for convenience. The effective lagrangian can then be expressed as $\mathcal{L}_{eff} = \mathcal{L}_{em} + \mathcal{L}_{corr}$, where $\mathcal{L}_{em}$ is the lagrangian density for the free electromagnetic field, *viz.* the third term under the integral in action (1), and $\mathcal{L}_{corr}$ is given by

$$\exp\, i \int d^4x\, \mathcal{L}_{corr}(A^\mu) = \int \mathcal{D}\Phi \int \mathcal{D}\Phi^* \exp\, i \int d^4x \Big\{ (\partial_\mu \Phi + iqA_\mu \Phi)\, (\partial^\mu \Phi^* - iqA^\mu \Phi^*) \\ - m^2 \Phi\Phi^* \Big\}. \tag{4}$$

Integrating the action for the scalar field in the above equation by parts and dropping the resulting surface terms, we obtain that [19]

$$\exp\, i \int d^4x\, \mathcal{L}_{corr}(A^\mu) = \int \mathcal{D}\Phi \int \mathcal{D}\Phi^*\, \exp -i \int d^4x\, \Phi^* \hat{D} \Phi = \left(\det \hat{D}\right)^{-1}, \tag{5}$$

where the operator $\hat{D}$ is given by

$$\hat{D} \equiv D_\mu D^\mu + m^2 \qquad \text{and} \qquad D_\mu \equiv \partial_\mu + iqA_\mu. \tag{6}$$

The determinant in equation (5) can be expressed as follows

$$\exp i \int d^4x\, \mathcal{L}_{corr} = \left(\det \hat{D}\right)^{-1} = \exp - \mathrm{Tr}(\ln \hat{D}) = \exp - \int d^4x\, \langle t, \mathbf{x}|\, \ln \hat{D}\, |t, \mathbf{x}\rangle \tag{7}$$

and in arriving at the last expression, following Schwinger [1], we have chosen a complete and orthonormal set of basis vectors $|t, \mathbf{x}\rangle$ to evaluate the trace of the operator $\ln \hat{D}$. ¿From the above equation it is easy to identify that

$$\mathcal{L}_{corr} = i\, \langle t, \mathbf{x}| \ln \hat{D} |t, \mathbf{x}\rangle. \tag{8}$$

Using the following integral representation for the operator $\ln \hat{D}$,

$$\ln \hat{D} \equiv - \int_0^\infty \frac{ds}{s}\, \exp -i(\hat{D} - i\epsilon)s \tag{9}$$

(where $\epsilon \to 0^+$), the expression for $\mathcal{L}_{corr}$ can be written as

$$\mathcal{L}_{corr} = -i \int_0^\infty \frac{ds}{s}\, e^{-i(m^2 - i\epsilon)s}\, K(t, \mathbf{x}, s|t, \mathbf{x}, 0), \tag{10}$$

where

$$K(t, \mathbf{x}, s\,|\,t, \mathbf{x}, 0) = \langle t, \mathbf{x}|\, e^{-i\hat{H}s}\, |t, \mathbf{x}\rangle \qquad \text{and} \qquad \hat{H} \equiv D_\mu D^\mu. \tag{11}$$

That is, $K(t, \mathbf{x}, s\,|\,t, \mathbf{x}, 0)$ is the kernel for a quantum mechanical particle in the coincidence limit (in four dimensions) described by the time evolution operator $\hat{H}$. The variable s that was introduced in (9) when the operator $\ln \hat{D}$ was expressed in an integral form, acts as the time parameter for the quantum mechanical system.

