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M-matrix and inverse M-matrix extensions

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Abstract: A class of matrices that simultaneously generalizes the M-matrices and the inverse M-matrices is brought forward and its properties are reviewed. It is interesting to see how this class bridges the properties of the matrices it generalizes and provides a new perspective on their classical theory.

Keywords: M-matrix, Inverse M-matrix, Mime, P-matrix, Semipositive matrix, Hidden Z-matrix, Principal pivot transform, Positive stable matrix, Group inverse

MSC: 15A48, 15A23, 15A09, 15A18, 90C33

1 Introduction

In the study and applications of the Linear Complementarity Problem [3], the solution A of a matrix equation of the form $AX = Y$ is of interest; typically, X and Y have particular properties, e.g., they have non-positive off-diagonal entries (Z-matrices), in which case A (under an additional assumption) is designated as a *hidden* Z-matrix. In one particular instance, X and Y are invertible M-matrices with a common nonnegative vector u such that Xu and Yu are also positive. Such matrices A were introduced by Pang [12, 13], who extended notions by Mangasarian, Cottle and Dantzig. They were originally termed 'hidden Minkowski matrices' in [13]. Note then that $A = YX^{-1}$ is the product of an M-matrix and an inverse M-matrix. In that respect, A belongs to a class of matrices that simultaneously generalizes the classes of M-matrices and inverse M-matrices.

To be more specific, recall that an *M-matrix* is a matrix of the form $M = sI - B$, where B is an entrywise nonnegative matrix ($B \geq 0$) and $s \geq \rho(B)$ with $\rho(B)$ denoting the spectral radius of B. The M-matrix M is nonsingular if and only if $s > \rho(B)$. It is known that for every nonsingular M-matrix M, there exists an entrywise nonnegative vector u ($u \geq 0$) such that Mu is entrywise positive ($Mu > 0$); we refer to u as a *semipositivity* vector associated with M. This paper concerns matrices of the form $A = M_1 M_2^{-1}$, where M_1, M_2 are invertible M-matrices possessing a common semipositivity vector u. We adopt a newer name for this class of matrices, coined in [19], indicative of their matricial nature and origin:

Definition 1.1. We call $A \in M_n(\mathbb{R})$ a *mime* if

$$A = (s_1 I - B_1)(s_2 I - B_2)^{-1}, \tag{1.1}$$

where $B_1 \geq 0$, $B_2 \geq 0$, $s_1 > \rho(B_1)$, $s_2 > \rho(B_2)$, and there exists a vector $u \geq 0$ such that

$$(s_1 I - B_1)u > 0, \quad (s_2 I - B_2)u > 0.$$

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Read [12, Abstract and Theorem 2] and compare with [13, Introduction] to see that mimes coincide with Pang's hidden Minkowski matrices. Notice that when $B_2 = 0$ the matrix A in Definition 1.1 is an M-matrix, and when $B_1 = 0$, A is the inverse of an M-matrix. Thus, the class of mimes contains and generalizes the classes of M-matrices and inverse M-matrices, justifying the acronym 'mime'. For comprehensive references on M-matrices and inverse M-matrices; see [1, 6, 8]. In particular, inverse M-matrices are entrywise nonnegative. An immediate observation is that the class of mimes is indeed closed under inversion and permutational similarity.

Whereas mimes have been studied in the pioneering work of Pang and in [19], our aim herein is to review the properties of mimes, further develop their theory, as well as provide a self-contained, matrix-theoretic approach that unifies the theory of M-matrices and their inverses.

The presentation unfolds as follows. Section 2 contains most of the required notation, terminology and notions. The basic properties of mimes are proven in Section 3, including that mimes are semipositive and have positive principal minors. Schur complements, principal submatrices and principal pivot transforms of mimes are also shown to be mimes, and some subclasses of the mimes are identified. Counterexamples to properties that do not generalize from (inverse) M-matrices to mimes are provided in Section 4. In Section 5, the notion of a mime is generalized to allow the first factor be a general M-matrix and the second factor be replaced by the group inverse of a singular M-matrix.

2 Definitions and Notation

We group the contents of this section in several categories for convenience.

General notation

• Entrywise ordering of arrays of the same size is indicated by $\geq$. We write $X \geq Y$ ($X > Y$) if X, Y are real and every entry of $X - Y$ is nonnegative (positive). When $X \geq 0$ (resp., $X > 0$), we refer to X as *nonnegative* (resp., *positive*).

• Given matrices $Y \geq X$, the interval $[X, Y]$ denotes the set of all matrices Z such that $X \leq Z \leq Y$.

Let n be a positive integer and $A \in M_n(\mathbb{C})$. The following notation is used:

• $\langle n \rangle = \{1, \ldots, n\}$.

• For $x \in \mathbb{R}^n$, $x \in \mathbb{R}^n_+$ is equivalent to saying $x \geq 0$.

• $\sigma(A)$ denotes the spectrum of A.

• $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ is the spectral radius of A.

• A is called *positive stable* if $\sigma(A)$ lies in the open right-half complex plane.

• $R(A)$ and $N(A)$ respectively denote the range and nullspace of A.

• $\operatorname{diag}(d_1, \ldots, d_n)$ is the diagonal matrix with diagonal entries $d_1, \ldots, d_n$.

• For $\alpha \subseteq \langle n \rangle$, $|\alpha|$ denotes the cardinality of α and $\overline{\alpha} = \langle n \rangle \setminus \alpha$.

• $A[\alpha, \beta]$ is the submatrix of A whose rows and columns are indexed by $\alpha, \beta \subseteq \langle n \rangle$, respectively; the elements of α, β are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered vacuous and by convention has determinant equal to 1. We abbreviate $A[\alpha, \alpha]$ by $A[\alpha]$. Similar notation is used to denote vector partitions.

Matrix Transforms

Given $A \in M_n(\mathbb{C})$ and $\alpha \subseteq \langle n \rangle$ such that $A[\alpha]$ is invertible, $A/A[\alpha]$ denotes the *Schur complement* of $A[\alpha]$ in A, that is,

$$A/A[\alpha] = A[\overline{\alpha}] - A[\overline{\alpha}, \alpha]A[\alpha]^{-1}A[\alpha, \overline{\alpha}].$$

Definition 2.1. Given a nonempty $\alpha \subseteq \langle n \rangle$ and provided that $A[\alpha]$ is invertible, we define the *principal pivot transform of* $A \in M_n(\mathbb{C})$ *relative to* α as the matrix $\operatorname{ppt}(A, \alpha)$ obtained from A by replacing

$$\begin{array}{llll} A[\alpha] & \text{by } A[\alpha]^{-1}, & A[\alpha, \overline{\alpha}] & \text{by } -A[\alpha]^{-1}A[\alpha, \overline{\alpha}], \\ A[\overline{\alpha}, \alpha] & \text{by } A[\overline{\alpha}, \alpha]A[\alpha]^{-1} & A[\overline{\alpha}] & \text{by } A/A[\alpha]. \end{array}$$

By convention, $\operatorname{ppt}(A, \emptyset) = A$.

Given $A \in M_n(\mathbb{C})$ and $\alpha \subseteq \langle n \rangle$, the matrix $B = \mathrm{ppt}(A, \alpha)$ is the unique linear transformation which for any pair $x, y \in \mathbb{C}^n$ related by $Ax = y$, relates $\hat{x}$ (obtained from x by replacing $x[\alpha]$ by $y[\alpha]$) to $\hat{y}$ (obtained from y by replacing $y[\alpha]$ by $x[\alpha]$) via $B\hat{x} = \hat{y}$. In the case $\alpha = \{1, \ldots, k\}$ $(0 < k < n)$, we have

$$\mathrm{ppt}(A, \alpha) = \begin{bmatrix} A[\alpha]^{-1} & -A[\alpha]^{-1}A[\alpha, \overline{\alpha}] \\ A[\overline{\alpha}, \alpha]A[\alpha]^{-1} & A/A[\alpha] \end{bmatrix} \tag{2.2}$$

and

$$A \begin{bmatrix} x[\alpha] \\ x[\overline{\alpha}] \end{bmatrix} = \begin{bmatrix} y[\alpha] \\ y[\overline{\alpha}] \end{bmatrix} \quad \text{if and only if} \quad B \begin{bmatrix} y[\alpha] \\ x[\overline{\alpha}] \end{bmatrix} = \begin{bmatrix} x[\alpha] \\ y[\overline{\alpha}] \end{bmatrix}.$$

For the properties and history of the principal pivot transform, see [17].

