

Neumann eigenvalue problems on the exterior domains

T. V. Anoop*, Nirjan Biswas

Abstract

For $p \in (1, \infty)$, we consider the following weighted Neumann eigenvalue problem on B_1^c , the exterior of the closed unit ball in \mathbb{R}^N :

$$\begin{aligned} -\Delta_p \phi &= \lambda g |\phi|^{p-2} \phi \text{ in } B_1^c, \\ \frac{\partial \phi}{\partial \nu} &= 0 \text{ on } \partial B_1, \end{aligned} \tag{0.1}$$

where Δ_p is the p -Laplace operator and $g \in L^1_{loc}(B_1^c)$ is an indefinite weight function. Depending on the values of p and the dimension N , we take g in certain Lorentz spaces or weighted Lebesgue spaces and show that (0.1) admits an unbounded sequence of positive eigenvalues that includes a unique principal eigenvalue. For this purpose, we establish the compact embeddings of $W^{1,p}(B_1^c)$ into $L^p(B_1^c, |g|)$ for g in certain weighted Lebesgue spaces. For $N > p$, we also provide an alternate proof for the embedding of $W^{1,p}(B_1^c)$ into $L^{p^*}(B_1^c)$. Further, we show that the set of all eigenvalues of (0.1) is closed.

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1 Introduction

Let Ω be a smooth domain in \mathbb{R}^N and $g \in L^1_{loc}(\Omega)$. For $p \in (1, \infty)$, we consider the following nonlinear weighted eigenvalue problem:

$$\begin{aligned} -\Delta_p \phi &= \lambda g |\phi|^{p-2} \phi \text{ in } \Omega, \\ \frac{\partial \phi}{\partial \nu} &= 0 \text{ on } \partial \Omega, \end{aligned} \tag{1.1}$$

where Δ_p is the p -Laplace operator defined as $\Delta_p(\phi) = \operatorname{div}(|\nabla \phi|^{p-2} \nabla \phi)$. We say a real number λ is an eigenvalue of (1.1), if there exists $\phi \in W^{1,p}(\Omega) \setminus \{0\}$ satisfying the following:

$$\int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla v = \lambda \int_{\Omega} g |\phi|^{p-2} \phi v, \quad \forall v \in W^{1,p}(\Omega). \tag{1.2}$$

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In this case, we also say ϕ is an eigenfunction of (1.1) corresponding to λ . An eigenvalue λ is called a *principal eigenvalue*, if there exists an eigenfunction corresponding to λ that does not change sign on Ω .

If Ω is bounded, then zero is always a principal eigenvalue of (1.1) (nonzero constants as corresponding eigenfunctions). If $\int_{\Omega} g(x)dx \geq 0$, then zero is the only nonnegative principal eigenvalue. Thus when Ω is bounded, for the existence of a positive principal eigenvalue of (1.1), $\int_{\Omega} g(x)dx < 0$ is necessary. This condition alone does not ensure the existence of a positive principal eigenvalue for (1.1). Under the additional assumptions such as $g \in L^{\infty}(\Omega)$ ([10]), $g \in C(\bar{\Omega})$ ([16]), or $g \in L^d(\Omega)$ with $d > \frac{N}{p}$ ([14]), the eigenvalue problem (1.1) does admit a principal eigenvalue and it is unique.

If $\Omega = \mathbb{R}^N$, then (1.2) corresponds to the weak formulation of the Dirichlet eigenvalue problem. In this context, for $N > p$, (1.1) admits a positive principal eigenvalue even for certain g with $\int_{\Omega} g(x)dx \geq 0$. For example, smooth g with g^- is bounded away from zero at infinity ([9, 17]), g with $g^+ \in L^{\frac{N}{2}}(\mathbb{R}^N)$ ([2, 3]). Further, if the eigenfunctions are allowed to be in Beppo-Levi space $\mathcal{D}_0^{1,p}(\mathbb{R}^N) :=$ completion of $C_c^{\infty}(\mathbb{R}^N)$ with respect to the $\|\nabla \cdot\|_p$, then (1.1) admits a positive principal eigenvalue for weights in bigger classes of function spaces, see [5, 7] and the references therein. For $N = p$, if $\int_{\mathbb{R}^N} g(x)dx > 0$, the non-existence of positive principal eigenvalue for (1.1) is proved in [9] for $p = 2$ and in [17] for general p .

In this article, we study the existence of a positive principal eigenvalue of (1.1) on $\Omega = B_1^c$. The Dirichlet eigenvalue problem for p -Laplacian on the exterior domain is considered in [6]. We enlarge the class of weight functions that admits a positive principal eigenvalue by providing two distinct categories of function spaces. The first category contains certain closed subspace of the Lorentz space $L^{\frac{N}{p},\infty}(B_1^c)$ for $N > p$ and the second one contains certain weighted Lebesgue spaces for all choices of p and N .

We consider the following closed subspace (introduced in [7]) of the Lorentz space $L^{\frac{N}{p},\infty}(B_1^c)$:

$$\mathcal{F}_{\frac{N}{p}} := \text{closure of } C_c^{\infty}(B_1^c) \text{ in } L^{\frac{N}{p},\infty}(B_1^c).$$

For details of the space $\mathcal{F}_{\frac{N}{p}}$, we refer to [7].

Theorem 1.1. *Let $p \in (1, \infty)$ and $N > p$. If $g \in \mathcal{F}_{\frac{N}{p}}$ and $g^+ \not\equiv 0$, then*

$$\lambda_1 = \inf \left\{ \int_{B_1^c} |\nabla \phi|^p : \phi \in W^{1,p}(B_1^c), \int_{B_1^c} g|\phi|^p = 1 \right\}$$

is the unique positive principal eigenvalue of (1.1). Furthermore, λ_1 is simple and isolated.

Our proof for the above theorem uses the continuous embedding of the Sobolev space $W^{1,p}(B_1^c)$ into the Lorentz space $L^{p^*,p}(B_1^c)$. This embedding can be obtained from the embedding of $W^{1,p}(\mathbb{R}^N)$ into $L^{p^*,p}(\mathbb{R}^N)$ due to Tartar [24]. However, we give a simple proof for this embedding using the Pólya-Szegő and the Hardy-Littlewood inequalities for the Schwarz symmetrization and the Muckenhoupt condition (Theorem 2 of [22]) for the one-dimensional weighted Hardy inequalities.