The integral representation for the operator $\ln \hat{D}$ we have used above is divergent in the lower limit of the integral, *i.e.* near $s = 0$. This divergence should be regularized by subtracting from it another divergent integral, *viz.* the integral representation of an operator $\ln \hat{D}_0$, where $\hat{D}_0 = (\partial^\mu \partial_\mu + m^2)$, the operator corresponding to that of a free quantum field. That is, to avoid the divergence, the integral representation for $\ln \hat{D}$ should actually be considered as

$$\ln \hat{D} - \ln \hat{D}_0 \equiv -\int_0^\infty \frac{ds}{s} \left\{ \exp -i(\hat{D} - i\epsilon)s \;-\; \exp -i(\hat{D}_0 - i\epsilon)s \right\}. \tag{12}$$

Or equivalently, the quantity $\mathcal{L}^0_{corr}$, which corresponds to the case of a free quantum field, can be subtracted from $\mathcal{L}_{corr}$ to obtain finite results. The quantum mechanical kernel $K(t, \mathbf{x}, s|t, \mathbf{x}, 0)$ corresponding to the operator $\hat{D}_0$ is the kernel for a free particle in four dimensions, *i.e.* $K(t, \mathbf{x}, s|t, \mathbf{x}, 0) = (1/16\pi^2 i s^2)$. Substituting this quantity in the expression for $\mathcal{L}_{corr}$ above, we obtain that

$$\mathcal{L}^0_{corr} = -\left(\frac{1}{16\pi^2}\right) \int_0^\infty \frac{ds}{s^3}\, e^{-i(m^2 - i\epsilon)s}. \tag{13}$$

This is the expression which has to be subtracted from $\mathcal{L}_{corr}$ to yield a finite result. (It turns out that such a simple regularization scheme works for the cases we consider in this paper. In general, it may be necessary to use more complicated regularization schemes.)

B. Evaluation of the effective lagrangian

Now, consider a time independent electromagnetic background described by the vector potential

$$A^\mu = (\phi(x,y), 0, 0, \phi(x,y)), \tag{14}$$

where $\phi(x,y)$ is an arbitrary function of the coordinates x and y. The resulting electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are then given by

$$\mathbf{E} = -\left(\frac{\partial\phi}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{y}}\right) \qquad ; \qquad \mathbf{B} = \left(\frac{\partial\phi}{\partial y}\,\hat{\mathbf{x}} - \frac{\partial\phi}{\partial x}\,\hat{\mathbf{y}}\right), \tag{15}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors along the positive x and y axes repectively. According to Maxwell's equations, in the absence of time dependence, the charge and the current densities, *viz.* ρ and $\mathbf{j}$ that give rise to the above field configuration are

$$\rho = \nabla.\mathbf{E} = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \qquad ; \qquad \mathbf{j} = \nabla\times\mathbf{B} = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\hat{\mathbf{z}}, \tag{16}$$

where $\hat{\mathbf{z}}$ is the unit vector along the positive z axis. Therefore, if the functions ρ and $\mathbf{j}$ are chosen such that they are finite and continuous everywhere and also vanish as $(x^2+y^2)\rightarrow\infty$, then the corresponding electric and magnetic fields given by equation (15) will be confined to a finite extent in the $x-y$ plane.

It is obvious from equation (15) that $\mathcal{G} = 2\,(\mathbf{B}^2 - \mathbf{E}^2) = 0$ and $\mathcal{F} = -8\,(\mathbf{E}.\mathbf{B}) = 0$ for this background field configuration. (As an aside, note that this is an example of a field configuration other than that of a wave, for which $\mathbf{E}^2 - \mathbf{B}^2$ as well as $\mathbf{E}.\mathbf{B}$ are zero.) It is therefore a good candidate to test our conjecture. The operator $\hat{H}$ that corresponds to the vector potential (14) is given by

$$\hat{H} \equiv \partial_t^2 - \nabla^2 + 2iq\phi(\partial_t + \partial_z). \tag{17}$$

The kernel for the quantum mechanical particle described by the hamiltonian operator above can then be formally written as

$$K(t,\mathbf{x},s\,|\,t,\mathbf{x},0) = \langle t,\mathbf{x}|\exp -i\left[\left(\partial_t{}^2 - \nabla^2 + 2iq\phi(\partial_t+\partial_z)\right)s\right]|t,\mathbf{x}\rangle. \tag{18}$$