Definition 2.2. For $A \in M_n(\mathbb{C})$ with $-1 \notin \sigma(A)$, consider the fractional linear map $F_A \equiv (I + A)^{-1}(I - A)$. This map is an involution, namely, $A = (I + F_A)^{-1}(I - F_A)$. The matrix F_A is referred to as the *Cayley transform* of A.

The Cayley transform for various matrix positivity classes is treated in [4].

Matrix classes of interest

- We call $A = [a_{ij}] \in M_n(\mathbb{C})$ *row diagonally dominant* if for all $i \in \langle n \rangle$,

  $$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

  Note that in our terminology the diagonal dominance is strict. Due to the Geršgorin Theorem [5, Theorem 6.1.1], row diagonally dominant matrices with positive diagonal entries are positive stable.
- A matrix $A \in M_n(\mathbb{C})$ is *reducible* if $PAP^T = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$ for some permutation matrix P and non-vacuous square matrices B, D. Otherwise, A is *irreducible*.
- $A \in M_n(\mathbb{C})$ is a *P-matrix* if for all $\alpha \in \langle n \rangle$, $\det A[\alpha] > 0$. For real matrices, denoting the Hadamard (entrywise) product by $\circ$, $A \in M_n(\mathbb{R})$ is a P-matrix if and only if

  $$x \in \mathbb{R}^n, \ x \circ (Ax) \leq 0 \implies x = 0.$$

- A singular class of real P-matrices, namely, the *$P_\#$-matrices* was introduced in [10]. The idea is to use the sign non-reversal property of the P-matrices and restrict it to vectors in the range space of A. The precise definition is as follows: Matrix $A \in M_n(\mathbb{R})$ is said to be a $P_\#$-matrix if

  $$x \in R(A), \ x \circ (Ax) \leq 0 \implies x = 0.$$

  It is interesting to note that the group inverse of a $P_\#$-matrix always exists and is also a $P_\#$-matrix. A brief study of $P_\#$-matrices is carried out in [15].
- $A \in M_n(\mathbb{R})$ is *semipositive* if there exists $x \geq 0$ such that $Ax > 0$. We refer to x as a *semipositivity vector* of A. Notice that by continuity of the map $x \mapsto Ax$, semipositivity of A is equivalent to the existence of $u > 0$ such that $Au > 0$.

  The following notions of monotonicity are mostly classical; see [1].
- $A \in M_n(\mathbb{R})$ is *monotone* if $Ax \geq 0 \implies x \geq 0$.
- $A \in M_n(\mathbb{R})$ is *range monotone* if $x \in R(A)$, $Ax \geq 0 \implies x \geq 0$.
- $A \in M_n(\mathbb{R})$ is *almost monotone* if $Ax \geq 0 \implies Ax = 0$. An example of an almost monotone matrix is given by

  $$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

- $A \in M_n(\mathbb{R})$ is called a *Z-matrix* if all of its off-diagonal entries are non-positive. A nonsingular *M-matrix* is a positive stable Z-matrix. An *inverse M-matrix* is the inverse of an M-matrix and is entrywise nonnegative.

- It is known that an M-matrix A takes the form $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B)$. The Perron-Frobenius Theorem applied to B implies that A possesses a nonnegative eigenvector x corresponding to the eigenvalue $s - \rho(B)$. When B is also irreducible, it is known that $s - \rho(B)$ is a simple eigenvalue of A and that $x > 0$.
  All invertible M-matrices and inverse M-matrices are P-matrices, monotone and semipositive. See [1, 5, 6] for general background on nonnegative matrices, Z-matrices and M-matrices.
- $A \in M_n(\mathbb{C})$ is said to be *convergent*, if $\lim_{k\to\infty} A^k$ exists and is equal to zero. Convergence is with respect to any matrix norm. Note that A is convergent if and only if $\rho(A) < 1$.
- $A \in M_n(\mathbb{C})$ is said to be *semiconvergent* if $\lim_{k\to\infty} A^k$ exists.
- An M-matrix $A = sI - B$ with $B \geq 0$ and $s \geq \rho(B)$ is said to have *"property c"* if $\frac{B}{s}$ is semiconvergent. This notion was introduced in [14], where several properties of such matrices were proved. For instance, every invertible M-matrix is semiconvergent. The M-matrix $\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ is not semiconvergent, while $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is an M-matrix with "property c."

3 Basic Properties of Mimes

It has been shown in [12] that the notion of a mime A as in Definition 1.1 is tantamount to A being 'hidden Z' and a P-matrix at the same time. We revisit these facts in Theorem 3.1 and Proposition 3.4 below, providing proofs that use the language and properties of M-matrices and P-matrices.

Theorem 3.1. *Let $A \in M_n(\mathbb{R})$. Then A is a mime if and only if*
(1) *$AX = Y$ for some Z-matrices X and Y, and*
(2) *A and X are semipositive.*

Proof. Clearly, if A is a mime as in Definition 1.1, then (1) holds with the roles of X and Y being played by $(s_2I - B_2)$ and $(s_1I - B_1)$, respectively. That (2) holds follows from the fact that $z = Xu > 0$ (i.e., X is semipositive) and $Yu > 0$, where $u \geq 0$ is a common semipositivity vector associated with A. We then have that $Az = YX^{-1}Xu = Yu > 0$; that is, A is also semipositive.
For the converse, suppose (1) and (2) hold. Then X is an invertible M-matrix and $X^{-1} \geq 0$ [6, Theorem 2.5.3]. As A is assumed semipositive, $Ax > 0$ for some $x > 0$. Let then $u = X^{-1}x$ so that $Yu = Ax > 0$; that is, Y is also semipositive and so an invertible M-matrix as well. In fact, u is a common semipositivity vector of X and Y and thus A is a mime. □

Corollary 3.2. *Let $A \in M_n(\mathbb{R})$ be a mime and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ with $d_k > 0$ $(k = 1, 2, \ldots, n)$. Then the following matrices are also mimes:*

$$(1)\ A + D, \quad (2)\ AD, \quad (3)\ DA.$$

Proof. Since A is a mime, by Theorem 3.1 there exist Z-matrices X and Y such that $AX = Y$ where A and X are semipositive.
To prove (1) note that $(A + D)X = Y + DX$, and $Y + DX$ is a Z-matrix since Z-matrices are closed under addition and multiplication by a positive diagonal matrix. Also note $A + D$ is semipositive since A is semipositive. Thus $A + D$ is a mime.

To prove (2) note that $(AD)(D^{-1}X) = Y$. Now $D^{-1}X$ and Y are both Z-matrices, and AD, $D^{-1}X$ are semipositive. Thus AD is a mime by Theorem 3.1.

To prove (3) note that $(DA)X = DY$ where DY is a Z-matrix and DA is semipositive. Thus DA is a mime. □

Corollary 3.3. *Let $A \in M_n(\mathbb{R})$ be a mime and $P \in M_n(\mathbb{R})$ a permutation matrix. Then PAP^T is also a mime.*

Proof. Since A is a mime by Theorem 3.1 there exist Z-matrices X and Y such that $AX = Y$ where A and X are semipositive. Then

$$PAP^T(PXP^T) = PYP^T,$$

where clearly PXP^T, PYP^T are Z-matrices, and PAP^T, PXP^T are indeed semipositive. Thus PAP^T is a mime by Theorem 3.1. □

Proposition 3.4. *Let $A \in M_n(\mathbb{R})$ be a mime. Then A is a P-matrix.*

Proof. Let A be a mime and s_1, s_2, B_1, B_2 and $u \geq 0$ be as in Definition 1.1. Then, for every $T \in [0, I]$, the matrix

$$C = T(s_2 I - B_2) + (I - T)(s_1 I - B_1)$$

is a Z-matrix and $Cu > 0$ (i.e., C is a semipositive Z-matrix). This means that C is an M-matrix. In particular, C is invertible for every $T \in [0, I]$. By [9, Theorem 3.3], we can now conclude that $A = (s_1 I - B_1)(s_2 I - B_2)^{-1}$ is a P-matrix. □

In order to argue next that every principal submatrix of a mime is a mime, we will need the following two known lemmata with proofs provided for completeness.