To state our next result, we associate a radial function with g as below. For $r \in [1, \infty)$,

$$\tilde{g}(r) = \text{ess sup}\{|g(r\omega)| : \omega \in S^{N-1}\},$$

where the essential supremum is taken with respect to the $(N-1)$ -dimensional surface measure. Since $g \in L^1_{loc}(B_1^c)$, we get $\tilde{g}(r)$ is finite a.e. in $(1, \infty)$ (Theorem 2.49 of [15]). Now we consider the following weighted Lebesgue spaces:

$$X = \begin{cases} L^1((1, \infty); r^{p-1}), & N \neq p, \\ L^1((1, \infty); (r(1 + \log r))^{N-1}), & N = p. \end{cases}$$

Theorem 1.2. *Let $p \in (1, \infty)$ and let $g \in L^1_{loc}(B_1^c)$ with $g^+ \not\equiv 0$. If $\tilde{g} \in X$, then*

$$\lambda_1 = \inf \left\{ \int_{B_1^c} |\nabla \phi|^p : \phi \in W^{1,p}(B_1^c), \int_{B_1^c} g|\phi|^p = 1 \right\}$$

is the unique positive principal eigenvalue of (1.1). Furthermore, λ_1 is simple and isolated.

The Dirichlet eigenvalue problem for g for which \tilde{g} lies in an analogous weighted Lebesgue space has been considered in [6]. For $\tilde{g} \in X$, we show that $W^{1,p}(B_1^c)$ is continuously and compactly embedded into the weighted Lebesgue space $L^p(B_1^c, |g|)$. A similar embedding for $\mathcal{D}_0^{1,p}(B_1^c)$ is obtained in [6].

We also study the existence of infinitely many positive eigenvalues of (1.1). A complete characterization of the set of all eigenvalues of (1.1) with $p \neq 2$ is a challenging open problem. However, there are many ways to produce infinite set of eigenvalues of (1.1), for example, see [14, 20]. In [20], An L e proved that, for Ω bounded and $g \equiv 1$, the set of all eigenvalues of (1.1) is closed. We extend these results as below:

Theorem 1.3. *Let $p \in (1, \infty)$ and g be in Theorem 1.1 or Theorem 1.2. Then*

- (i) *there exists a sequence of positive eigenvalues of (1.1) tending to infinity,*
- (ii) *the set of all eigenvalues of (1.1) is closed.*

The rest of the paper is organized as follows. In Section 2, we briefly define symmetrization, Lorentz spaces and state the Muckenhoupt conditions for the weighted Hardy inequalities. In Section 3, we prove the required continuous embeddings and its compactness. Section 4 contains the functional settings. In the last section, we give the proofs of the above theorems.

2 Preliminaries

We define the one-dimensional rearrangement and then define the Lorentz spaces. Further, we state some important results such as Muckenhoupt condition, maximum principle for p -Laplacian that will be used subsequently.

2.1 Symmetrization

Let $\Omega \subset \mathbb{R}^N$ be a Lebesgue measurable set and let $\mathcal{M}(\Omega)$ be the set of all extended real valued Lebesgue measurable functions that are finite a.e. in Ω . Given a function $f \in \mathcal{M}(\Omega)$ and for $s > 0$, we define $E_f(s) = \{x \in \Omega : |f(x)| > s\}$. The *distribution function* α_f of f is defined as $\alpha_f(s) = |E_f(s)|$ for $s > 0$, where $|\cdot|$ denotes the Lebesgue measure. We define the *one dimensional decreasing rearrangement* f^* of f as

$$f^*(t) = \inf\{s > 0 : \alpha_f(s) < t\}, \quad \text{for } t > 0.$$

The map $f \mapsto f^*$ is not sub-additive. However, we obtain a sub-additive function from f^* , namely the maximal function f^{**} of f^* , defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau) d\tau, \quad t > 0.$$

The *Schwarz symmetrization* of f is defined by

$$f^*(x) = f^*(\omega_N |x|^N), \quad \forall x \in \Omega^*,$$

where ω_N is the measure of the unit ball in \mathbb{R}^N and Ω^* is the open ball centered at the origin with same measure as Ω .

Next we state two important inequalities concerning the symmetrization.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$.*

(a) *Hardy-Littlewood inequality: Let f and g be nonnegative measurable functions. Then*

$$\int_{\Omega} f(x)g(x)dx \leq \int_{\Omega^*} f^*(x)g^*(x)dx = \int_0^{|\Omega|} f^*(t)g^*(t)dt.$$

(b) *Pólya-Szegő inequality: Let $\phi \in W^{1,p}(\mathbb{R}^N)$. Then*

$$\int_{\Omega^*} |\nabla \phi^*(x)|^p dx = N^p \omega_N^{\frac{p}{N}} \int_0^{\infty} s^{(p-\frac{p}{N})} |\phi^{*'}(s)|^p ds \leq \int_{\mathbb{R}^N} |\nabla \phi(x)|^p dx.$$

2.2 Lorentz Space

The Lorentz spaces are introduced by Lorentz in [21] and these are a refinement of the classical Lebesgue spaces. For more details on Lorentz spaces, we refer to the books [1, 13].

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in \mathcal{M}(\Omega)$ and $(p, q) \in [1, \infty) \times [1, \infty]$. Consider the following quantity:

$$|f|_{(p,q)} := \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right\|_{L^q((0,\infty))} = \begin{cases} \left(\int_0^{\infty} \left[t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right]^q dt \right)^{\frac{1}{q}}, & 1 \leq q < \infty; \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & q = \infty. \end{cases}$$

The Lorentz space $L^{p,q}(\Omega)$ is defined as

$$L^{p,q}(\Omega) := \{f \in \mathcal{M}(\Omega) : |f|_{(p,q)} < \infty\},$$

where $|f|_{(p,q)}$ is a complete quasi norm on $L^{p,q}(\Omega)$. For $p \in (1, \infty)$, $L^{p,p}(\Omega) = L^p(\Omega)$ and $L^{p,\infty}(\Omega)$ coincides with the weak- L^p space $:= \left\{f \in \mathcal{M}(\Omega) : \sup_{s>0} s(\alpha_f(s))^{\frac{1}{p}} < \infty\right\}$. Indeed, one can define a norm on $L^{p,q}(\Omega)$ for certain values of p and q as in the following proposition (Lemma 3.4.6 of [13]).