Using the translational invariance of the hamiltonian operator $\hat{H}$ along the time coordinate t and the spatial coordinate z, we can express the above kernel as follows

$$K(t,\mathbf{x},s\,|\,t,\mathbf{x},0) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\; e^{i(\omega^2-p_z^2)s} \times \langle x,y|\,\exp -i\left[\left(-\partial_x^2-\partial_y^2+2q(\omega-p_z)\phi\right)s\right]|x,y\rangle.$$

Changing variables of integration in the expression above to $p_u = (p_z-\omega)/2$ and $p_v = (p_z+\omega)/2$, we find that

$$K(t,\mathbf{x},s\,|\,t,\mathbf{x},0) = \left(\frac{1}{2\pi^2}\right)\int_{-\infty}^{\infty}dp_u\int_{-\infty}^{\infty}dp_v\; e^{-4ip_up_vs} \times \langle x,y|\,\exp -i\left[\left(-\partial_x^2-\partial_y^2-4qp_u\phi\right)s\right]|x,y\rangle. \tag{19}$$

Performing the integrations over p_v and the p_u in that order, we obtain that

$$\begin{aligned} K(t,\mathbf{x},s\,|\,t,\mathbf{x},0) &= \left(\frac{1}{\pi}\right)\int_{-\infty}^{\infty}dp_u\;\delta(4p_us)\;\langle x,y|\,\exp -i\left[\left(-\partial_x^2-\partial_y^2-4qp_u\phi\right)s\right]|x,y\rangle \\ &= \left(\frac{1}{4\pi s}\right)\langle x,y|\,\exp -i\left[\left(-\partial_x^2-\partial_y^2\right)s\right]|x,y\rangle = \left(\frac{1}{16\pi^2 i s^2}\right). \end{aligned} \tag{20}$$

(In arriving at the above result we have carried out the p_v and the p_u integrals first and then evaluated the matrix element. We show in Appendix A that such an interchange of operations is valid by testing it in a specific example.) Substituting this expression for $K(t,\mathbf{x},s\,|\,t,\mathbf{x},0)$ in (10) we find that the resulting $\mathcal{L}_{corr}$ is the same as that of a free field. So, on regularization $\mathcal{L}_{corr}$ identically reduces to zero. This result then implies that in the time independent electromagnetic background we have considered here neither any particle production nor any vacuum polarization takes place.

As mentioned in the introduction, the effective lagrangian Schwinger had obtained for the constant electromagnetic background identically vanishes when the gauge invariant quantities $\mathcal{G}$ and $\mathcal{F}$ are set to zero [1]. Our result above agrees with Schwinger's result since a constant electromagnetic background would just correspond to choosing the function $\phi(x,y)$ above to be linear in the coordinates x and/or y. Having said that, we would like to stress here the following fact. *In evaluating the effective lagrangian above we have not made any assumptions at all on the form of the function $\phi(x,y)$.* Hence, our result above holds good for *any* time independent electromagnetic background with vanishing $\mathcal{G}$ and $\mathcal{F}$. Thus, in a way, our result here is more generic than Schwinger's result.

III. EFFECTIVE LAGRANGIAN FOR THE EXAMPLE FROM GRAVITY

A. Preliminaries

The system we shall consider in this section consists of a massive, real scalar field Φ coupled minimally to gravity. It is described by the action

$$\mathcal{S}[g_{\mu\nu},\Phi] = \int d^4x\,\sqrt{-g}\;\mathcal{L}(g_{\mu\nu},\Phi) = \int d^4x\,\sqrt{-g}\;\left\{\frac{R}{16\pi} + \frac{1}{2}\,g_{\mu\nu}\partial^\mu\Phi\partial^\nu\Phi - \frac{1}{2}\,m^2\Phi^2\right\}, \quad (21)$$

where m is the mass of a single quantum of the scalar field and $g_{\mu\nu}$ is the metric tensor describing the gravitational background and we have set $G=1$ for convenience. As it was done for the electromagnetic background in the last section, an effective lagrangian can be defined for the gravitational background as follows

$$\exp\, i\int d^4x\,\sqrt{-g}\;\mathcal{L}_{eff}(g_{\mu\nu}) \equiv \int \mathcal{D}\Phi\;\exp\, i\,\mathcal{S}[\Phi,g_{\mu\nu}]. \quad (22)$$