Lemma 3.5. *Let $A \in M_n(\mathbb{R})$ be a nonsingular M-matrix and let $\alpha \subseteq \langle n \rangle$. Then the Schur complement $A/A[\alpha]$ is a nonsingular M-matrix.*

Proof. Let $A = sI - B \in M_n(\mathbb{R})$ be a nonsingular M-matrix ($B \geq 0$, $s > \rho(B)$) and let $\alpha \subseteq \langle n \rangle$. Note that $A[\overline{\alpha}, \alpha] \leq 0$ and $A[\alpha, \overline{\alpha}] \leq 0$. Note also that $A[\alpha] = (sI - B)[\alpha]$ is a nonsingular M-matrix since $s > \rho(B) \geq \rho(B[\alpha])$), the latter inequality following by the monotonicity of the spectral radius in the order of nonnegative matrices; see [5, Corollary 8.1.20]). Thus $A[\alpha]^{-1} \geq 0$ and so

$$A/A[\alpha] = A[\overline{\alpha}] - A[\overline{\alpha}, \alpha]A[\alpha]^{-1}A[\alpha, \overline{\alpha}]$$

is a Z-matrix. $A/A[\alpha]$ is also a P-matrix because A is a P-matrix and Schur complementation preserves P-matrices; see, e.g., [8, Theorem 4.3.2]). Thus, $A/A[\alpha]$ is a nonsingular M-matrix. □

Lemma 3.6. *Suppose $A \in M_n(\mathbb{R})$ is a nonsingular M-matrix with a semipositivity vector u. Let $\alpha \subseteq \langle n \rangle$. Then $A[\alpha]$ is a nonsingular M-matrix with semipositivity vector $u[\alpha]$.*

Proof. Let $A = [a_{ij}]$ be a nonsingular M-matrix with a semipositivity vector u and let $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \langle n \rangle$. Note that $A[\alpha]$ is also a nonsingular M-matrix as argued in the proof of Lemma 3.5. Also, note that

$$A[\alpha]u[\alpha] = \begin{bmatrix} a_{\alpha_1,\alpha_1}u_{\alpha_1} + a_{\alpha_1,\alpha_2}u_{\alpha_2} + \cdots + a_{\alpha_1,\alpha_k}u_{\alpha_k} \\ \vdots \\ a_{\alpha_k,\alpha_1}u_{\alpha_1} + a_{\alpha_k,\alpha_2}u_{\alpha_2} + \cdots + a_{\alpha_k,\alpha_k}u_{\alpha_k} \end{bmatrix}.$$

Since $a_{ij} \leq 0$ if $i \neq j$, $u > 0$, and $Au > 0$, we get that

$$A[\alpha]u[\alpha] \geq \begin{bmatrix} a_{\alpha_1,1}u_1 + a_{\alpha_1,2}u_2 + \cdots + a_{\alpha_1,n}u_n \\ \vdots \\ a_{\alpha_k,1}u_1 + a_{\alpha_k,2}u_2 + \cdots + a_{\alpha_k,n}u_n \end{bmatrix} > 0.$$

Hence, $u[\alpha]$ is a semipositivity vector for $A[\alpha]$. □

Theorem 3.7. *Let $A \in M_n(\mathbb{R})$ be a mime. Then every principal submatrix of A is also a mime.*

Proof. Let $A \in M_n(\mathbb{R})$ be a mime and $\alpha \subseteq \langle n \rangle$. Then $A = M_1 M_2^{-1}$ for some nonsingular M-matrices M_1 and M_2 that share a common semipositivity vector u. Note that

$$\begin{bmatrix} A[\alpha] & A[\alpha, \overline{\alpha}] \end{bmatrix} \begin{bmatrix} M_2[\alpha] & M_2[\alpha, \overline{\alpha}] \\ M_2[\overline{\alpha}, \alpha] & M_2[\overline{\alpha}] \end{bmatrix} = \begin{bmatrix} M_1[\alpha] & M_1[\alpha, \overline{\alpha}] \end{bmatrix}.$$

Hence it follows that

$$A[\alpha]M_2[\alpha] + A[\alpha,\overline{\alpha}]M_2[\overline{\alpha},\alpha] = M_1[\alpha] \tag{3.3}$$

and

$$A[\alpha]M_2[\alpha,\overline{\alpha}] + A[\alpha,\overline{\alpha}]M_2[\overline{\alpha}] = M_1[\alpha,\overline{\alpha}]. \tag{3.4}$$

Upon multiplying (3.4) from the right by $M_2[\overline{\alpha}]^{-1}M_2[\overline{\alpha},\alpha]$ and subtracting the outcome from (3.3), we obtain $A[\alpha]X = Y$, where

$$X = M_2/M_2[\overline{\alpha}] \quad \text{and} \quad Y = M_1[\alpha] - M_1[\alpha,\overline{\alpha}]M_2[\overline{\alpha}]^{-1}M_2[\overline{\alpha},\alpha].$$

Notice that

$$X = M_2/M_2[\overline{\alpha}] \quad \text{and} \quad Y = W/W[\overline{\alpha}],$$

where

$$W = \begin{bmatrix} M_1[\alpha] & M_1[\alpha,\overline{\alpha}] \\ M_2[\overline{\alpha},\alpha] & M_2[\overline{\alpha}] \end{bmatrix}$$

is evidently a Z-matrix. As $M_1u > 0$ and $M_2u > 0$, we have $Wu > 0$ and so W is also an invertible M-matrix. From Lemma 3.5, it follows that X and Y are nonsingular M-matrices.

To conclude the proof, we have shown that $A[\alpha]X = Y$, where X and Y are, in particular, Z-matrices. By Proposition 3.4, A is P-matrix and so $A[\alpha]$ is also a P-matrix and thus it is semipositive; see e.g., [8, Theorem 4.3.6]. Finally, X is semipositive since it is a nonsingular M-matrix. It now follows from Theorem 3.1 that $A[\alpha]$ is a mime. □

Given a mime A and any $\alpha \subseteq \langle n\rangle$, it is clear from Theorem 3.7 that $A[\alpha]$ is invertible and so $\mathrm{ppt}(A,\alpha)$ is well-defined. It is observed in [12, 17] that $\mathrm{ppt}(A,\alpha)$ is also a mime. We include a proof of this result below that utilizes the language and the observations herein and in [9].

Theorem 3.8. *Let $A \in M_n(\mathbb{R})$ be a mime and $\alpha \subseteq \langle n\rangle$. Then $ppt(A,\alpha)$ is a mime.*

Proof. Let A be a mime as in Definition 1.1 and denote $Y = (s_1I - B_1)$, $X = (s_2I - B_2)$ so that $AX = Y$. Let u be a common semipositivity vector of X, Y. Let T be the matrix obtained from the identity by setting the diagonal entries indexed by $\alpha \subseteq \langle n\rangle$ equal to 0 and consider the matrices

$$U = TX + (I-T)Y, \quad V = (I-T)X + TY.$$

That is, U and V are obtained from X and Y, respectively, by exchanging their rows indexed by α. Thus, on letting $B = \mathrm{ppt}(A,\alpha)$, we observe that $AX = Y$ implies $BU = V$, where U and V are Z-matrices satisfying $Uu > 0$ and $Vu > 0$. It follows that $B = VU^{-1}$ is a mime. □

Corollary 3.9. *Let $\alpha \subseteq \langle n\rangle$. If $A \in M_n(\mathbb{R})$ is a mime, then the Schur complement $A/A[\alpha]$ is also a mime.*

Proof. This follows from Theorem 3.7, Theorem 3.8 and the fact that $A/A[\alpha]$ is a principal submatrix of $\mathrm{ppt}(A,\alpha)$. □

We proceed with more relations among mimes and other matrix classes. Included is another result from [12] and a method for constructing entrywise nonnegative mimes, whose original proofs in [12] rely on (hidden) Leontief matrices and principal pivot transforms. In particular, the proof of the latter construction method is attributed to Mangasarian [11], who also used ideas from mathematical programming. The proofs presented below are shorter and based on standard P-matrix and M-matrix theory.

It is known that P-matrices are semipositive; see e.g., [8, Theorem 4.3.6]. As a consequence of Proposition 3.4, we thus have that mimes are semipositive. A direct proof is offered below that also identifies a semipositivity vector.

Corollary 3.10. *Let $A \in M_n(\mathbb{R})$ be a mime. Then A is semipositive.*

Proof. Let A be a mime as in Definition 1.1 with common semipositivity vector $u \geq 0$. Letting $y = (s_2 I - B_2)u > 0$, we have $Ay = (s_1 I - B_1)u > 0$. □

Semipositivity of mimes leads to an interesting conclusion, especially in light of Theorem 3.1 and the definition of a mime. Namely, a mime is the product of an M-matrix and an inverse M-matrix, but also the product of a positive matrix and an inverse positive matrix.