Proposition 2.2. *For $(p, q) \in (1, \infty) \times [1, \infty]$, let*

$$\|f\|_{(p,q)} := \left\| t^{\frac{1}{p} - \frac{1}{q}} f^{**}(t) \right\|_{L^q((0, \infty))}.$$

Then $\|f\|_{(p,q)}$ is a norm in $L^{p,q}(\Omega)$ and it is equivalent to the quasi-norm $|f|_{(p,q)}$.

2.3 Some important results

The following result is a sufficient condition for the one-dimensional weighted Hardy inequalities (4.17 of [19]).

Proposition 2.3 (Muckenhoupt condition). *Let u, v be nonnegative measurable functions such that $v > 0$. Let $p \in (1, \infty)$ and let p' be the Hölder conjugate of p . If*

$$A = \sup_{t>0} \left(\int_0^t u(s) ds \right)^{\frac{1}{p}} \left(\int_t^\infty v(s)^{1-p'} ds \right)^{\frac{1}{p'}} < \infty,$$

then

$$\left(\int_0^\infty \left| \int_s^\infty f(t) dt \right|^p u(s) ds \right)^{\frac{1}{p}} \leq p^{\frac{1}{p}} (p')^{\frac{1}{p'}} A \left(\int_0^\infty |f(s)|^p v(s) ds \right)^{\frac{1}{p}} \quad (2.1)$$

holds for any measurable function f on $(0, \infty)$.

In this article we use the following version of strong maximal principle due to Kawohl, Lucia and Prashanth (Proposition 3.2 of [18]).

Theorem 2.4 (Strong Maximum Principle for p -Laplacian). *Let ϕ be a non negative function in $W^{1,p}(B_1^c)$ and $V \in L^1_{loc}(B_1^c)$ with $V \geq 0$ a.e. in B_1^c . Assume that $V\phi^{p-1} \in L^1_{loc}(B_1^c)$. Consider the inequality*

$$\int_{B_1^c} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla v + \int_{B_1^c} V |\phi|^{p-2} \phi v \geq 0, \quad \forall v \in C_c^\infty(B_1^c), v \geq 0.$$

Then either $\phi \equiv 0$ or $\phi > 0$ a.e. in B_1^c .

3 The embeddings of $W^{1,p}(B_1^c)$

For $N > p$, we prove the continuous embeddings of $W^{1,p}(B_1^c)$ into $L^{p^*}(B_1^c)$, where $p^* = \frac{Np}{N-p}$. For g as in Theorem 1.2, we prove $W^{1,p}(B_1^c)$ is continuously and compactly embedded in $L^p(B_1^c, |g|)$.

3.1 The embeddings into Lorentz spaces

First we prove a lemma using the Muckenhoupt condition.

Lemma 3.1. *Let $N > p$. If $g \in L^{\frac{N}{p}, \infty}(\mathbb{R}^N)$, then*

$$\int_0^\infty g^*(s) \phi^*(s)^p ds \leq p(p^*)^{p-1} \|g\|_{(\frac{N}{p}, \infty)} \int_0^\infty s^{(p-\frac{p}{N})} |\phi^{*'}(s)|^p ds, \quad \forall \phi \in C_c^1(\mathbb{R}^N). \quad (3.1)$$

Proof. In Proposition 2.3, set $f = \phi^{*'}$, $u = g^*$ and $v(s) = s^{p-\frac{p}{N}}$. Then

$$\int_0^t u(s) ds = \int_0^t g^*(s) ds = t g^{**}(t)$$

and

$$\int_t^\infty v(s)^{1-p'} ds = \int_t^\infty s^{-\frac{p'}{N'}} ds = \frac{N'}{p' - N'} t^{\frac{N' - p'}{N'}} = \frac{N(p-1)}{N-p} t^{\frac{p-N}{N(p-1)}}.$$

Now,

$$\begin{aligned} A &= \sup_{t>0} \left(\int_0^t u(s) ds \right)^{\frac{1}{p}} \left(\int_t^\infty v(s)^{1-p'} ds \right)^{\frac{1}{p'}} \\ &= \left(\frac{N(p-1)}{N-p} \right)^{\frac{1}{p'}} \sup_{t>0} \{ t g^{**}(t) \}^{\frac{1}{p}} t^{\frac{p-N}{Np}} = \left(\frac{N(p-1)}{N-p} \right)^{\frac{1}{p'}} \|g\|_{(\frac{N}{p}, \infty)}^{\frac{1}{p}}. \end{aligned}$$

Therefore, by the Muckenhoupt condition we have for all $\phi \in C_c^1(\mathbb{R}^N)$,

$$\left(\int_0^\infty g^*(s) \phi^*(s)^p ds \right)^{\frac{1}{p}} \leq p^{\frac{1}{p}} \left(\frac{N(p-1)p'}{N-p} \right)^{\frac{1}{p'}} \|g\|_{(\frac{N}{p}, \infty)}^{\frac{1}{p}} \left(\int_0^\infty s^{(p-\frac{p}{N})} |\phi^{*'}(s)|^p ds \right)^{\frac{1}{p}}.$$

Now (3.1) follows by noting that $p(p^*)^{p-1}$ is precisely the p^{th} power of the constant in the right hand side of the above inequality. \square

Theorem 3.2. *For $N > p$, there exists $C > 0$ such that*

$$\|\phi\|_{(p^*, p)}^p \leq C \|\phi\|_{W^{1,p}(\mathbb{R}^N)}^p, \quad \forall \phi \in W^{1,p}(\mathbb{R}^N). \quad (3.2)$$