The effective lagrangian can then be expressed as $\mathcal{L}_{eff} = \mathcal{L}_{grav} + \mathcal{L}_{corr}$, where $\mathcal{L}_{grav} = (R/16\pi)$, the lagrangian density for the gravitational background. Integrating the action for the scalar field in the above equation by parts and dropping the resulting surface terms, we find that $\mathcal{L}_{corr}$ can then be expressed as [20]

$$\begin{aligned}\exp\, i\int d^4x\,\sqrt{-g}\;\mathcal{L}_{corr}(g_{\mu\nu}) &= \int \mathcal{D}\Phi\;\exp -i\int d^4x\,\sqrt{-g}\;\left(\Phi\hat{D}\Phi\right)\\ &= \left(\det\hat{D}\right)^{-1/2} = \exp -\frac{1}{2}\,\mathrm{Tr}(\ln\hat{D})\\ &= \exp -\frac{1}{2}\int d^4x\,\sqrt{-g}\;\langle t,\mathbf{x}|\,\ln\hat{D}\,|t,\mathbf{x}\rangle, \quad (23)\end{aligned}$$

where the operator $\hat{D}$ is given by

$$\hat{D} \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-g} \partial_\nu \right) + m^2. \tag{24}$$

and, as it was done in the last section, we have introduced a complete set of orthonormal vectors $|t, \mathbf{x}\rangle$, to evaluate the trace. ¿From equation (23) it is easy to identify that $\mathcal{L}_{corr} = (i/2) \langle t, \mathbf{x}| \ln \hat{D} |t, \mathbf{x}\rangle$. Using equation (9) $\mathcal{L}_{corr}$ above can then be written as

$$\mathcal{L}_{corr} = -\frac{i}{2} \int_0^\infty \frac{ds}{s} \, e^{-i(m^2 - i\epsilon)s} \, K(t, \mathbf{x}, s | t, \mathbf{x}, 0), \tag{25}$$

where

$$K(t, \mathbf{x}, s | t, \mathbf{x}, 0) = \langle t, \mathbf{x} | e^{-i\hat{H}s} | t, \mathbf{x} \rangle \tag{26}$$

and the operator $\hat{H}$ is now given by

$$\hat{H} \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-g} \partial_\nu \right). \tag{27}$$

To obtain finite results, the quantity that has to be subtracted from $\mathcal{L}_{corr}$, is then given by

$$\mathcal{L}^0_{corr} = -\left(\frac{1}{32\pi^2} \right) \int_0^\infty \frac{ds}{s^3} \, e^{-i(m^2 - i\epsilon)s}, \tag{28}$$

which corresponds to setting $g_{\mu\nu} = \eta_{\mu\nu}$ in the operator $\hat{H}$ above. ($\mathcal{L}^0_{corr}$ given by equation (13) is twice the $\mathcal{L}^0_{corr}$ above because the complex scalar field we had considered in the last section has twice the number of degrees of freedom as a real scalar field we are considering here.)

B. Evaluation of the effective lagrangian

A gravitational background can be described by fourteen independent scalar invariants constructed out of the Riemann curvature tensor [21]. To verify our conjecture, we should evaluate $\mathcal{L}_{corr}$ defined in equation (25) for a background for which all these invariants vanish. And, of course, we need a background which is sufficiently simple for allowing the evaluation of $\mathcal{L}_{corr}$ in a closed form.

One such example is given by the spacetime described by the line element

$$ds^2 = (1 + f(x, y))dt^2 - 2f(x, y)dtdz - (1 - f(x, y))dz^2 - dx^2 - dy^2, \tag{29}$$

where $f(x, y)$ is an arbitrary function of the coordinates x and y. (This metric is a special case of the metric that appears in [22]. It can be shown that all the fourteen algebraic invariants for this metric vanish identically [23].) The non-zero components of the Ricci tensor for the above metric are

$$R^{00} = R^{33} = R^{30} = \left(\frac{1}{2}\right)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \tag{30}$$