Theorem 3.11. *Let $A \in M_n(\mathbb{R})$ be a mime. Then there exists a positive matrix Y and an invertible positive matrix X such that $A = YX^{-1}$.*

Proof. By [18, Theorem 3.1], A is semipositive if and only if there exists a positive matrix Y and an invertible positive matrix X such that $A = YX^{-1}$. The result now follows by Corollary 3.10. □

Theorem 3.12. *Let $A \in M_n(\mathbb{R})$ be a mime. Then A can be factored into $A = BC^{-1}$, where $B, C \in M_n(\mathbb{R})$ are row diagonally dominant matrices with positive diagonal entries.*

Proof. Suppose that $A = (s_1 I - B_1)(s_2 I - B_2)^{-1}$ is a mime as in Definition 1.1. By continuity, the common semipositivity vector u can be taken to be positive, i.e., $u > 0$. Let then $D = \mathrm{diag}(u_1, u_2, \ldots, u_n)$ and define $B = (s_1 I - B_1)D$ and $C = (s_2 I - B_2)D$ so that $Be > 0$ and $Ce > 0$, where e is the all ones vector. Notice that as B and C are Z-matrices with positive diagonal entries, they are indeed row diagonally dominant and $A = BC^{-1}$. □

Note that M-matrices are monotone, i.e., $Ax \geq 0$ implies that $x \geq 0$ by [6, 2.5.3]. Inverse M-matrices, and thus mimes, are not necessarily monotone and it is easy to produce counterexamples (e.g., by considering mimes that are inverse M-matrices). However, the following generalization of monotonicity holds for mimes.

Proposition 3.13. *Let $A \in M_n(\mathbb{R})$ be a mime as in Definition 1.1 and $x \in \mathbb{R}^n$. If $Ax \geq 0$, then $x \in (s_2 I - B_2)\mathbb{R}^n_+$.*

Proof. Let A be a mime as in Definition 1.1 and suppose

$$Ax = (s_1 I - B_1)(s_2 I - B_2)^{-1} x \geq 0.$$

As $(s_1 I - B_1)$ is an M-matrix and thus monotone, we have that $y = (s_2 I - B_2)^{-1} x \geq 0$. That is, $x = (s_2 I - B_2)y$ with $y \geq 0$. □

Next, we show that the converse of Proposition 3.13 is not true.

Example 3.14. Let $B_1 = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $s_1 = 2$ and $s_2 = 2$. Then $s_2 I - B_2$ is an invertible M-matrix. Let $A = (s_1 I - B_1)(s_2 I - B_2)^{-1} = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$. Then A is not a P-matrix and so by Proposition 3.4, A is not a mime. We claim that

$$Ax \geq 0 \Longrightarrow x \in (s_2 I - B_2)\mathbb{R}^n_+.$$

Let $Ax \geq 0$. Then $-x_1 - 2x_2 \geq 0$ and $x_1 + x_2 \geq 0$. Adding, we get $x_2 \leq 0$. Also, $x_1 \geq -x_2 \geq 0$. Thus, $Ax \geq 0 \Longrightarrow x_2 \leq 0$, $x_1 + x_2 \geq 0$ and $x_1 \geq 0$. Now, set $u = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} \geq 0$. Then

$$(s_2 I - B_2)u = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x,$$

showing that $x \in (s_2 I - B_2)\mathbb{R}^n_+$, as was claimed.

Nevertheless, we have the following characterization.

Proposition 3.15. *Let $A \in M_n(\mathbb{R})$ and let $s_2 I - B_2$ be invertible. Then*

$$Ax \geq 0 \Longrightarrow x \in (s_2 I - B_2)\mathbb{R}^n_+.$$

if and only if A is invertible and $(s_2I - B_2)^{-1}A^{-1} \geq 0$.

Proof. First, we show the forward implication. Let $Ax = 0$. Then $x = (s_2I - B_2)u$, $u \geq 0$. We have $A(-x) = 0$. So $-x = (s_2I - B_2)v$, $v \geq 0$.
It follows by invertibility of $s_2I - B_2$ that $u = -v$ and so $u = 0$. Thus $x = 0$, proving that A is invertible.

Next, let $x \geq 0$ and $y = (s_2I - B_2)^{-1}A^{-1}x$. Then $A(s_2I - B_2)y = x \geq 0$ and so $(s_2I - B_2)y \in (s_2I - B_2)\mathbb{R}^n_+$. By the invertibility of $s_2I - B_2$, one has $y \geq 0$. Since a matrix maps every $x \geq 0$ into a nonnegative vector if and only if it is itself nonnegative, we get $(s_2I - B_2)^{-1}A^{-1} \geq 0$ as desired.

Conversely, suppose that $Ax \geq 0$. Then $0 \leq u = (s_2I - B_2)^{-1}A^{-1}Ax = (s_2I - B_2)^{-1}x$. So $x = (s_2I - B_2)u \in (s_2I - B_2)\mathbb{R}^n_+$. ◻

Definition 3.16. The *semipositive cone* of a semipositive matrix $A \in M_n(\mathbb{R})$ is defined as

$$K_+(A) = \{x \in \mathbb{R}^n : x \geq 0 \text{ and } Ax \geq 0\}.$$

This set is studied in [16] and [18], where it is shown (see [16, Corollary 3.4]) that $K_+(A)$ is a proper polyhedral cone of $\mathbb{R}^n$. Also, when A is invertible,

$$AK_+(A) = K_+(A^{-1}).$$

The following holds for mimes.

Theorem 3.17. *Let $A = M_1M_2^{-1} \in M_n(\mathbb{R})$ be a mime, where M_1 and M_2 are invertible M-matrices. Then*

$$K_+(A) = M_2K_+(M_1) \cap \mathbb{R}^n_+.$$

Proof. Let $A = M_1M_2^{-1}$ and let $x \in M_2K_+(M_1) \cap \mathbb{R}^n_+$. Note that $M_2^{-1}x \in K_+(M_1)$ and thus $Ax = M_1M_2^{-1}x \geq 0$. Also, since $x \in \mathbb{R}^n_+$, we have $x \geq 0$ and hence $x \in K_+(A)$. Thus, $M_2K_+(M_1) \cap \mathbb{R}^n_+ \subseteq K_+(A)$.
Now assume $x \in K_+(A)$. Thus, $x \geq 0$ and $Ax \geq 0$. Clearly, $x \in \mathbb{R}^n_+$. Next notice $M_1M_2^{-1}x \geq 0$. Further, since $M_2^{-1} \geq 0$, $M_2^{-1}x \in K_+(M_1)$ and hence $x \in M_2K_+(M_1)$. Thus, we have shown that $x \in M_2K_+(M_1) \cap \mathbb{R}^n_+$. ◻

Next, we turn our attention to necessary or sufficient conditions for a matrix to be a mime, starting with the Cayley transform. The following result fits into the investigations done in [4].

Theorem 3.18. *Let $A \in M_n(\mathbb{R})$ be a mime and let $F = C(A)$ be the Cayley transform of A. Then $I + F$ and $I - F$ are mimes.*

Proof. Suppose that $A \in M_n(\mathbb{R})$ is a mime. Notice that since A is a P-matrix by Proposition 3.4, -1 is not an eigenvalue of A (see [8, Theorem 4.4.2]) and hence $F = C(A) = (I + A)^{-1}(I - A)$ exists. Next, notice that

$$\begin{aligned} I + F &= I + (I + A)^{-1}(I - A) \\ &= (I + A)^{-1}(I + A) + (I + A)^{-1}(I - A) \\ &= 2(I + A)^{-1} \end{aligned}$$

and

$$\begin{aligned} I - F &= (I + A)^{-1}(I + A) - (I + A)^{-1}(I - A) \\ &= 2(I + A)^{-1}A \\ &= 2(I + A^{-1})^{-1}. \end{aligned}$$

It follows from Corollary 3.2, and the fact that inverses of mimes are mimes, that $I + F$ and $I - F$ are mimes. ◻
The following is a large class of mimes mentioned by Pang [13, p. 238] for which we give a different proof.

Theorem 3.19. *Let $A \in M_n(\mathbb{R})$ be a triangular P-matrix. Then A is a mime.*

Proof. We prove the claim by induction on the order n of A. If $n = 1$ the result is obviously true. Assume the claim is true for $n = k - 1$; we will prove it for $n = k$. For this purpose, consider the triangular P-matrix

$$A = \begin{bmatrix} A_{11} & a \\ 0 & a_{22} \end{bmatrix},$$

where A_{11} is a $(k-1) \times (k-1)$ P-matrix, $a \in \mathbb{R}^{k-1}$ and $a_{22} > 0$. By the inductive hypothesis and Theorem 3.1, there exist Z-matrices X_{11}, Y_{11} and nonnegative vector $u_1 \in \mathbb{R}^{k-1}$ such that $A_{11}X_{11} = Y_{11}$, $Y_{11}u_1 > 0$, and $X_{11}u_1 > 0$. Consider then the Z-matrix

$$X = \begin{bmatrix} X_{11} & -X_{11}u_1 \\ 0 & x_{22} \end{bmatrix},$$

where $x_{22} > 0$ is to be chosen. Then let

$$Y = AX = \begin{bmatrix} A_{11}X_{11} & -Y_{11}u_1 + x_{22}a \\ 0 & a_{22}x_{22} \end{bmatrix}.$$

Notice that $x_{22} > 0$ can be chosen so that Y is a Z-matrix. Let also $u^T = [u_1^T \ u_2]$. Choosing $u_2 > 0$ small enough, we have that $Xu > 0$ and $Yu > 0$. Thus A is a mime by Theorem 3.1. The case of a lower triangular matrix is similar. □

Remark 3.20. In view of Proposition 3.4 and Theorem 3.19, it is natural to wonder if every P-matrix is a mime. The answer is in the negative, as shown by the construction of a counterexample in Pang [13]; see Example 4.2.