Proof. Let $g \in L^{\frac{N}{p}, \infty}(\mathbb{R}^N)$. Then by Pólya-Szegő inequality (part (b) of Proposition 2.1) and by the above lemma we have

$$\int_0^\infty g^*(s) \phi^*(s)^p ds \leq \tilde{C} \|g\|_{(\frac{N}{p}, \infty)} \int_{\mathbb{R}^N} |\nabla \phi(x)|^p dx, \quad \forall \phi \in C_c^1(\mathbb{R}^N), \quad (3.3)$$

with $\tilde{C} = p(p^*)^{p-1} N^{-p} \omega_N^{-\frac{p}{N}}$. As $g(x) = \frac{1}{|x|^p}$ is in $L^{\frac{N}{p}, \infty}(\mathbb{R}^N)$ with $g^*(s) = \left(\frac{\omega_N}{s}\right)^{\frac{p}{N}}$ and $\|g\|_{(\frac{N}{p}, \infty)} = \frac{N \omega_N^{\frac{p}{N}}}{N-p}$, from (3.3) we have

$$\int_0^\infty s^{-\frac{p}{N}} \phi^*(s)^p ds \leq \tilde{C} \frac{N}{N-p} \int_{\mathbb{R}^N} |\nabla \phi(x)|^p dx \leq \tilde{C} \frac{N}{N-p} \|\phi\|_{W^{1,p}(\mathbb{R}^N)}^p, \quad \forall \phi \in C_c^1(\mathbb{R}^N).$$

The integral in the left hand side of the inequality is equivalent to $\|\phi\|_{(p^*,p)}$ (Proposition 2.2) and hence by the density of $C_c^1(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$ we obtain (3.2) with $C = D \left(\frac{p^*}{N\omega_N^{\frac{1}{N}}} \right)^p$, where D is the equivalence constant. \square

Remark 3.3. For $g \in L^{\frac{N}{p},\infty}(\mathbb{R}^N)$, from (3.3) and Hardy-Littlewood inequality, we have the following generalized Hardy-Sobolev inequality:

$$\int_{\mathbb{R}^N} g(x)|\phi(x)|^p dx \leq \frac{p(p^*)^{p-1}}{N^p \omega_N^{\frac{p}{N}}} \|g\|_{(L^{\frac{N}{p},\infty})} \int_{\mathbb{R}^N} |\nabla \phi(x)|^p dx, \quad \forall \phi \in W^{1,p}(\mathbb{R}^N).$$

In particular, by taking $g(x) = \frac{1}{|x|^p}$, we get the classical Hardy-Sobolev inequality

$$\int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^p} dx \leq \left(\frac{p}{N-p} \right)^p \int_{\mathbb{R}^N} |\nabla \phi(x)|^p dx, \quad \forall \phi \in W^{1,p}(\mathbb{R}^N).$$

Corollary 3.4. Let $N > p$. Then $\|\phi\|_{(p^*,p)} \leq C \|\phi\|_{W^{1,p}(B_1^c)}$.

Proof. Since the boundary of B_1^c is smooth, it has the extension property (Theorem 9.7 of [8], page 272), i.e. , there exists a positive constant C such that

$$\|\phi\|_{W^{1,p}(\mathbb{R}^N)} \leq C \|\phi\|_{W^{1,p}(B_1^c)}.$$

Now by the previous theorem we get the required embedding. \square

3.2 The embeddings into weighted Lebesgue spaces

Proposition 3.5. Let $N > p$. If $\tilde{g} \in L^1((1,\infty);r^{p-1})$, then there exists $C = C(N,p) > 0$ such that

$$\int_{B_1^c} \tilde{g}(|x|)|\phi(x)|^p dx \leq C \|\tilde{g}\|_{L^1((1,\infty);r^{p-1})} \|\phi\|_{W^{1,p}(B_1^c)}^p, \quad \forall \phi \in W^{1,p}(B_1^c). \quad (3.4)$$

Proof. Let $\psi \in C_c^1(\overline{B_1^c})$. For $\omega \in S^{N-1}$, set $\varphi(t) = \psi(t\omega)$ where $t \geq 1$. Using the fundamental theorem of calculus we have

$$\varphi(r) = - \int_r^\infty \varphi'(t) dt = - \int_r^\infty \varphi'(t) t^{\frac{1-N}{p}} t^{\frac{N-1}{p}} dt.$$

By Hölder inequality,

$$|\varphi(r)|^p \leq \left(\frac{p-1}{N-p} \right)^{p-1} r^{p-N} \left(\int_r^\infty t^{N-1} |\varphi'(t)|^p dt \right).$$

As $\varphi'(t) = \nabla \psi(t\omega) \cdot \omega$, for each $\omega \in S^{N-1}$ we have $|\varphi'(t)| = |\nabla \psi(t\omega)|$. Hence

$$|\psi(r\omega)|^p \leq \left(\frac{p-1}{N-p} \right)^{p-1} r^{p-N} \left(\int_r^\infty t^{N-1} |\nabla \psi(t\omega)|^p dt \right).$$

Set $C = \left(\frac{p-1}{N-p}\right)^{p-1}$. We multiply both sides by $r^{N-1}\tilde{g}(r)$ and integrate over $S^{N-1} \times (1, \infty)$ to get

$$\int_1^\infty \int_{S^{N-1}} |\psi(r\omega)|^p r^{N-1} \tilde{g}(r) dS dr \leq C \int_1^\infty r^{p-1} \tilde{g}(r) dr \left(\int_1^\infty \int_{S^{N-1}} r^{N-1} |\nabla \psi(r\omega)|^p dS dr \right).$$

Thus we obtain

$$\int_{B_1^c} \tilde{g}(|x|) |\psi(x)|^p dx \leq C \left(\int_1^\infty r^{p-1} \tilde{g}(r) dr \right) \|\psi\|_{W^{1,p}(B_1^c)}^p, \quad \forall \psi \in C_c^1(\overline{B_1^c}).$$

Now (3.4) follows by the density of $C_c^1(\overline{B_1^c})$ in $W^{1,p}(B_1^c)$. \square

Proposition 3.6. *Let $N = p$. If $\tilde{g} \in L^1((1, \infty); (r(1 + \log r))^{N-1})$, then there exists $C = C(N) > 0$ such that*

$$\int_{B_1^c} \tilde{g}(|x|) |\phi(x)|^N dx \leq C \|\tilde{g}\|_{L^1((1, \infty); (r(1 + \log r))^{N-1})} \|\phi\|_{W^{1,N}(B_1^c)}^N, \quad \forall \phi \in W^{1,N}(B_1^c).$$

Proof. Let $\psi \in C_c^1(\overline{B_1^c})$. As before, set $\varphi(t) = \psi(t\omega)$ where $t \geq 1$. Then

$$\varphi(r) - \varphi(1) = \int_1^r \varphi'(t) dt = \int_1^r t^{\frac{1-N}{N}} t^{\frac{N-1}{N}} \varphi'(t) dt.$$

