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Nonnegative Moore–Penrose inverses of Gram operators

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Abstract

This paper is concerned with necessary and sufficient conditions for the nonnegativity of Moore–Penrose inverses of Gram operators between real Hilbert spaces. These conditions include statements on acuteness (or obtuseness) of certain closed convex cones. The main result generalizes a well known result for inverses in the finite dimensional case over the nonnegative orthant to Moore–Penrose inverses in (possibly) infinite dimensional Hilbert spaces over any general closed convex cone.

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Keywords: Gram operator; Moore–Penrose inverse; Obtuse cones

1. Introduction

A square real matrix A is called monotone if $Ax \geqslant 0 \Rightarrow x \geqslant 0$. Here $x = (x_i) \geqslant 0$ means that $x_i \geqslant 0$ for all i. Collatz (see for example, [5]) has shown that a matrix is monotone iff it is invertible and the inverse is nonnegative. Mangasarian [13] studied rectangular real matrices while Berman and Plemmons [2,3] presented a plethora of generalizations. Extensions of some of these results to spaces, not necessarily finite dimensional, were considered by Kulkarni and Sivakumar [12,15,16]. Gil gave sufficient conditions on the entries of a matrix A in order for A^{-1} to be nonnegative (refer to [6,8] for finite matrices and [7] for infinite matrices). The book by Berman and Plemmons [3] has numerous examples of applications of nonnegative generalized inverses that include numerical analysis and linear economic models.

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Monotonicity of Gram matrices has received a lot of attention in recent years. This has been primarily motivated by applications in convex optimization problems. In this connection, there is a well known result that characterizes nonnegative invertibility of Gram matrices in terms of obtuseness (or acuteness) of certain polyhedral cones. (See for instance Lemma 1.6 in [4,9] and Section 3 in [11].) The sole objective of this paper is to generalize the characterization of nonnegativity of inverses of Gram matrices in two directions; from finite dimensional real Euclidean spaces to (possibly) infinite dimensional real Hilbert spaces and from classical inverses to Moore–Penrose inverses. It will be interesting to consider applications of our results to convex optimization, similar to [4]. This is postponed for a future study. The paper is organized as follows. In Section 2, we introduce some basic notations, definitions and results. In Section 3, we present some preliminary results and prove the main theorem. We conclude with some observations.

2. Notations, definitions and preliminaries

We first introduce notations and definitions that will be used in the rest of the paper.

$\mathbb{R}^n$, $\mathbb{R}^n_+$ denote the n dimensional real Euclidean space, non-negative orthant respectively. H, H_1, H_2 denote Hilbert spaces over $\mathbb{R}$. $BL(H_1, H_2)$ denotes the set of all bounded linear operators from H_1 into H_2. When $H = H_1 = H_2$, $BL(H_1, H_2)$ will be denoted by $BL(H)$. $P_{R(A^*)}$ denotes the projection of H onto $R(A^*)$, where $R(X)$ denotes the range space of the operator X. $\langle x, y\rangle$ denotes the inner product of x and y.

For $A \in BL(H_1, H_2)$, A^* denotes the adjoint of A. For a subset K of a Hilbert space H, the polar of K denoted K° is defined as $K^\circ = \{x \in H_1: \langle x, t\rangle \leqslant 0, \forall t \in K\}$. $K^{\circ\circ}$ denotes $(K^\circ)^\circ$. Note that in general, $K^{\circ\circ} = \mathrm{cl}K$, where $\mathrm{cl}K$ denotes the closure of K. If $H = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$ then $K^\circ = -\mathbb{R}^n_+$ and so $K^{\circ\circ} = K$. If $K = \mathbb{R}^n_+ \cap R(B^*)$ for some $m \times n$ real matrix B, then $K^\circ = -\mathbb{R}^n_+ + N(B)$, where $N(B)$ denotes the null space of the matrix B. Again $K^{\circ\circ} = K$. If $H = \ell^2$, the Hilbert space of all square summable real sequences and $K = \ell^2_+ = \{x \in \ell^2: x_i \geqslant 0, \forall i\}$, then $K^\circ = -\ell^2_+$ and hence $K^{\circ\circ} = \ell^2_+$.