Next is a remarkable method for constructing nonnegative mimes. A proof can be found in [12, Corollary 3] and is attributed to Mangasarian [11] who used ideas from mathematical programming. We include here a shorter proof that is based on standard M-matrix and P-matrix theory.

Proposition 3.21. *Let $B \geq 0$ with $\rho(B) < 1$. Let $\{a_k\}_{k=1}^m$ be a sequence such that $0 \leq a_{k+1} \leq a_k \leq 1$ for all $k = 1, \dots, m-1$. Then*

$$A = I + \sum_{k=1}^{m} a_k B^k$$

is a mime. If m is infinite, under the additional assumption that $\sum_{k=1}^{\infty} a_k$ is convergent, we can still conclude that A is a mime.

Proof. Consider the matrix $C = A(I - B)$ and notice that C can be written as

$$C = I - G, \ \text{ where } \ G \equiv B - \sum_{k=1}^{m} a_k(B^k - B^{k+1}).$$

First we show that G is nonnegative: Indeed, as $0 \leq a_{k+1} \leq a_k \leq 1$, we have

$$\begin{aligned} G \quad &\geq \quad B - \sum_{k=1}^{m} a_k B^k + \sum_{k=1}^{m-1} a_{k+1} B^{k+1} + a_m B^{m+1} \\ &= \quad B - \sum_{k=1}^{m} a_k B^k + \sum_{k=2}^{m} a_k B^k + a_m B^{m+1} \\ &= \quad B - a_1 B + a_m B^{m+1} \ = \ (1 - a_1)B + a_m B^{m+1} \ \geq \ 0. \end{aligned}$$

Next we show that $\rho(G) < 1$. For this purpose, consider the function

$$\begin{aligned} g(z) \quad &= \quad z - \sum_{k=1}^{m} a_k(z^k - z^{k+1}) \\ &= \quad z(1 - a_1) + z^2(a_1 - a_2) + \dots + z^m(a_{m-1} - a_m) + a_m z^{m+1}, \end{aligned}$$

in which all the coefficients are by assumption nonnegative. Thus $|g(z)| \le g(|z|)$. However, for $|z| < 1$, we have

$$g(|z|) = |z| - \sum_{k=1}^{m} a_k(|z|^k - |z|^{k+1}) \le |z|.$$

That is, for all $|z| < 1$,

$$|g(z)| \le g(|z|) \le |z|.$$

We can now conclude that for every $\lambda \in \sigma(B)$,

$$|g(\lambda)| \le |\lambda| \le \rho(B) < 1;$$

that is, $\rho(G) < 1$. We have thus shown that

$$A = (I - G)(I - B)^{-1},$$

where $B,\ G \ge 0$ and $\rho(B) < 1,\ \rho(G) < 1$. Also, as $B \ge 0$, we may consider its eigenvector $u \ge 0$ corresponding to $\rho(B)$, as guaranteed by the Perron-Frobenius Theorem. By construction, u is also an eigenvector of G corresponding to $\rho(G)$, because $\sigma(G) = \{g(\lambda)\ :\ \lambda \in \sigma(G)\}$, as is well known [5, Theorem 1.1.6]. That is, there exists a vector $u \ge 0$ such that

$$Bu = \rho(B)\,u < u \quad \text{and} \quad Gu = \rho(G)\,u < u\,.$$

Thus A is a mime. It is clear how to adapt this proof to the case that m is infinite. □

Proposition 3.22. *Let $B \ge 0$. Then e^{tB} is a mime for every $t \in [0, 1/\rho(B))$.*

Proof. It follows from Proposition 3.21, taking $m = \infty$ and $a_k = \frac{1}{k!}$. □
We conclude this section by a special case of mimes.

Proposition 3.23. *Let $A = (s_1 I - B)\,(s_2 I - B)^{-1}$, where $B \ge 0$ and $s_1,\ s_2 > \rho(B)$. The following hold:*
(a) *If $s_1 = s_2$, then $A = I$ is both an M-matrix and an inverse M-matrix.*
(b) *If $s_1 > s_2$, then A is an inverse M-matrix.*
(c) *If $s_1 < s_2$, then A is an M-matrix.*

Proof. Observe that

$$\begin{aligned} A &= (s_1 I - B)(s_2 I - B)^{-1} \\ &= (s_1 I - s_2 I + s_2 I - B)(s_2 I - B)^{-1} \\ &= (s_1 - s_2)(s_2 I - B)^{-1} + I \end{aligned}$$

(a) If $s_1 = s_2$, then $A = I$.
(b) Since $(s_2 I - B)^{-1}$ is an inverse M-matrix, which is preserved under positive scaling and positive diagonal shift (see [6, pp. 119-120] or [8, Section 5.2]), we conclude that A is also an inverse M-matrix.
(c) Applying the same argument as in (b) to A^{-1}, it follows that A^{-1} is an inverse M-matrix and hence A is an M-matrix. □

Remark 3.24. The original designation of a mime A as a ‘hidden Minkowski matrix’ is surely indicative of the non-trivial effort required to detect them, namely, discover whether or not $AX = Y$ for some invertible M-matrices X and Y. In that respect, a finite procedure to recognize mimes is proposed in [13].

4 Counterexamples

In this section, we use counterexamples to examine certain features of mimes, as well as to disprove some properties that hold for either M-matrices or inverse M-matrices, but not generally for mimes.

It is evident in our analysis that the common semipositivity vector in the definition of a mime plays a crucial role in establishing many of its properties. What if we remove that condition? First, the conclusion that the product of an M-matrix and an inverse M-matrix is a P-matrix may fail, as shown in the following example.

Example 4.1 (Necessity of a common semipositivity vector).
Let $A = (s_1 I - B_1)(s_2 I - B_2)^{-1}$, where $B_1 = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Take $s_1 = 3 > 2 = \rho(B_1)$ and $s_2 = 2 > 1 = \rho(B_2)$. Clearly $s_1 I - B_1$ and $s_2 I - B_2$ do not have a common semipositivity vector. In this case, $A = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$ is not a P-matrix.

Example 4.2 (Counterexample to every P-matrix being a mime).
In Pang [13, p. 239], the following matrix which is a P-matrix but not a mime is given.

$$M = \begin{bmatrix} 2 & -1 & 2 \\ -3 & 2 & -2 \\ -4 & 3 & 5 \end{bmatrix}.$$

Example 4.3 (Counterexample to Hadamard-Fischer for mimes).
The Hadamard-Fischer inequalities hold for $A \in M_n(\mathbb{R})$ if for every $\alpha \subseteq \langle n \rangle$,

$$\det(A) \le \det(A[\alpha]) \det(A[\overline{\alpha}]).$$

M-matrices and inverse M-matrices satisfy these inequalities [6, Theorem 2.5.4 and Exercise 9, p. 127]. Let now

$$B_1 = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and consider

$$A = (5I - B_1)(3I - B_2)^{-1} = \begin{bmatrix} 1.5 & -1 & -0.125 & 0.125 \\ 2.25 & 2.5 & 1.6875 & 1.3125 \\ 1 & 0 & 2.25 & 0.75 \\ 1.5 & 0 & 1.375 & 2.625 \end{bmatrix}.$$

Since $\rho(B_1) = 2.855 < 5$ and $\rho(B_2) = 2.4142 < 3$, and because for $u = [\,1.01 \quad 1.01 \quad 1 \quad 1\,]^T$ we have

$$(5I - B_1)\,u = \begin{bmatrix} 0.02 \\ 3.04 \\ 2.99 \\ 3.99 \end{bmatrix} > 0 \quad \text{and} \quad (3I - B_2)\,u = \begin{bmatrix} 0.02 \\ 0.01 \\ 0.99 \\ 0.99 \end{bmatrix} > 0,$$

we conclude that A is a mime. However, for $\alpha = \{2, 3\}$, we have

$$\det(A) - \det(A[\alpha]) \det(A[\overline{\alpha}]) = 23.9375 - (5.625 \times 3.75) > 0.$$

Thus the Hadamard-Fischer inequality does not generally hold for mimes.