By Hölder inequality,

$$|\varphi(r) - \varphi(1)| \leq \left(\int_1^r \frac{1}{t} dt \right)^{\frac{1}{N'}} \left(\int_1^r t^{N-1} |\varphi'(t)|^N dt \right)^{\frac{1}{N}},$$

where N' is the Hölder conjugate of N . Thus

$$|\varphi(r)|^N \leq 2^{N-1} \left\{ |\varphi(1)|^N + (\log r)^{N-1} \int_1^r t^{N-1} |\varphi'(t)|^N dt \right\}$$

and hence for $\omega \in S^{N-1}$, we have

$$|\psi(r\omega)|^N \leq 2^{N-1} \left\{ |\psi(\omega)|^N + (\log r)^{N-1} \int_1^r t^{N-1} |\nabla \psi(t\omega)|^N dt \right\}.$$

Now multiply both sides by $r^{N-1}\tilde{g}(r)$ and integrate over $S^{N-1} \times (1, \infty)$ to get

$$\begin{aligned} \frac{1}{2^{N-1}} \int_1^\infty \int_{S^{N-1}} |\psi(r\omega)|^N r^{N-1} \tilde{g}(r) dS dr &\leq \int_1^\infty \int_{S^{N-1}} |\psi(\omega)|^N r^{N-1} \tilde{g}(r) dS dr \\ &+ \left(\int_1^\infty (r \log r)^{N-1} \tilde{g}(r) dr \right) \left(\int_1^\infty \int_{S^{N-1}} r^{N-1} |\nabla \psi(r\omega)|^N dS dr \right). \end{aligned} \quad (3.5)$$

Using the trace embedding of $W^{1,N}(B_1^c)$ into $L^N(\partial B_1^c)$ (Theorem 2.86 of [11], page 100) we estimate the following integral:

$$\int_1^\infty \int_{S^{N-1}} |\psi(\omega)|^N r^{N-1} \tilde{g}(r) dS dr = \left(\int_1^\infty r^{N-1} \tilde{g}(r) dr \right) \left(\int_{S^{N-1}} |\psi(\omega)|^N dS \right)$$

$$\leq C_1 \left(\int_1^\infty r^{N-1} \tilde{g}(r) dr \right) \int_{B_1^c} (|\psi(x)|^N + |\nabla \psi(x)|^N) dx, \quad (3.6)$$

where $C_1 = C_1(N) > 0$ is the embedding constant. By combining the above inequalities and using the density argument we obtain

$$\int_{B_1^c} \tilde{g}(|x|) |\psi(x)|^N dx \leq C \left(\int_1^\infty (r + r \log r)^{N-1} \tilde{g}(r) dr \right) \|\psi\|_{W^{1,N}(B_1^c)}^N, \quad \forall \psi \in W^{1,N}(B_1^c).$$

□

Proposition 3.7. *Let $N < p$. If $\tilde{g} \in L^1((1, \infty); r^{p-1})$, then there exists $C = C(N, p) > 0$ such that*

$$\int_{B_1^c} \tilde{g}(|x|) |\phi(x)|^p dx \leq C \|\tilde{g}\|_{L^1((1, \infty); r^{p-1})} \|\phi\|_{W^{1,p}(B_1^c)}^p, \quad \forall \phi \in W^{1,p}(B_1^c). \quad (3.7)$$

Proof. For φ as in the above proof we have

$$\varphi(r) - \varphi(1) = \int_1^r \varphi'(t) dt = \int_1^r t^{\frac{1-N}{p}} t^{\frac{N-1}{p}} \varphi'(t) dt.$$

By Hölder inequality we get

$$|\varphi(r) - \varphi(1)| \leq \left(\int_1^r t^{\frac{1-N}{p-1}} dt \right)^{\frac{1}{p'}} \left(\int_1^r t^{N-1} |\varphi'(t)|^p dt \right)^{\frac{1}{p}},$$

where p' is the Hölder conjugate of p . Thus

$$|\varphi(r)|^p \leq 2^{p-1} \left\{ |\varphi(1)|^p + \left(\frac{p-1}{p-N} \right)^{p-1} r^{p-N} \left(\int_1^r t^{N-1} |\varphi'(t)|^p dt \right) \right\}. \quad (3.8)$$

As before, we multiply both sides of (3.8) by $r^{N-1} \tilde{g}(r)$ and integrate over $S^{N-1} \times (1, \infty)$ to get

$$\begin{aligned} \frac{1}{2^{p-1}} \int_1^\infty \int_{S^{N-1}} |\psi(r\omega)|^p r^{N-1} \tilde{g}(r) dS dr &\leq \int_1^\infty \int_{S^{N-1}} |\psi(\omega)|^p r^{N-1} \tilde{g}(r) dS dr \\ &+ \left(\frac{p-1}{p-N} \right)^{p-1} \left(\int_1^\infty r^{p-1} \tilde{g}(r) dr \right) \left(\int_1^\infty \int_{S^{N-1}} r^{N-1} |\nabla \psi(r\omega)|^p dS dr \right). \end{aligned}$$

The rest of the proof follows as in the proof of Proposition 3.6. □

Next, we prove the embeddings given above are indeed compact.

Proposition 3.8. *Let $\tilde{g} \in X$. Then $W^{1,p}(B_1^c)$ embedded compactly into $L^p(B_1^c, |g|)$.*

Proof. Let $\phi_n \rightharpoonup \phi$ in $W^{1,p}(B_1^c)$. Set $M = \sup\{\|\phi_n - \phi\|_{W^{1,p}(B_1^c)}\}$. Let $\epsilon > 0$ be arbitrary. By density of $C_c^\infty((1, \infty))$ in X , there exists $\tilde{g}_\epsilon \in C_c^\infty((1, \infty))$ such that $\|\tilde{g} - \tilde{g}_\epsilon\|_X < \frac{\epsilon}{M^p}$. Now,

$$\int_{B_1^c} |g| |\phi_n - \phi|^p \leq \int_{B_1^c} \tilde{g} |\phi_n - \phi|^p \leq \int_{B_1^c} |\tilde{g} - \tilde{g}_\epsilon| |\phi_n - \phi|^p + \int_{B_1^c} \tilde{g}_\epsilon |\phi_n - \phi|^p. \quad (3.9)$$