A cone C is said to be acute if $\langle x, y\rangle \geqslant 0$, for all $x, y \in C$. C is said to obtuse if $C^\circ \cap \{cl\ span\ C\}$ is acute, where *span* C denotes the linear subspace spanned by C. In particular, if $A \in BL(H_1, H_2)$, $K \subseteq H_1$, a closed convex cone with $C = AK$, then the obtuseness of C is equivalent to the acuteness of $C^\circ \cap R(A)$. The notion of obtuseness of a cone in $\mathbb{R}^n$ was first proposed by Goffin [9].

For a linear map $A : H_1 \longrightarrow H_2$, the operator A^*A is said to be the Gram operator of A. Let A be bounded with closed range. Then the Moore–Penrose inverse of A is the unique operator $A^\dagger$ in $BL(H_2, H_1)$ which satisfies the following equations:

$$AA^\dagger A = A, \tag{1}$$

$$A^\dagger AA^\dagger = A^\dagger, \tag{2}$$

$$(AA^\dagger)^* = AA^\dagger, \tag{3}$$

$$(A^\dagger A)^* = A^\dagger A. \tag{4}$$

The following properties of $A^\dagger$ are well known [1,10]: $R(A^*) = R(A^\dagger)$; $N(A^*) = N(A^\dagger)$; $AA^\dagger = P_{R(A)}$; $A^\dagger A = P_{R(A^*)}$. In particular, if $x \in R(A^*)$ then $x = A^\dagger Ax$. This will be used frequently in our proofs.

Let $A \in BL(H)$ with $R(A)$ closed. The group inverse of A is the unique operator $A^\# \in BL(H)$ which satisfies the following equations:

$$AA^{\#}A = A, \tag{5}$$
$$A^{\#}AA^{\#} = A^{\#}, \tag{6}$$
$$AA^{\#} = A^{\#}A. \tag{7}$$

It is a well known result in finite dimensional spaces that $A^{\#}$ exists if and only if $R(A) = R(A^2)$. Equivalently, $A^{\#}$ exists if and only if $N(A) = N(A^2)$. Another equivalent condition is that $R(A)$ and $N(A)$ are complementary subspaces. This, in particular means that every hermitian matrix has group inverse. In infinite dimensional spaces $A^{\#}$ exists if and only if $R(A) = R(A^2)$ and $N(A) = N(A^2)$ [14].

The following result is fundamental in studying linear equations. Its proof is easy.

Lemma 2.1. *Let $A \in BL(H_1, H_2)$ and $b \in H_2$. Then the linear equation $Ax = b$ has a solution iff $b \in R(A)$. In this case the general solution is given by $x = Tb + z$ for some T satisfying $ATA = A$ and for arbitrary $z \in N(A)$.*

3. Main results

For proving the main theorem (Theorem 3.6) we consider the following results. Let H_1 and H_2 be real Hilbert spaces, $A \in BL(H_1, H_2)$ be with closed range, K be a closed convex cone in H_1, $C = AK$ and $D = (A^{\dagger})^* K^{\circ}$.

Lemma 3.1 (Theorem 2.1.5, [10]). *Let $A \in BL(H_1, H_2)$. Then*

$$A^{\dagger} = (A^*A)^{\dagger}A^* = A^*(AA^*)^{\dagger}.$$

Remark 3.2. From the first equation, it follows that $A^{\dagger}(A^{\dagger})^* = (A^*A)^{\dagger}$.

Lemma 3.3. $u \in C^{\circ} \Longrightarrow A^*u \in K^{\circ}$.

Proof. Let $u \in C^{\circ}$ and $r \in K$. Then $0 \geqslant \langle u, Ar\rangle = \langle A^*u, r\rangle$. □

Lemma 3.4. *The following are equivalent*:

(a) $C^{\circ} \cap R(A)$ *is acute*.
(b) *For all x, y with $A^*Ax \in K^{\circ}$, $A^*Ay \in K^{\circ}$, the inequality $\langle A^*Ax, y\rangle \geqslant 0$ holds*.