and the Ricci scalar R is zero. Since the Ricci scalar R is zero, the Einstein tensor is given by $G^{\mu\nu} = R^{\mu\nu}$ and the Einstein's equations reduce to $R^{\mu\nu} = 8\pi T^{\mu\nu}$. A pressureless steady flow of null dust with energy density $\rho = R^{00}$ traveling along the z-direction satisfies the above Einstein's equations and therefore gives rise to the metric (29). Since $det(g_{\mu\nu}) = -1$, the operator $\hat{H}$ corresponding to this metric is given by

$$\hat{H} = \partial_t^2 - \partial_z^2 - \partial_x^2 - \partial_y^2 - f(\partial_t^2 + \partial_z^2 + 2\partial_t\partial_z). \tag{31}$$

Using the translational invariance along the t and z directions the kernel for the time evolution operator above can be written as

$$\begin{aligned} K(t, \mathbf{x}, s|t, \mathbf{x}, 0) = & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi}\, e^{i(\omega^2 - p_z^2)s} \\ & \times \langle x, y| \exp -i\left[\left(-\partial_x^2 - \partial_y^2 + (\omega - p_z)^2 f\right) s\right] |x, y\rangle. \end{aligned} \tag{32}$$

Changing the variables of integration to $p_u = (p_z - \omega)/2$ and $p_v = (p_z + \omega)/2$, we obtain that

$$\begin{aligned} K(t, \mathbf{x}, s|t, \mathbf{x}, 0) &= \left(\frac{1}{2\pi^2}\right) \int_{-\infty}^{\infty} dp_u \int_{-\infty}^{\infty} dp_v\, e^{-4ip_u p_v s} \\ & \qquad \times \langle x, y| \exp -i\left[\left(-\partial_x^2 - \partial_y^2 + 4p_u^2 f\right) s\right] |x, y\rangle \\ &= \left(\frac{1}{\pi}\right) \int_{-\infty}^{\infty} dp_u\, \delta(4p_u s)\, \langle x, y| \exp -i\left[\left(-\partial_x^2 - \partial_y^2 + 4p_u^2 f\right) s\right] |x, y\rangle \\ &= \left(\frac{1}{4\pi s}\right) \langle x, y| \exp -i\left[\left(-\partial_x^2 - \partial_y^2\right) s\right] |x, y\rangle = \left(\frac{1}{16\pi^2 i s^2}\right). \end{aligned} \tag{33}$$

Substituting the above result in equation (25) we find that

$$\mathcal{L}_{corr} = -\left(\frac{1}{32\pi^2}\right) \int_0^\infty \frac{ds}{s}\, e^{-i(m^2-i\epsilon)s}. \tag{34}$$

which on subtracting the quantity $\mathcal{L}^0_{corr}$ given by equation (28) reduces to zero. This result again implies that in the gravitational background we have considered here neither any particle production nor any vacuum polarization takes place.

IV. DISCUSSION

The effective lagrangian provides a simple way of estimating the amount of vacuum polarization and particle production in a classical background. For example, the background field is expected to induce vacuum instability and produce particles if and only if the effective lagrangian has an imaginary part. If the effective lagrangian vanishes for a particular background field, then no vacuum polarization or particle production takes place in such a field configuration.

In principle, this is an observable phenomenon since physical effects occur if the effective lagrangian happens to be non-zero. For example, consider a constant electric field confined in space, say, the electric field between a pair of capacitor plates. In such a case, the imaginary part of effective lagrangian will be nonzero and the particle production will take place. These particles that have been produced will get attracted towards the capacitor plates thereby reducing the strength of the electric field between the plates. To maintain the original configuration intact, an external agency has to correct for this effect. We can therefore conclude that the above configuration—*viz.*, that of a constant electric field in a confined region—is not immune to quantum backreaction effects. Such, physically observable, effects do occur even if the effective lagrangian does not have an imaginary part. A typical example would be Casimir effect in flat spacetime. It can be shown that for such a case the effective lagrangian is non-zero and real; the real part, which depends on the separation between the plates, can be related to the Casimir energy. The resulting observable physical effect is the attraction between the Casimir plates. Left to themselves, the Casimir plates will move towards each other because of a force which is a quantum backreaction effect arising from the non-zero real part of effective lagrangian. Once again, to maintain the original configuration—*viz.*, the original separation between the plates—an external agency has to correct for the quantum backreaction effect.