From Proposition 3.6-3.8, we have

$$\int_{B_1^c} |\tilde{g} - \tilde{g}_\epsilon| |\phi_n - \phi|^p \leq C \|\tilde{g} - \tilde{g}_\epsilon\|_X \|\phi_n - \phi\|_{W^{1,p}(B_1^c)}^p, \quad (3.10)$$

where $C > 0$ is the embedding constant. By the compactness of the embedding of $W^{1,p}(B_1^c)$ into $L_{loc}^p(B_1^c)$, there exists $n_0 \in \mathbb{N}$ such that $\int_{B_1^c} |\tilde{g}_\epsilon| |\phi_n - \phi|^p < \epsilon, \forall n > n_0$. Now by the above inequalities we obtain

$$\int_{B_1^c} \tilde{g} |\phi_n - \phi|^p < C\epsilon, \quad \forall n \geq n_0.$$

Thus ϕ_n converges strongly in $L^p(B_1^c, |g|)$ as required. \square

4 The variational settings

Now we develop the functional settings for proving our main theorems. For g as in Theorem 1.1 or Theorem 1.2, we consider the following functionals on $W^{1,p}(B_1^c)$:

$$J(\phi) = \int_{B_1^c} |\nabla \phi|^p; \quad G(\phi) = \int_{B_1^c} g |\phi|^p.$$

One can easily verify that $J, G \in C^1(W^{1,p}(B_1^c); \mathbb{R})$ and for $\phi, u \in W^{1,p}(B_1^c)$,

$$\langle J'(\phi), u \rangle = p \int_{B_1^c} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla u; \quad \langle G'(\phi), u \rangle = p \int_{B_1^c} g |\phi|^{p-2} \phi u,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality action.

Definition 4.1. We say a function g belongs to the class \mathcal{A} , if $g \in L_{loc}^1(B_1^c)$, $\text{supp}(g^+)$ has a positive measure and

(i) $g \in \mathcal{F}_{\frac{N}{p}}$ with $N > p$, or

(ii) $\tilde{g} \in \begin{cases} L^1((1, \infty); r^{p-1}), & N \neq p, \\ L^1((1, \infty); (r(1 + \log r))^{N-1}), & N = p. \end{cases}$

Proposition 4.2. If $g \in \mathcal{A}$, then G and G' are compact on $W^{1,p}(B_1^c)$.

Proof. Compactness of G : If $g \in \mathcal{F}_{\frac{N}{p}}$, then G is compact using the density of $C_c^\infty(B_1^c)$ in $\mathcal{F}_{\frac{N}{p}}$ and the arguments as in the proof of Proposition 3.8. If $\tilde{g} \in X$, then the compactness of G follows from Proposition 3.8.

Compactness of G' : Let $g \in \mathcal{F}_{\frac{N}{p}}$ and let $\phi_n \rightharpoonup \phi$ in $W^{1,p}(B_1^c)$. For $v \in W^{1,p}(B_1^c)$,

$$\begin{aligned} |\langle G'(\phi_n) - G'(\phi), v \rangle| &\leq \int_{B_1^c} |g| (|\phi_n|^{p-2} \phi_n - |\phi|^{p-2} \phi) |v| \\ &\leq \left(\int_{B_1^c} |g| (|\phi_n|^{p-2} \phi_n - |\phi|^{p-2} \phi)^{p'} \right)^{\frac{1}{p'}} \left(\int_{B_1^c} |g| |v|^p \right)^{\frac{1}{p}} \end{aligned}$$

$$\leq C \left(\int_{B_1^c} |g| (|\phi_n|^{p-2}\phi_n - |\phi|^{p-2}\phi)^{p'} \right)^{\frac{1}{p'}} \|g\|_{\left(\frac{N}{p}, \infty\right)}^{\frac{1}{p}} \|v\|_{W^{1,p}(B_1^c)},$$

where $C > 0$ is the embedding constant. Therefore,

$$\|G'(\phi_n) - G'(\phi)\| \leq C \left(\int_{B_1^c} |g| (|\phi_n|^{p-2}\phi_n - |\phi|^{p-2}\phi)^{p'} \right)^{\frac{1}{p'}} \|g\|_{\left(\frac{N}{p}, \infty\right)}^{\frac{1}{p}}.$$

Now consider the map K defined on $W^{1,p}(B_1^c)$ as $K(\phi) = |g|^{\frac{1}{p'}} |\phi|^{p-2}\phi$. Clearly K maps $W^{1,p}(B_1^c)$ into $L^{p'}(B_1^c)$ and using a similar set of arguments as in the proof of Proposition 3.8, one can prove that K is compact. Hence we conclude $\|G'(\phi_n) - G'(\phi)\| \rightarrow 0$, as $n \rightarrow \infty$. For $\tilde{g} \in X$, the proof is similar. \square

For g as before, consider the set

$$N_g := \{\phi \in W^{1,p}(B_1^c) : \int_{B_1^c} g|\phi|^p = 1\} = G^{-1}(1).$$

Since $g^+ \not\equiv 0$, one can show that the set N_g is nonempty (Proposition 4.2 of [18]). The functional J is not coercive on $W^{1,p}(B_1^c)$. We prove a Poincaré type inequality for functions in N_g that will ensure $\int_{B_1^c} |\nabla\phi|^p$ is coercive on N_g .

Lemma 4.3. *Let $g \in \mathcal{A}$. Then there exists $\epsilon > 0$ such that*

$$\int_{B_1^c} |\nabla\phi|^p \geq \epsilon \int_{B_1^c} |\phi|^p, \quad \forall \phi \in N_g. \quad (4.1)$$

Proof. We prove by the method of contradiction. If (4.1) is not true, then there exists a sequence (ϕ_n) in N_g such that

$$\int_{B_1^c} |\phi_n|^p = 1, \quad \int_{B_1^c} |\nabla\phi_n|^p \leq \frac{1}{n}. \quad (4.2)$$

Thus (ϕ_n) is bounded in $W^{1,p}(B_1^c)$ and hence by the reflexivity we get a subsequence (ϕ_{n_k}) of (ϕ_n) such that as $k \rightarrow \infty$, $\phi_{n_k} \rightharpoonup \phi$ in $W^{1,p}(B_1^c)$. Thus $\phi \in N_g$, as G is compact. Further, from (4.2), by weak lowersemicontinuity of $\|\cdot\|_p$ and J , we have:

$$\int_{B_1^c} |\phi|^p \leq 1 \quad \text{and} \quad \int_{B_1^c} |\nabla\phi|^p = 0.$$