Proof. (a) ⇒ (b) Let x, y satisfy $A^*Ax \in K^{\circ}$ and $A^*Ay \in K^{\circ}$. For $r \in K$, we have

$$\langle Ax, Ar\rangle = \langle A^*Ax, r\rangle \leqslant 0.$$

So, $Ax \in C^{\circ}$. Similarly $Ay \in C^{\circ}$. Since $C^{\circ} \cap R(A)$ is acute, we have

$$0 \leqslant \langle Ax, Ay\rangle = \langle A^*Ax, y\rangle.$$

(b) $\Longrightarrow$ (a) Let $u, v \in C^{\circ} \cap R(A)$; $u = Ax$, $v = Ay$, $x, y \in H_1$. Since $u \in C^{\circ}$, for $r \in K$ we have

$$0 \geqslant \langle Ax, Ar\rangle = \langle A^*Ax, r\rangle.$$

Thus $A^*Ax \in K^\circ$. Similarly $A^*Ay \in K^\circ$. By assumption,

$$0 \leqslant \langle A^*Ax, y\rangle = \langle Ax, Ay\rangle = \langle u, v\rangle. \qquad \Box$$

Lemma 3.5. *D is acute iff $\langle r, (A^*A)^\dagger s\rangle \geqslant 0$, for every $r, s \in K^\circ$.*

Proof. Let $x, y \in D$; $x = (A^\dagger)^* r$, $y = (A^\dagger)^* s$ with $r, s \in K^\circ$. Then D is acute iff $0 \leqslant \langle x, y\rangle = \langle (A^\dagger)^* r, (A^\dagger)^* s\rangle = \langle r, A^\dagger (A^\dagger)^* s\rangle = \langle r, (A^*A)^\dagger s\rangle$. $\Box$

We are now in a position to prove the main result of this paper.

Theorem 3.6. *Let $A \in BL(H_1, H_2)$ with $R(A)$ closed, K be a closed convex cone of H_1 with $A^\dagger AK \subseteq K$. Let $C = AK$ and $D = (A^\dagger)^* K^\circ$. Then the following conditions are equivalent*:

(a) $(A^*A)^\dagger(-K^\circ) \subseteq K$.
(b) $C^\circ \cap R(A) \subseteq -C$.
(c) *D is acute.*
(d) *C is obtuse.*
(e) $A^*Ax \in P_{R(A^*)}(-K^\circ)$, $x \in R(A^*) \Longrightarrow x \in K$.
(f) $A^*Ax \in -K^\circ$, $x \in R(A^*) \Longrightarrow x \in K$.

Proof. (a) $\Longrightarrow$ (b) Let $u \in C^\circ \cap R(A)$; $u = Ap$, $p \in H_1$. Then by Lemmas 2.1 and 3.1,

$$p = A^\dagger u + w = (A^*A)^\dagger A^* u + w, \quad w \in N(A).$$

Set $z = (A^*A)^\dagger A^* u$. Then $u = Ap = Az$. Also $A^*u \in K^\circ$, by Lemma 3.3, so that $A^*(-u) \in -K^\circ$. So by assumption, $-z = (A^*A)^\dagger A^*(-u) \in K$. Thus $u \in -C$.

(b) $\Longrightarrow$ (c) Let $x = (A^\dagger)^* u$ and $y = (A^\dagger)^* v$ with $u, v \in K^\circ$. Then $x, y \in R((A^\dagger)^*) = R((A^*)^\dagger) = R((A^*)^*) = R(A)$. Let $r \in K$. We have $r' = A^\dagger Ar \in K$ (as $A^\dagger AK \subseteq K$). Then $\langle x, Ar\rangle = \langle (A^\dagger)^* u, Ar\rangle = \langle u, A^\dagger Ar\rangle = \langle u, r'\rangle \leqslant 0$. Thus $x \in C^\circ$. Since $C^\circ \cap R(A) \subseteq -C$, we have $x \in -C$. Thus $x = A(-p)$, $p \in K$. Finally, with $p' = A^\dagger Ap \in K$, we have $\langle x, y\rangle = \langle A(-p), (A^\dagger)^* v\rangle = -\langle A^\dagger Ap, v\rangle = -\langle p', v\rangle \geqslant 0$. Hence D is acute.