In contrast to the above examples, backgrounds with vanishing effective lagrangian are 'self-consistent' in the sense that no backreaction of the quantum field on the classical background occurs in these configurations. This is a feature of certain backgrounds which, at least as far as the authors know, does not seem to have been noted in literature before. This aspect seems to be worthy of further study.

It should be possible to express the determinant of the operator $\hat{D}$ (and hence the quantity $\mathcal{L}_{corr}$) appearing in equations (5) and (23), at least formally, in terms of the invariant quantities describing the background. In particular, one would expect the effective lagrangian to contain only those terms that are simple algebraic functions of the scalar invariants (otherwise renormalization would not be possible). If so, the effective lagrangian would prove to be zero if all the invariants describing the background vanish identically. Motivated by this fact, we put forward the conjecture that the regularized $\mathcal{L}_{corr}$ will prove to be zero for background field configurations for which all scalar invariants are zero. In other words, our conjecture implies that integrating out the degrees of freedom corresponding to the quantum field does not introduce any quantum corrections to the lagrangian describing classical backgrounds with vanishing scalar invariants.

We had also tested our conjecture with some specific examples. For the electromagnetic background we have considered in section II we had pointed out that the gauge invariant quantities $\mathcal{G}$ and $\mathcal{F}$ are zero and it can be easily shown that quantities such as $\partial_\lambda F^{\mu\nu}\partial^\lambda F_{\mu\nu}$ and $\epsilon^{\lambda\rho\mu\nu}\partial_\eta F_{\lambda\rho}\partial^\eta F_{\mu\nu}$ also vanish identically. It is likely that all the gauge invariant quantities that can be constructed out of the vector potential (14) vanish identically. For the gravitational example considered in section III, as mentioned before, it can be shown that all the fourteen algebraic invariants that can be constructed out of the Riemann tensor for the metric (29) vanish identically [23]. Therefore, the vanishing of $\mathcal{L}_{corr}$ for these backgrounds is consistent with—and supports—our conjecture.

We would like to point out here the following fact. The classical backgrounds we have presented in sections II and III are quite non-trivial though all the scalar invariants may vanish. They are not just flat space presented in an arbitrary gauge or a coordinate system. The fact that a particle in these backgrounds will experience non-trivial forces acting on it ascertains this fact.

The examples that we had presented in sections II and III are time independent examples. As mentioned in the introduction, an example of a time dependent background for which the effective lagrangian proves to be zero is for that of a plane electromagnetic wave [1]. (In appendix B we rederive Schwinger's result using our technique.) For the electromagnetic wave too it can be easily shown that apart from the gauge invariant quantities $\mathcal{G}$ and $\mathcal{F}$, quantities such as $\partial_\lambda F^{\mu\nu}\partial^\lambda F_{\mu\nu}$ and $\epsilon^{\lambda\rho\mu\nu}\partial_\eta F_{\lambda\rho}\partial^\eta F_{\mu\nu}$ also vanish identically. It is, in fact, quite likely that all possible gauge invariant quantities vanish for the electromagnetic wave background thereby confirming our conjecture.

Ideally, one would have liked to evaluate the effective lagrangian for an arbitrary classical field configuration, vary the resulting effective lagrangian with respect to the classical fields and obtain the equations of motion for the classical background, thereby even taking into account the backreaction of the quantum field on the classical background. Since evaluating the effective lagrangian for an arbitrary classical background proves to be an impossible task, our approach to this entire problem has been a more practical one. The conjecture we have put forward in this paper is but the first step in this approach. There exist deeper reasons in proposing this conjecture (with the danger of sounding obvious) and attempting to establish its validity with some specific examples. These motivations are as follows. The effective lagrangian may indeed prove to be zero for classical backgrounds for which all the scalar invariants are zero, but the converse need not be true. That is, the effective lagrangian may prove to be zero even though some of the scalar invariants describing the background are non-zero. Backgrounds with vanishing effective lagrangians but non-vanishing scalar invariants can help us identify the terms that will appear in the effective lagrangian for the most general case. Classifying such backgrounds will certainly prove to be a worthwhile exercise when evaluating the effective lagrangian for an arbitrary background is proving to be an impossible task.