Now the connectedness yields $\phi = 0$, a contradiction as $\phi \in N_g$. \square

Remark 4.4. For $\phi \in N_g$, $\langle G'(\phi), \phi \rangle = p$. Thus 1 is a regular point of G and hence N_g gets a C^1 manifold structure. For $\phi \in N_g$, the tangent space at ϕ is given by (Proposition 4.3.33 and Remark 4.3.40 of [12])

$$T_\phi N_g = \text{Ker}(G'(\phi)).$$

Further,

$$\|dJ(\phi)\| = \sup_{\substack{v \in \text{Ker}(G'(\phi)) \\ \|v\|=1}} \langle J'(\phi), v \rangle = \min_{\lambda \in \mathbb{R}} \|J'(\phi) - \lambda G'(\phi)\| \quad (4.3)$$

(Proposition 6.4.35 of [12]). In particular, if ϕ is critical point of J on N_g , then ϕ is an eigenfunction of (1.1) corresponding to the eigenvalue $J(\phi)$.

Definition 4.5. We say a map $J \in C^1(Y, \mathbb{R})$ satisfies Palais-Smale (P. S.) condition on a C^1 manifold $M \subset Y$, if (ϕ_n) in M such that $J(\phi_n) \rightarrow c \in \mathbb{R}$ and $\|dJ(\phi_n)\| \rightarrow 0$, then (ϕ_n) has a subsequence that converges in M .

Lemma 4.6. Let $g \in \mathcal{A}$. Then J satisfies the P. S. condition on N_g .

Proof. Let (ϕ_n) be a sequence in N_g such that $J(\phi_n) \rightarrow \lambda$ and $\|dJ(\phi_n)\| \rightarrow 0$. For $\lambda \in \mathbb{R}$, set $A_\lambda = J' - \lambda G'$. Then by (4.3), there exists a sequence (λ_n) such that

$$A_{\lambda_n}(\phi_n) = J'(\phi_n) - \lambda_n G'(\phi_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.4)$$

Using Lemma 4.3 and by the reflexivity of $W^{1,p}(B_1^c)$, up to a subsequence, $\phi_n \rightharpoonup \phi$ in $W^{1,p}(B_1^c)$. Further, $\langle J'(\phi_n) - \lambda_n G'(\phi_n), \phi_n \rangle = p(J(\phi_n) - \lambda_n)$. Thus $\lambda_n \rightarrow \lambda$. Observe that

$$\langle J'(\phi_n) - J'(\phi), \phi_n - \phi \rangle = \langle A_{\lambda_n}(\phi_n) - A_\lambda(\phi), \phi_n - \phi \rangle + \langle \lambda_n G'(\phi_n) - \lambda G'(\phi), \phi_n - \phi \rangle.$$

From the weak convergence of (ϕ_n) and the compactness of G' , we get

$$\langle J'(\phi_n) - J'(\phi), \phi_n - \phi \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Now

$$\begin{aligned} \frac{1}{p} \langle J'(\phi_n) - J'(\phi), \phi_n - \phi \rangle &= \|\nabla \phi_n\|_p^p + \|\nabla \phi\|_p^p - \int_{B_1^c} |\nabla \phi_n|^{p-2} \nabla \phi_n \cdot \nabla \phi - \int_{B_1^c} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla \phi_n \\ &\geq \|\nabla \phi_n\|_p^p + \|\nabla \phi\|_p^p - \|\nabla \phi_n\|_p^{p-1} \|\nabla \phi\|_p - \|\nabla \phi\|_p^{p-1} \|\nabla \phi_n\|_p \\ &= (\|\nabla \phi_n\|_p^{p-1} - \|\nabla \phi\|_p^{p-1}) (\|\nabla \phi_n\|_p - \|\nabla \phi\|_p). \end{aligned}$$

Hence $\|\nabla \phi_n\|_p \rightarrow \|\nabla \phi\|_p$. Thus the weak convergence of (ϕ_n) in $W^{1,p}(B_1^c)$ and the uniform convexity of $(L^p(B_1^c))^N$ gives $\nabla \phi_n \rightarrow \nabla \phi$ in $(L^p(B_1^c))^N$. Now using Lemma 4.3, we conclude that $\phi_n \rightarrow \phi$ in $W^{1,p}(B_1^c)$. This completes the proof. \square

5 Proof of main theorems

Proof of Theorem 1.1 and 1.2:

The existence: Recall that

$$\lambda_1 = \inf_{\phi \in N_g} \int_{B_1^c} |\nabla \phi|^p.$$

Let (ϕ_n) be a minimizing sequence for J on N_g . As before, using Lemma 4.3 we get sequence (ϕ_n) is bounded in $W^{1,p}(B_1^c)$. Thus by the reflexivity, (ϕ_n) has a subsequence (ϕ_{n_k}) that converges

weakly to some $\Phi \in W^{1,p}(B_1^c)$. Since the set N_g is weakly closed, $\Phi \in N_g$. Further, by weak lowersemicontinuity of J ,

$$\lambda_1 \leq J(\Phi) \leq \liminf J(\phi_{n_k}) = \lambda_1.$$

Thus λ_1 is attained and hence Φ is a critical point of J on N_g . Therefore, from Remark 4.4 we see that λ_1 is an eigenvalue of (1.1) and Φ is an eigenfunction corresponding to λ_1 .

The principality: Clearly $|\Phi|$ is also an eigenfunction corresponding to λ_1 . Thus for $v \in C_c^\infty(B_1^c)$ with $v \geq 0$,

$$\int_{B_1^c} |\nabla(|\Phi|)|^{p-2} \nabla(|\Phi|) \cdot \nabla v + \lambda_1 \int_{B_1^c} g^- |\Phi|^{p-1} v = \lambda_1 \int_{B_1^c} g^+ |\Phi|^{p-1} v \geq 0.$$

Using Hölder inequality, one can verify that $g^- |\Phi|^{p-1} \in L_{loc}^1(B_1^c)$. Thus $|\Phi|$ satisfies all the conditions of Theorem 2.4. Hence $|\Phi| > 0$ a.e. in B_1^c .