(c) $\Longrightarrow$ (d) Let x, y be such that $r = A^*Ax \in K^\circ$ and $s = A^*Ay \in K^\circ$. Since D is acute, by Lemma 3.5, $0 \leqslant \langle r, (A^*A)^\dagger s\rangle = \langle A^*Ax, (A^*A)^\dagger A^*Ay\rangle = \langle x, (A^*A)(A^*A)^\dagger (A^*A)y\rangle = \langle x, (A^*A)y\rangle = \langle A^*Ax, y\rangle$. By Lemma 3.3, $C^\circ \cap R(A)$ is acute.

(d) $\Longrightarrow$ (e) Let $A^*Ax = P_{R(A^*)} w = A^\dagger Aw$ for some $w \in -K^\circ$. Then by Lemma 2.1, for some $h \in N(A^*) = N(A^\dagger)$, $Ax = (A^\dagger)^* A^\dagger Aw + h = (A^\dagger)^*(A^\dagger A)^* w + h = (A^\dagger AA^\dagger)^* w + h = (A^\dagger)^* w + h$. Thus $A^\dagger Ax = A^\dagger (A^\dagger)^* w$. If $x \in R(A^*)$, then $x = A^\dagger Ax$. Let $r \in K^\circ$. Then $\langle x, r\rangle = \langle A^\dagger Ax, r\rangle = \langle A^\dagger (A^\dagger)^* w, r\rangle = \langle (A^\dagger)^* w, (A^\dagger)^* r\rangle$. Set $u = (A^\dagger)^* w$, $v = (A^\dagger)^* r$. Then, as was shown earlier, $u, v \in R(A)$. For $t \in K$, with $t' = A^\dagger At \in K$, we have $\langle -u, At\rangle = \langle (A^\dagger)^*(-w), At\rangle = \langle -w, A^\dagger At\rangle = \langle -w, t'\rangle \leqslant 0$. So $-u \in C^\circ$. Along similar lines it can be shown that $v \in C^\circ$. Thus for all $r \in K^\circ$, $\langle x, r\rangle = \langle u, v\rangle \leqslant 0$. So $x \in (K^\circ)^\circ = K$.

(e) $\Longrightarrow$ (f) Suppose that $A^*Ax \in -K^\circ$. Since $A^*Ax \in R(A^*) = R(A^\dagger A)$, we have $A^*Ax = P_{R(A^*)}(A^*Ax) \in P_{R(A^*)}(-K^\circ)$.

(f) $\Longrightarrow$ (a) Let $u = (A^*A)^\dagger v$ with $v \in -K^\circ$. Then $u \in R((A^*A)^\dagger) = R((A^*A)^*) = R(A^*A) = R(A^*)$. Also, $A^*Au = A^*A(A^*A)^\dagger v = P_{R(A^*)} v = A^\dagger Av$. Then for $r \in K$ with $r' = A^\dagger Ar \in K$, we have $\langle A^*A(-u), r\rangle = \langle A^\dagger A(-v), r\rangle = \langle -v, A^\dagger Ar\rangle = -\langle v, r'\rangle \leqslant 0$. Thus $A^*A(-u) \in K^\circ$. As (f) holds, $u \in K$. Thus $(A^*A)^\dagger(-K^\circ) \subseteq K$.

This completes the proof of the theorem. $\Box$

Corollary 3.7. *Let* $H_1 = \mathbb{R}^n$, $H_2 = \mathbb{R}^m$ *and* $K = \mathbb{R}^n_+ \cap R(A^*)$. *Then the conditions* (b)–(f) *are equivalent to* $(A^*A)^\dagger(\mathbb{R}^n_+) \subseteq \mathbb{R}^n_+$, *that is entrywise non-negativity of* $(A^*A)^\dagger$.

Proof. Clearly, $A^\dagger AK \subseteq K$, $K^\circ = -\mathbb{R}^n_+ + N(A)$ and $K^{\circ\circ} = K$. By Theorem 3.6, (b)–(f) are equivalent to (a): $(A^*A)^\dagger(\mathbb{R}^n_+ + N(A)) \subseteq \mathbb{R}^n_+ \cap R(A^*)$. The conclusion now follows, since $N((A^*A)^\dagger) = N(A)$ and $R((A^*A)^\dagger) = R(A^*)$. □

The next corollary includes Cegielski's result (Lemma 1.6, [4]) as a particular case. Note that if A is of full column rank, then $A^\dagger A = I$, so that $A^\dagger AK \subseteq K$ holds trivially.