Example 4.4 (Scalability to diagonally dominance).
Let $A \in M_n(\mathbb{R})$ be an invertible M-matrix. It is known that the columns of A can be scaled by positive numbers so that the resulting matrix is row diagonally dominant; i.e., there exists diagonal matrix D with positive diagonal entries such that AD is row diagonally dominant. As a consequence, we then have that its inverse $D^{-1}A^{-1} = [c_{ij}]$ is an inverse M-matrix that is (strictly) *diagonally dominant of its column entries*, namely, $|c_{ii}| >$

$|c_{ji}|$ for all $i = 1, 2, \ldots, n$ and all $j \neq i$; see [6, Definition 2.5.11 and Theorem 2.5.12]. In the following example we display a mime A such that AD is not row diagonally dominant for any diagonal matrix D with positive diagonal entries. Nevertheless, A^{-1} is strictly diagonally dominant of its column entries.

Let

$$B_1 = \begin{bmatrix} 1 & 8 & 4 & 6 \\ 8 & 9 & 9 & 3 \\ 4 & 6 & 6 & 1 \\ 8 & 3 & 8 & 1 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 3 & 6 & 5 & 4 \\ 9 & 6 & 10 & 1 \\ 2 & 8 & 2 & 7 \\ 2 & 2 & 10 & 8 \end{bmatrix}.$$

The spectral radii are aprroximately $\rho(B_1) = 21.75$ and $\rho(B_2) = 21.37$. Let $s_1 = 25$ and $s_2 = 21.5$ and $u = [5 \quad 6.8 \quad 5.44 \quad 5.9]^T$. It can be shown that $(s_1 I - B_1)u > 0$ and $(s_2 I - B_2)u > 0$. Thus $A = (s_1 I - B_1)(s_2 I - B_2)^{-1}$ is a mime. Using Matlab, we find that

$$A = \begin{bmatrix} 3.2493 & 2.7460 & 3.2780 & 2.4214 \\ 0.6437 & 1.8610 & 0.9705 & 0.6096 \\ 9.4889 & 13.4914 & 16.6768 & 12.3841 \\ 9.0804 & 13.0951 & 15.5598 & 13.5063 \end{bmatrix}.$$

Note that $A = [a_{ij}]$ satisfies $a_{ii} < \sum_{j \neq i} a_{ij}$ for all $i \in \{1, 2, 3, 4\}$. Thus there cannot exist a diagonal matrix D with positive diagonal entries such that AD is a row diagonally dominant matrix. To see this, let $F = AD = [f_{ij}]$, where $D = \operatorname{diag}(d_1, d_2, d_3, d_4)$, $d_j > 0$ for $j = 1, 2, 3, 4$. Let $d_m = \min\{d_1, d_2, d_3, d_4\}$. Then

$$|f_{mm}| = f_{mm} = d_m a_{mm} < \sum_{j \neq m} d_m a_{mj} \leq \sum_{j \neq m} d_j a_{mj} = \sum_{j \neq m} f_{mj} = \sum_{j \neq m} |f_{mj}|.$$

We also find that

$$A^{-1} = \begin{bmatrix} 0.7265 & -0.0641 & -0.1401 & 0.0011 \\ -0.0687 & 0.9112 & -0.0875 & 0.0514 \\ -0.3083 & -0.5303 & 0.5364 & -0.4127 \\ -0.0667 & -0.2294 & -0.4390 & 0.4989 \end{bmatrix},$$

which is strictly diagonally dominant of its column entries.

Example 4.5 (Counterexample to positive stability).
The eigenvalues of an M-matrix and its inverse lie in the open right-half of the complex plane; that is every M-matrix and its inverse is a positive stable matrix. This is not true, in general, for mimes. To see this, consider B_1 to be the adjacency matrix of the directed 5-cycle and $B_2 = B_1^T$; that is,

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\rho(B_1) = \rho(B_2) = 1$. Let $s_1 = s_2 = 1.1$. Then $e = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$ serves as the common semipositivity vector for $(s_1 I - B_1)$ and $(s_2 I - B_2)$. Thus $A = (s_1 I - B_1)(s_2 I - B_2)^{-1}$ is a mime. However, the eigenvalues of A are approximately $\lambda_1 = 1, \lambda_2 = 0.826 - 0.562i, \lambda_3 = 0.826 + 0.562i, \lambda_4 = -0.182 - 0.983i$ and $\lambda_5 = -0.182 + 0.983i$, so that λ_4 and λ_5 lie on the left half-plane. Hence this mime A is not positive stable.

Remark 4.6. Note that, in contrast to Example 4.5, every symmetric mime is indeed positive stable. This follows from the facts that, by Theorem 3.4, every mime is a P-matrix and that symmetric P-matrices are indeed positive definite and hence positive stable [5, Theorem 7.2.1]. Another instance of a positive stable mime A is when $B_1 = B_2$. Then, by Proposition 3.23, A is either an M-matrix or an inverse M-matrix and therefore A is positive stable [6, Theorem 2.5.3].

5 Generalizing Mimes By Allowing Singularity

To further our theoretical understanding of mimes, we initiate a generalization in which the M-matrix factors in the definition of a mime are not (necessarily) invertible. We delay the formal definition of “singular mimes" until Definition 5.6 in order to introduce notation, notions and results that justify our considerations.

– For a singular matrix or even a rectangular matrix $A \in M_{m,n}(\mathbb{R})$, there are many ways that a generalized inverse for A has been defined in the literature. The Moore-Penrose inverse of A, is the unique solution $X \in M_{n,m}(\mathbb{R})$ that satisfies the equations $AXA = A, XAX = X, (AX)^T = AX$ and $(XA)^T = XA$. However, the most pertinent generalized inverse for our discussion is the group inverse. The *group inverse* of $A \in M_n(\mathbb{R})$, if it exists, is the unique solution X to the matrix equations $AXA = A, XAX = X$ and $XA = AX$. When the group inverse exists, it is denoted by $A^\#$. A necessary and sufficient condition for $A^\#$ to exist is that $\text{rank}(A^2) = \text{rank}(A)$. This is equivalent to the statement $N(A^2) = N(A)$. For instance, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then $A^\#$ exists and is equal to A, whereas if

$$A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

then $A^\#$ does not exist. By using, for instance, the core-nilpotent decomposition of a matrix, it can be shown that the group inverse of a matrix is a polynomial in that matrix.
The following are useful facts. We refer the reader to [2] for proofs and more details.
– Let E be an idempotent matrix (i.e., $E^2 = E$). Then $E^\#$ exists and $E^\# = E$. In particular, if A is a square matrix such that $A^\#$ exists, and if $B = AA^\#$, then B is idempotent and so one has $B^\# = B$. One of the results proved in [15], that will be useful later, is that every idempotent matrix E is a $P_\#$-matrix. For the sake of completeness, let $x \in R(E)$ so that $x = Ey$ for some $y \in \mathbb{R}^n$. Now $Ex = E^2y = Ey = x$. Thus $x_i(Ex)_i = x_i^2 \le 0$ for all i implies $x = 0$.
– The following hold: $R(A) = R(AA^\#) = R(A^\#)$ and $N(A) = N(A^\#A) = N(A^\#)$. Also, $AA^\#x = x$ if and only if $x \in R(A)$.

Remark 5.1.
(a) Invertible matrices are not almost monotone. This is because a matrix is almost monotone if and only if $R(A) \cap \mathbb{R}^n_+ = \{0\}$. But for an invertible matrix $R(A) \cap \mathbb{R}^n_+ = \mathbb{R}^n_+$.
(b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then A is a group invertible matrix that is not almost monotone. If $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, then A is an almost monotone matrix whose group inverse does not exist, since $A^2 = 0$.

The following result will be needed subsequently.

Lemma 5.2. *Let $A \in M_n(\mathbb{R})$ be almost monotone. The following hold:*
(a) *If $A^\#$ exists, then $A^\#$ and $AA^\#$ are almost monotone.*
(b) *If $C \in M_n(\mathbb{R})$ is invertible and $C^{-1} \ge 0$, then CA is almost monotone.*

Proof. (a) Let A be almost monotone and let $A^\#y \ge 0$. Then

$$0 \le A^\#y = A^\#AA^\#y = A(A^\#A^\#y).$$

By almost monotonicity of A, we then have $AA^\#A^\#y = 0$. Rearranging, one has $A^\#y = A^\#AA^\#y = 0$, proving that $A^\#$ is almost monotone. Furthermore, suppose that $AA^\#x \ge 0$. If we set $y = A^\#x$, then $Ay \ge 0$ and so $Ay = 0$. Thus $AA^\#x = 0$, proving the almost monotonicity of $AA^\#$.