The uniqueness and the simplicity: The uniqueness of the principal eigenvalue can be obtained using the Picone's identity (Theorem 1.1 of [4]). The simplicity follows using the same arguments as in Theorem 1.3 of [18].

Isolatedness: Suppose (λ_n) is a sequence of eigenvalues of (1.1) converging to λ_1 . For each n , let $\phi_n \in N_g$ be an eigenfunction corresponding to λ_n . Then $J(\phi_n) = \lambda_n$ and for each $n \in \mathbb{N}$, we have

$$\langle J'(\phi_n) - \lambda_n G'(\phi_n), v \rangle = \int_{B_1^c} |\nabla \phi|^{p-2} \nabla \phi_n \cdot \nabla v - \lambda_n \int_{B_1^c} g |\phi_n|^{p-2} \phi_n v = 0,$$

i.e., $\|dJ(\phi_n)\| = 0$. Hence using Lemma 4.6 we conclude that $\phi_n \rightarrow \pm|\Phi|$ in $W^{1,p}(B_1^c)$. Assume that $\phi_n \rightarrow |\Phi|$. Thus by Egorov's theorem there exists $A \subset B_1^c$ with $|A| > 0$ and ϕ_n converges to $|\Phi|$ uniformly on A . Thus there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\phi_n^- = 0$ a.e. in A . Further, from (1.2),

$$\int_{B_1^c} |\nabla \phi_n^-|^p = \lambda_n \int_{B_1^c} g |\phi_n^-|^p.$$

For $v_n = \left(\int_{B_1^c} g |\phi_n^-|^p \right)^{-\frac{1}{p}} \phi_n^-$, observe that $\int_{B_1^c} g |v_n|^p = 1$ and $\int_{B_1^c} |\nabla v_n|^p = \lambda_n \rightarrow \lambda_1$. Therefore, $v_n \rightarrow |\Phi|$. A contradiction, as $v_n = 0$ a.e. in A for $n \geq n_0$. Thus such a sequence (λ_n) does not exist. \square

Remark 5.1. (a) For $g \in \mathcal{A}$, we have $\lambda_1 > 0$. Hence $\frac{1}{\lambda_1}$ is the best constant in the following Hardy-Sobolev inequality

$$\int_{B_1^c} g(x) |\phi(x)|^p dx \leq C \int_{B_1^c} |\nabla \phi(x)|^p dx, \quad \forall \phi \in W^{1,p}(B_1^c)$$

and it is attained.

(b) For $N > p$, Theorem 1.1 holds for any unbounded domain in \mathbb{R}^N and holds for any bounded domain with the additional assumption $\int g dx < 0$. Since $L^{\frac{N}{p}}(B_1^c)$ is strictly contained in $\mathcal{F}_{\frac{N}{p}}$ (Proposition 3.1 of [7]), Theorem 1.1 with the additional assumption $\int g dx < 0$, extends the results of [10, 14, 16].

(c) The spaces $\mathcal{F}_{\frac{N}{p}}$ and $L^1((1, \infty); r^{p-1})$ are not comparable. For $N > p$, we consider the following two functions:

$$g_1(x) = \frac{1}{|x|^q}, \quad \text{for } p < q < \infty; \quad g_2(x) = \begin{cases} 0, & 1 \leq |x| \leq 2; \\ \frac{1}{|x|^p \log |x|}, & |x| > 2. \end{cases}$$

The function $g_1 \in L^1((1, \infty); r^{p-1})$ but does not belong to $L^{\frac{N}{p}, \infty}(B_1^c)$, whereas $g_2 \in L^{\frac{N}{p}}(B_1^c)$ but does not belong to $L^1((1, \infty); r^{p-1})$.

5.1 The existence of an infinite set of eigenvalues

For the existence of a sequence of eigenvalues of (1.1), we use the Ljusternik-Schnirelmann theory on C^1 manifold due to Szulkin [23]. The next theorem follows from Corollary 4.1 of [23].

Theorem 5.2 (Szulkin's Theorem). *Let M be a closed symmetric C^1 submanifold of a real Banach space X and $0 \notin M$. Let $f \in C^1(M, \mathbb{R})$ be even and bounded below. Let*

$$\lambda_j = \inf_{A \in \Gamma_j} \sup_{\phi \in A} f(\phi),$$

and $\Gamma_j = \{A \subset M : A \text{ compact, symmetric and } \gamma(A) \geq j\}$. If $\Gamma_n \neq \emptyset$ for some $n \geq 1$ and f satisfies the $(P. S.)_c$ condition for all $c = \lambda_j$, where $j = 1, 2, \dots, n$, then λ_j are the critical values of f .

Proof of Theorem 1.3.

(i) The set N_g and the functional J satisfy all the properties of Szulkin's theorem. Using the arguments as in the proof of Lemma 5.9 of [5], one can show that, for each $n \in \mathbb{N}$, the set $\Gamma_n = \{A \subset N_g : A \text{ compact, symmetric and } \gamma(A) \geq n\}$ is nonempty. Hence by Theorem 5.2, there exists $\phi_n \in N_g$ such that $\|dJ(\phi_n)\| = 0$ and $J(\phi_n) = \lambda_n$. Therefore, λ_n is an eigenvalue of (1.1) and ϕ_n is an eigenfunction corresponding to λ_n . Further, (λ_n) is unbounded by the same arguments as in the proof of Theorem 2 of [17].

(ii) Let (λ_n) be a sequence of eigenvalues of (1.1) such that $\lambda_n \rightarrow \lambda$. Let ϕ_n be an eigenfunction corresponding to λ_n satisfying $\int_{B_1^c} g|\phi_n|^p = 1$. Thus $J(\phi_n) = \lambda_n$ and $\|dJ(\phi_n)\| = 0$. Hence by Lemma 4.6, there exists a subsequence of (ϕ_n) that converges to ϕ in $W^{1,p}(B_1^c)$. Now the continuity of J' and G' ensures that λ is an eigenvalue of (1.1). \square

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T. V. Anoop

Department of Mathematics,
 Indian Institute of Technology Madras,
 Chennai, 600036, India.
Email: anoop@iitm.ac.in

Nirjan Biswas

Department of Mathematics,
 Indian Institute of Technology Madras,
 Chennai, 600036, India.
Email: nirjaniitm@gmail.com