Corollary 3.8. *In addition to the conditions of Theorem* 3.6, *suppose that* A *is of full column rank. Then the conditions* (b)–(f) *are equivalent to* $(A^*A)^{-1} \geqslant 0$.

Remark 3.9

(i) The following example illustrates Theorem 3.6. Let $H_1 = H_2 = \ell^2$, and $K = \ell^2_+$. Then $K^\circ = -\ell^2_+$. Let A be the left-shift operator on ℓ^2 defined by $A(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Then $A^\dagger = A^* = B$, the right-shift operator on ℓ^2 defined by $B(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Here $C = A\ell^2_+ = \ell^2_+$ and $D = (A^\dagger)^*(\ell^2_+)^\circ = -\ell^2_+$. We have $A^\dagger A \geqslant 0$ and $(A^*A)^\dagger = A^*A \geqslant 0$.

(ii) Since A^*A is Hermitian, it follows that $(A^*A)^\dagger = (A^*A)^\#$.

(iii) If K is such that $K \subseteq R(A^*)$ then it follows that $A^\dagger AK \subseteq K$; but the latter condition is more general. This can be seen as follows: Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $A^\dagger = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. We have $A^\dagger A \geqslant 0$, but $\mathbb{R}^2_+ \nsubseteq R(A^*)$.

(iv) Conditions (e) and (f) with A^*A replaced by A and with $K = \mathbb{R}^n_+$ were shown to be equivalent to each other and also equivalent to the nonnegativity of $A^\dagger$, by Berman and Plemmons [2]. A generalization of this to real Hilbert spaces was obtained in [12]. In Theorem 3.6, we have more general equivalences.

(v) The condition $A^\dagger A \geqslant 0$ does not imply $(A^*A)^\dagger \geqslant 0$. This is shown as follows: Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, so that $A^\dagger A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Also, $(A^*A)^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. Thus $A^\dagger A \geqslant 0$ whereas $(A^*A)^\dagger \ngeqslant 0$.

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References

[1] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Springer Verlag, New York, 2003.
[2] A. Berman, R.J. Plemmons, Monotonocity and the generalized inverse, SIAM J. Appl. Math. 22 (1972) 155–161.
[3] A. Berman, R.J. Plemmons, Nonnegative matrices in the mathematical sciences, Classics in Applied Mathematics, SIAM, 1994.
[4] A. Cegielski, Obtuse cones and Gram matrices with non-negative inverse, Linear Algebra Appl. 335 (2001) 167–181.

[5] L. Collatz, Functional Analysis and Numerical Mathematics, Academic, New York, 1966.
[6] M.I. Gil, On positive invertibility of matrices, Positivity 2 (1998) 165–170.
[7] M.I. Gil, Stability of Finite and Infinite Dimensional systems, Kluwer Academic, 1998.
[8] M.I. Gil, On invertibility and positive invertibility of matrices, Linear Algebra Appl. 327 (2001) 95–104.
[9] J.L. Goffin, The relaxation method for solving systems of linear inequalities, Math. Oper. Res. 5 (3) (1980) 388–414.
[10] C.W. Groetsch, Generalized Inverses of Linear Operators: Representation and Approximation, Mercel Dekker, 1974.
[11] K.C. Kiwiel, Monotone gram matrices and deepest surrogate inequalities in accelerated relaxation methods for convex feasibility problems, Linear Algebra Appl. 252 (1997) 27–33.
[12] S.H. Kulkarni, K.C. Sivakumar, Three types of operator monotonicity, J. Anal. 12 (2004) 153–163.
[13] O.L. Mangasarian, Characterizations of real matrices of monotone kind, SIAM. Rev. 10 (1968) 439–441.
[14] P. Robert, On the group-inverse of a linear transformation, J. Math. Anal. Appl. 22 (1968) 658–669.
[15] K.C. Sivakumar, Nonnegative generalized inverses, Indian J. Pure Appl. Math. 28 (7) (1997) 939–942.
[16] K.C. Sivakumar, Range and group monotonicity of operators, Indian J. Pure Appl. Math. 32 (1) (2001) 85–89.