(b) Let $CAy \geq 0$. Then $Ay \geq 0$, since $C^{-1} \geq 0$. Since A is almost monotone, $Ay = 0$ and hence $CAy = 0$. □

Next, we recall certain notions that are useful in understanding the properties that the group inverse of a singular irreducible M-matrix possesses.

Theorem 5.3. ([1, Chapter 6, Theorem 4.16]) *Let $A \in \mathbb{R}^{n\times n}$ be a singular irreducible M-matrix. Then*
(a) *A has rank $n-1$.*
(b) *There exists $x > 0$ such that $Ax = 0$.*
(c) *A has "property c."*
(d) *Each proper principal submatrix of A is a nonsingular M-matrix.*
(e) *A is almost monotone.*

Theorem 5.4. ([14, Theorem 2]) *Let A be a Z-matrix. Then the following statements are equivalent:*
(a) *A is an M-matrix with "property c".*
(b) *$A^\#$ exists and $A^\#$ is nonnegative on $R(A)$.*

Remark 5.5. Note that if $A \in M_n(\mathbb{R})$ is a Z-matrix such that $Au \geq 0$ for some $u > 0$, then A is an M-matrix with "property c"; see [1, Exercise 4.14, Chapter 6].

To generalize mimes, we propose a representation

$$A = (sI - B_1)(\rho(B_2)I - B_2)^\#,$$

where B_1 and B_2 are nonnegative, in which we require the first factor $sI - B_1$ be an M-matrix whose *group inverse is nonnegative on its range space*. In view of Theorem 5.4, we must thus assume that $sI - B_1$ is an M-matrix with "property c." The second factor in a nonsingular mime is the inverse of an M-matrix and so it is a nonnegative matrix. Motivated by this, we require that the second factor $(\rho(B_2)I - B_2)^\#$ above is *nonnegative on its range space*. Now, if we also assume that B_2 is irreducible, then by Theorem 5.3 and Theorem 5.4, it follows that the group inverse $(\rho(B_2)I - B_2)^\#$ exists and is nonnegative on its range space. These considerations culminate in the following formal definition.

Recall that according to [5], if $A \in M_n(\mathbb{R})$ is a nonnegative irreducible matrix, then there exists a unique vector $x > 0$ so that $Ax = \rho(A)x$ and $\sum_{i=1}^n x_i = 1$. Then $\rho(A)$ is called the *Perron root* of A and x is its *Perron vector*.

Definition 5.6 (A singular mime)**.** Let $B_1 \geq 0$, $B_2 \geq 0$ with B_2 irreducible. Let p be the Perron vector of B_2. Suppose that $(sI - B_1)p \geq 0$. Then $A = (sI - B_1)(\rho(B_2)I - B_2)^\#$ is called a *singular mime*.

Furthermore, a singular mime will be referred to as *type I* if the first factor $(sI - B_1)$ is an invertible M-matrix; otherwise it is a singular mime of *type II*.

A few more clarifying remarks on the above definition are in order.

Remark 5.7.

- The condition $(sI - B_1)p \geq 0$ with $p > 0$ and $B_1 \geq 0$ implies that $\rho(B_1) \leq s$. Here is a sketch of proof: By the Perron-Frobenius theorem, B_1 and hence B_1^T has $\rho(B_1)$ as an eigenvalue and $x \geq 0$ is the Perron vector associated to B_1^T. Hence, $0 \leq x^T((sI - B_1)p) = sx^Tp - x^TB_1p = sx^Tp - \rho(B_1)x^Tp = (s - \rho(B_1))x^Tp$. Since $x^Tp > 0$, the claim follows.
- If $(sI - B_1)p > 0$, then $sI - B_1$ is an invertible M-matrix. This is due to the fact that a semipositive Z-matrix is an M-matrix.
- If $(sI - B_1)p \geq 0$, then $(sI - B_1)$ is an M-matrix with "property c"; see Remark 5.5.
- To reiterate, a singular mime of type II has a representation given by $A = (\rho(B_1)I - B_1)(\rho(B_2)I - B_2)^\#$, where B_1, B_2 are nonnegative and B_2 is irreducible.

Analogously to Proposition 3.13, we obtain below Proposition 5.9 for singular mimes; a preliminary result is needed for its proof. Note that this lemma applies to any matrix X that satisfies the condition $AXA = A$ and the group inverse $A^{\#}$ is one such matrix.

Lemma 5.8. [2] *Let $A \in M_n(\mathbb{R})$ be group invertible. If $b \in \mathbb{R}^n$, then the linear equation $Ax = b$ has a solution if and only if $AA^{\#}b = b$. In such a case, the general solution is given by $x = A^{\#}b + z$ for some $z \in N(A)$.*

Proposition 5.9. *Let $A \in M_n(\mathbb{R})$ be a singular mime of type I as in Definition 5.6. The following implication holds:*

$$Ax \geq 0 \implies x \in (\rho(B_2)I - B_2)\mathbb{R}^n_+ + \alpha p \quad \text{for some} \ \ \alpha \in \mathbb{R}.$$

Proof. We know $A = (sI - B_1)(\rho(B_2)I - B_2)^{\#}$. Let $Ax \geq 0$. Set $y = (\rho(B_2)I - B_2)^{\#}x$. Since $(sI - B_1)^{-1}$ is an inverse M-matrix, it is nonnegative and hence $0 \leq (sI - B_1)^{-1}Ax = y$. We view this as a linear system with the right hand side vector y and invoke Lemma 5.8, to find the general solution. The general solution is given by $x = ((\rho(B_2)I - B_2)^{\#})^{\#}y + N((\rho(B_2)I - B_2)^{\#})$. We make use of the fact that for any matrix X, for which $X^{\#}$ exists, one has $(X^{\#})^{\#} = X$. Also, $N(X^{\#}) = N(X)$. By hypothesis, B_2 is irreducible and nonnegative; and therefore $N(\rho(B_2)I - B_2) = \{u : (\rho(B_2)I - B_2)u = 0\} = \mathbb{R}p$. These remarks imply the claimed representation for x. □

In the rest of the discussion, we set $B_1 = B_2$. We have the following characterization of a singular mime of type II, motivated by Theorem 3.1.

Theorem 5.10. *$A \in M_n(\mathbb{R})$ is a singular mime of type II if and only if there exists a Z-matrix $X \in M_n(\mathbb{R})$ such that*
(a) *X is irreducible;*
(b) *$AX = X$ and $AXX^{\#} = A$;*
(c) *there exists $u > 0$ such that $Xu = 0$.*

Proof. **Necessity**: Let $A = (\rho(B)I - B)(\rho(B)I - B)^{\#}$, where B is irreducible. Define $X = (\rho(B)I - B)$ so that $A = XX^{\#}$. Thus, $AX = XX^{\#}X = X$. Clearly $AXX^{\#} = (XX^{\#})^2 = XX^{\#} = A$. Also since B is irreducible, so is X and by Theorem 5.3, there exists $u > 0$ such that $Xu = 0$. Thus the conditions are necessary.

Sufficiency: Suppose that conditions (a)-(c) hold. Observe that due to condition (c) and in view of Remark 5.5, it follows that X is a *singular* M-matrix with "property c", so that $X^{\#}$ exists. We may write $X = rI - C$, where $C \geq 0$ and $r = \rho(C)$. Since X is irreducible, it follows that C is irreducible. As $AX = X$, one has $XX^{\#} = AXX^{\#} = A$, showing that A is a singular mime of type II. □

Theorem 5.11. *Let $A = (sI - B)(\rho(B)I - B)^{\#}$ be a singular mime as in Definition 5.6 (for the case $B_1 = B_2 =: B \geq 0$ irreducible and $s \geq \rho(B)$). Then:*
(a) *One has $A^{\#} = (\rho(B)I - B)(sI - B)^{\#}$. If A is a singular mime of type II, then $A = A^{\#}$.*
(b) *A has rank $n - 1$.*
(c) *If A is of type II, then A is idempotent and hence a $P_{\#}$-matrix.*
(d) *A and $A^{\#}$ are almost monotone. If A is of type II then, A and $A^{\#}$ are range monotone.*

Proof. (a) Set $X = (\rho(B)I - B)(sI - B)^{\#}$. We show that X satisfies the three conditions for the group inverse of A.