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APPENDIX A:

In this appendix we illustrate the validity of carrying out the p_v and the p_u integrals first and then evaluating the matrix element in equation (20) by testing it in a simple example.

Consider the case when $\phi(x,y) = x$. This corresponds to a constant electromagnetic background with the electric and magnetic fields given by $\mathbf{E} = -\hat{\mathbf{x}}$ and $\mathbf{B} = -\hat{\mathbf{y}}$. For this case, the operator $\hat{H}$ is given by

$$\hat{H} = \partial_t^2 - \nabla^2 + 2iqx(\partial_t + \partial_z). \tag{A1}$$

The translational invariance of the above operator along the t, y and z directions can then be exploited to express the quantum mechanical kernel for the above operator as follows

$$K(t,\mathbf{x},s\,|\,t,\mathbf{x},0) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\int_{-\infty}^{\infty}\frac{dp_y}{2\pi}\, e^{i(\omega^2-p_y^2-p_z^2)s} \times \langle x|\, \exp -i\left[\left(-d_x^2 + 2q(\omega - p_z)x\right) s\right]\, |x\rangle. \tag{A2}$$

Carrying out the p_y intergration and changing variables to $p_u = (p_z - \omega)/2$ and $p_v = (p_z + \omega)/2$, we obtain that

$$K(t,\mathbf{x},s\,|\,t,\mathbf{x},0) = \left(\frac{1}{2\pi^2(4\pi i s)^{1/2}}\right)\int_{-\infty}^{\infty}dp_v\int_{-\infty}^{\infty}dp_u\, e^{-4ip_up_vs} \times \langle x|\, \exp -i\left[\left(-d_x^2 - 4qp_ux\right) s\right]\, |x\rangle. \tag{A3}$$

The matrix element in the above equation corresponds to that of a quantum mechanical particle subjected to a constant force along the x-axis. The matrix element above is then given by (see [24])

$$\langle x|\ \exp -i\left[\left(-d_x^2 - 4qp_u x\right) s\right]\ |x\rangle = \left(\frac{1}{(4\pi i s)^{1/2}}\right)\ \exp -4i\left(qp_u xs + \frac{1}{3}q^2 p_u^2 s^3\right). \tag{A4}$$

Substituting this expression in the kernel (A3), we obtain that

$$\begin{aligned} K(t,\mathbf{x},s\,|\,t,\mathbf{x},0) &= \left(\frac{1}{8\pi^3 is}\right)\int_{-\infty}^{\infty} dp_v \int_{-\infty}^{\infty} dp_u\ \exp -4i\left(\frac{1}{3}q^2p_u^2s^3 + p_u s(qx+p_v)\right) \\ &= \left(\frac{1}{8\pi^3 is}\right)\left(\frac{3\pi}{4iq^2s^3}\right)^{1/2}\int_{-\infty}^{\infty} dp_v\ \exp\left(\frac{3i}{q^2s}(p_v+qx)^2\right) \\ &= \left(\frac{1}{16\pi^2 is^2}\right), \end{aligned} \tag{A5}$$

which is the result quoted in the text.