First, consider the case $s = \rho(B)$. Then $X = A = (\rho(B)I - B)(\rho(B)I - B)^{\#}$, an idempotent matrix, as was noted earlier. So, $A^{\#} = A$, in this case.

Next, we consider the case $s > \rho(B)$. Here, one has $(sI - B)^{\#} = (sI - B)^{-1}$ and so $X = (\rho(B)I - B)(sI - B)^{-1}$. Thus,

$$XA = (\rho(B)I - B)(\rho(B)I - B)^{\#},$$

so that upon pre-multiplying by A, one has

$$\begin{aligned} AXA &= (sI - B)(\rho(B)I - B)^{\#}(\rho(B)I - B)(\rho(B)I - B)^{\#} \\ &= (sI - B)(\rho(B)I - B)^{\#} \\ &= A. \end{aligned}$$

Further, upon post-multiplying by X, the expression for XA, one has

$$\begin{aligned} XAX &= (\rho(B)I - B)(\rho(B)I - B)^{\#}(\rho(B)I - B)(sI - B)^{-1} \\ &= (\rho(B)I - B)(sI - B)^{-1} \\ &= X. \end{aligned}$$

Finally, using the fact that $(\rho(B)I - B)^{\#}$ is a polynomial in B, one observes that the matrices $(sI - B)$, $(\rho(B)I - B)$, $(\rho(B)I - B)^{\#}$ and $(sI - B)^{-1}$ mutually commute. Hence, one has

$$\begin{aligned} AX &= (sI - B)(\rho(B)I - B)^{\#}(\rho(B)I - B)(sI - B)^{-1} \\ &= (\rho(B)I - B)(\rho(B)I - B)^{\#} \\ &= XA. \end{aligned}$$

This proves (a).
(b) By Theorem 5.3 we know that $\rho(B)I - B$ has rank $n - 1$. As was noted earlier, this equals the rank of $(\rho(B)I - B)^{\#}$, as well. If $s > \rho(B)$, then $sI - B$ is nonsingular and thus $(sI - B)(\rho(B)I - B)^{\#}$ again has rank $n - 1$. Thus, independent of its type, A has rank $n - 1$.
(c) If A is of type II, then $A = (\rho(B)I - B)(\rho(B)I - B)^{\#}$, which we know is idempotent. We had proved earlier that any idempotent matrix is a $P_{\#}$-matrix.
(d) By Theorem 5.3 again, the almost monotonicity of $\rho(B)I - B$ follows. By item (a) of Lemma 5.2, then $(\rho(B)I - B)^{\#}$ and $(\rho(B)I - B)(\rho(B)I - B)^{\#}$ are almost monotone. Also, if $s > \rho(B)$, then $(sI - B)^{-1}$ is a nonnegative matrix and so by item (b) of Lemma 5.2, then $(sI - B)(\rho(B)I - B)^{\#}$ is almost monotone. Thus, independent of its type, A is almost monotone, and by (a) of Lemma 5.2 again, $A^{\#}$ is almost monotone.
Next, assume A is of type II. Let $x \in R(A)$ and $Ax \geq 0$. Since $x = Au$ for some $u \in \mathbb{R}^n$, we have $Ax = AAu = Au = x$ by the idempotency of A proved in (c). So, $x \geq 0$, showing that A is range monotone. Since $A = A^{\#}$ as shown in (a), we see that $A^{\#}$ is also range monotone. □
The following example illustrates the above theorem.

Example 5.12. Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ so that B is irreducible and $\rho(B) = 2$.
(a) Take $s = 3 > 2 = \rho(B)$. Then

$$A = (sI - B)(\rho(B)I - B)^{\#} = \frac{1}{3}\begin{bmatrix} 3 & -2 & -1 \\ -1 & 3 & -2 \\ -2 & -1 & 3 \end{bmatrix}$$

and

$$A^{\#} = (\rho(B)I - B)(sI - B)^{-1} = \frac{1}{7}\begin{bmatrix} 3 & -1 & -2 \\ -2 & 3 & -1 \\ -1 & -2 & 3 \end{bmatrix}.$$

If $x = [x_1 \ \ x_2 \ \ x_3]^T$, then

$$Ax = \frac{1}{3}\left[(3x_1 - 2x_2 - x_3) \quad (-x_1 + 3x_2 - 2x_3) \quad (-2x_1 - x_2 + 3x_3)\right]^T.$$

Suppose that $Ax \geq 0$. Then one has the following inequalities:

$$3x_1 - 2x_2 - x_3 \geq 0; \quad -x_1 + 3x_2 - 2x_3 \geq 0; \quad -2x_1 - x_2 + 3x_3 \geq 0.$$

Adding any two of the above inequalities, one obtains the other inequality reversed. Thus $Ax = 0$ and so A is almost monotone. We have shown that $R(A) \cap \mathbb{R}^n_+ = \{0\}$, which also means that $R(A^\#) \cap \mathbb{R}^n_+ = \{0\}$ (due to $R(A) = R(A^\#)$), showing that $A^\#$ is almost monotone, as well.
(b) Take $s = \rho(B) = 2$. In this case,

$$A = (\rho(B)I - B)(\rho(B)I - B)^\# = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Then A is an idempotent matrix and hence (as noted earlier in the beginning of this section) $A = A^\#$. Let $x = [x_1 \quad x_2 \quad x_3]^T$ be such that $Ax \geq 0$, i.e.,

$$\frac{1}{3}[(2x_1 - x_2 - x_3) \quad (-x_1 + 2x_2 - x_3) \quad (-x_1 - x_2 + 2x_3)]^T \geq 0.$$

In an entirely similar manner, it follows that $Ax = 0$. Range monotonicity follows in a similar manner, too.
Let $x \in R(A)$. Then $x = [(-x_2 - x_3) \quad x_2 \quad x_3]^T$ and so

$$Ax = [(-x_2 - x_3) \quad x_2 \quad x_3]^T.$$

One has

$$x_1(Ax)_1 = (x_2 + x_3)^2; \; x_2(Ax)_2 = x_2^2; \; x_3(Ax)_3 = x_3^2.$$

If $x_i(Ax)_i \leq 0$ for all i, then one must have $x_2 = x_3 = 0$ and hence $x = 0$. Thus A is a $P_\#$-matrix.

Remark 5.13. While we have indicated the importance of the assumption on the irreducibility of B, it is interesting to observe what happens when this supposition is dropped: Consider $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Then B is reducible and $\rho(B) = 3$. Take $s = 4$. We then have

$$A = (sI - B)(\rho(B)I - B)^\# = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A^\# = (\rho(B)I - B)(sI - B)^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -1 & -3 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

We observe that $A[1 \;\; 0 \;\; 0]^T = [\frac{3}{2} \;\; 0 \;\; 0]^T$, which however is a nonzero vector. Hence A is not almost monotone. By Lemma 5.2, $A^\#$ cannot be almost monotone, otherwise $A = (A^\#)^\#$ would be.
Let $x = [1 \quad 16 \quad 0]^T$. Then one may verify that $x = A[-2 \quad 8 \quad 0]^T$ so that $x \geq 0$ and $x \in R(A)$. However, $\tilde{x} = A^\# x = [-2 \;\; 8 \;\; 0]^T \ngeq 0$. So, $\tilde{x} \in R(A^\#) = R(A)$, $A\tilde{x} = x \geq 0$, but $\tilde{x} \ngeq 0$. Thus, A is not range monotone. In this example $A^\#$ is range monotone. This is because any vector x in $R(A) = R(A^\#)$ is of the form $x = [\alpha \;\; \beta \;\; 0]^T$ for some $\alpha, \beta \in \mathbb{R}$. Thus $A^\# x = \frac{1}{6}[4\alpha - \beta \;\; 3\beta \;\; 0]^T \geq 0$ gives $\beta \geq 0$ and $4\alpha \geq \beta \geq 0$, so that $x \geq 0$.
For the case when $s = \rho(B)$, from the proof Theorem 5.11, we see that the irreducibility of B is used only to show that A is almost monotone. So dropping the irreducibility of B does not affect the range monotonicity of A or the fact that A is a $P_\#$-matrix. However, the above example with $s = \rho(B)$ can be used to show that if

B is not irreducible, then A need not be almost monotone.

$$A = (3I - B)(3I - B)^{\#} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

is not almost monotone because $A[1 \ \ 0 \ \ 0]^T = [1 \ \ 0 \ \ 0]^T \geq 0$, which however is nonzero.

Remark 5.14. In item (c) of Theorem 5.11, we have shown that if A is a singular mime of type II, then A is a $P_{\#}$-matrix. We believe that this should be that case for type I singular mimes, as well.

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