APPENDIX B:

In this appendix we rederive Schwinger's result for the electromagnetic wave background using our technique. The plane electromagnetic wave can be described by the vector potential

$$A_\mu = (0,1,0,0)f(t-z) \tag{B1}$$

where $f(t-z)$ is an arbitrary function of $(t-z)$. The operator $\hat{H}$ corresponding to this vector potential is then given by

$$\hat{H} = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 + 2iqf\partial_x + q^2f^2 \tag{B2}$$

and in terms of the null coordinates $u=(t-z)$ and $v=(t+z)$ the above operator reduces to

$$\hat{H} = 4\partial_u\partial_v - \partial_x^2 - \partial_y^2 + 2iqf(u)\partial_x + q^2f^2(u). \tag{B3}$$

The corresponding quantum mechanical kernel can then be formally expressed as

$$K(u,x,y,v,s|u,x,y,v,0)$$
$$= \langle u,x,y,v|\exp -i\left[\left(4\partial_u\partial_v - \partial_x^2 - \partial_y^2 + 2iqf\partial_x + q^2f^2\right)s\right]|u,x,y,v\rangle. \quad \text{(B4)}$$

Exploiting the translational invariance of the operator $\hat{H}$ along the x, y and the v coordinates we can write the above kernel as

$$K(u,x,y,v,s|u,x,y,v,0) = \int_{-\infty}^{\infty}\frac{dp_x}{2\pi}\int_{-\infty}^{\infty}\frac{dp_y}{2\pi}\int_{-\infty}^{\infty}\frac{dp_v}{2\pi}\ e^{-i(p_x^2+p_y^2)s}$$
$$\times\ 2\ \langle u|\exp -i\left[\left(-4ip_vd_u - 2qp_xf + q^2f^2\right)s\right]|u\rangle, \quad \text{(B5)}$$

where the factor 2 is the Jacobian of the transformation between the conjugate momenta (ω, p_x) and (p_u, p_v) corresponding to the coordinates (t, z) and (u, v) respectively.

The matrix element in the above equation corresponds to the quantum mechanical kernel for a time evolution operator given by

$$\hat{H}_1 = -4ip_vd_u - 2qp_xf(u) + q^2f^2(u). \quad \text{(B6)}$$

The normalized solution $\psi_E(u)$ to the time independent Schrödinger equation for the operator $\hat{H}_1$ corresponding to an energy eigen value E is then given by

$$\psi_E(u) = \left(\frac{1}{8\pi p_v}\right)^{1/2}\ e^{iqE/4p_v}\exp -i\left(h(u)/4p_v\right), \quad \text{(B7)}$$

where

$$h(u) = -\int du\left(2qp_xf(u) - q^2f^2(u)\right). \quad \text{(B8)}$$

The matrix element can now be evaluated with the help of the Feynman-Kac formula [25] as follows

$$\langle u|e^{-i\hat{H}_1s}|u'\rangle = \int_{-\infty}^{\infty}dE\ \psi_E(u)\ \psi_E^*(u')\ e^{-iEs}$$
$$= \left(\frac{1}{8\pi p_v}\right)\ \exp -i\Big\{(h(u)-h(u'))/4p_v\Big\}\ \int_{-\infty}^{\infty}dE\ \ \exp i\left(E(u-u')/4p_v\right)\ e^{-iEs}$$
$$= \exp -i\Big\{(h(u)-h(u'))/4p_v\Big\}\ \delta\left(u-u'-4p_vs\right) \quad \text{(B9)}$$

and in the coincidence limit $u = u'$, the matrix element reduces to a Dirac delta function *i.e.*

$$\langle u| \exp -i\hat{H}_1 s|u\rangle = \delta(4p_v s). \tag{B10}$$

Substituting this result in equation (B5), we obtain that

$$\begin{aligned} K(u,x,y,v,s|u,x,y,v,0) &= \left(\frac{2}{(4\pi i s)^{1/2}}\right) \int_{-\infty}^{\infty} \frac{dp_x}{2\pi}\, e^{-ip_x^2 s} \int_{-\infty}^{\infty} \frac{dp_v}{2\pi}\, \delta(4p_v s) \\ &= \left(\frac{2}{4\pi i s}\right) \int_{-\infty}^{\infty} \frac{dp_v}{2\pi} \left(\frac{1}{4s}\right) \delta(p_v) = \left(\frac{1}{16\pi^2 i s^2}\right). \end{aligned} \tag{B11}$$

When the above kernel is substituted in equation (10) we find that the resulting $\mathcal{L}_{corr}$ is the same as that of $\mathcal{L}^0_{corr}$, which on regularization reduces identically to zero.

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