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Linear stability of miscible two-fluid flow down an incline

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We show that miscible two-layer free-surface flows of varying viscosity down an inclined substrate are different in their stability characteristics from both immiscible two-layer flows, and flows with viscosity gradients spanning the entire flow. New instability modes arise when the critical layer of the viscosity transport equation overlaps the viscosity gradient. A lubricating configuration with a less viscous wall layer is identified to be the most stabilizing at moderate miscibility (moderate Peclet numbers). This also is in contrast with the immiscible case, where the lubrication configuration is always destabilizing. The co-existence that we find under certain circumstances, of several growing overlap modes, the usual surface mode, and a Tollmien-Schlichting mode, presents interesting new possibilities for nonlinear breakdown. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4823855]

I. BACKGROUND AND MOTIVATION

One of the most important models used to investigate the dynamics of open flows, such as surface waves, solitary waves, and transition from laminar to turbulent flow, is the flow of a liquid film down an inclined plane.[1–4] In many situations of interest, fluid properties, such as density and viscosity, are stratified across the film. One could have multiple layers, each consisting of a different fluid, or a continuous stratification. Multiple layer inclined plane geometries find application, to give just one example, in the manufacture of photographic film,[5,6] which can consist of more than ten different layers of emulsion. Such flows are known to be unstable, and the instabilities often manifest themselves as waves which travel along the fluid interfaces, down the inclined plane, in the direction of the bulk fluid flow.[7–15] As a consequence, thickness variations, often undesirable, are created. Environmental flows, such as rock glaciers,[11] could also be modeled in this manner. Continuous stratification across the flow, on the other hand, is typically manifested due to the wall temperature being maintained different from that well above the film.

Due to its importance to coating technology, the scientific literature related to instabilities in one- and multiple-layer free-surface flows on inclined planes is extensive. However, the fluids in different layers have always, to the best of our knowledge, been taken to be immiscible, with fluid properties varying discontinuously across their interfaces. In reality however, most pairs of fluids are miscible to some degree, so layers of pure fluid are separated by mixed layers of finite thickness, rather than by sharp interfaces. Consequently, fluid and flow properties would vary continuously, even if across a thin layer, rather than in jumps. The other limit, of viscosity gradient throughout the film, has been studied as well for decades, primarily in the context of temperature variations across the film. These studies stem, for the most part, from a need to control thermocapillary instabilities. However, with one or two exceptions, viscosity, whose variation is the primary focus of the present work, has been assumed constant across the film. Our flow, shown schematically in Fig. 1 and containing a thin mixed layer between the fluids, lies in between the two limits of a finite viscosity

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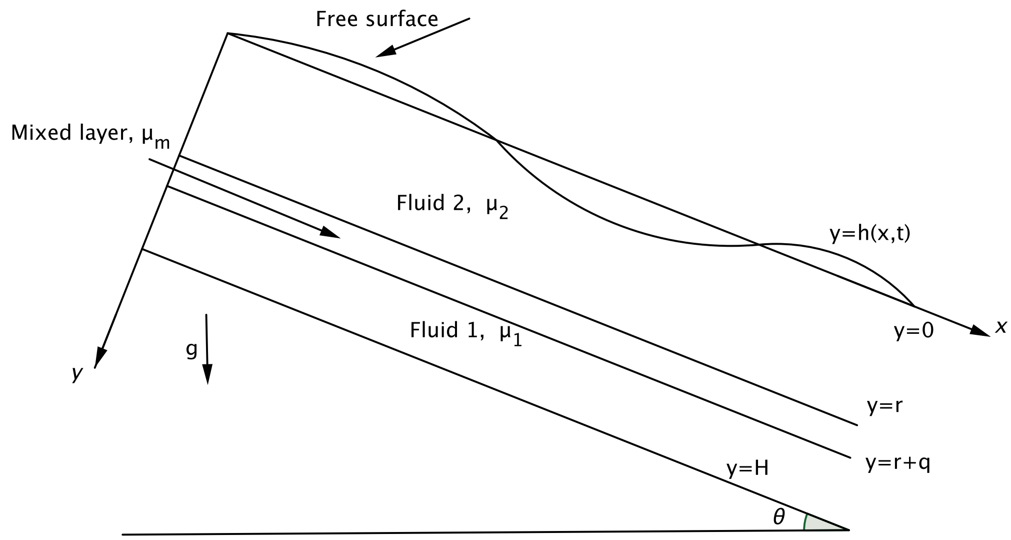

FIG. 1. Schematic of miscible two-layer flow down an inclined plane.

gradient throughout the film (that could be created by infinitely miscible fluids, for example), and sudden jumps (immiscible fluids). We will see that this flow differs from immiscible fluids and infinitely miscible fluids, in that some of the instabilities displayed require a finite miscibility, and stem from interactions of the mixed layer with the basic shear. Some modes of instability are thus unique to miscible multi-layer flow.

We use the following definitions: the (free) surface on top, next to air, will be called "surface," and the layer separating the two liquid layers will be called "interface" for immiscible fluids and "mixed layer" in the miscible case. Four configurations will be briefly reviewed here: the flow of two immiscible liquids with a viscosity jump at their interface, the free-surface flow of a single liquid, the free-surface flow of two immiscible liquids, and the free-surface flow where viscosity varies across the entire film, down an inclined substrate. We will then outline the effects of miscibility observed for two-layer wall-bounded flows.

It is known that a sharp viscosity jump in itself can create an instability at arbitrarily low Reynolds numbers. A long-wavelength instability was first discovered for wall-bounded channel and Couette flows,[16] originating in a hidden neutral mode in the single-fluid flow, destabilized by viscosity stratification. Later, a short-wavelength instability was shown to exist in unbounded Couette flow, also at all Reynolds numbers.[17] It was, however, argued[18] that the short-wavelength instability is stabilized by interfacial tension in all but sub-micron flow configurations. Later, a half-bounded configuration was studied,[19] and in addition to the above, a third instability regime was found at long wavelengths and long viscous length scales. This instability occurred when the viscosity of the upper layer is more than that of the lower bounded layer and it was not affected much by interfacial tension.

The investigations by Yih[20,21] sparked interest in the gravity-driven flow of a single layer of liquid film down an inclined substrate, and it is now one of the most studied free-surface problems in fluid mechanics.[1–4,20–25] This flow develops an instability in the form of surface waves when the Reynolds number exceeds a critical value.[21,22] This mode of instability is termed here as the surface, or S, mode. In immiscible multiple layer flows with density and/or viscosity jump across the interface, the number of wave solutions is equal to the number of interfaces.[7–9,11,15,26] There is thus an S mode, and one or more interfacial modes. Each interfacial mode owes its existence to a jump in liquid properties across a particular interface. While the surface mode requires inertia to exhibit wave growth, the interfacial modes can be unstable at any Reynolds number.[7–9,11,15]

Kao[9] was the first to examine the stability of a free-surface flow of two superposed liquid layers with different viscosities. He found that for a less viscous lower layer, a long-wave interfacial mode is unstable at all Reynolds numbers. For a more viscous lower layer, a non-zero critical Reynolds number for interfacial modes exists, which increases with increasing viscosity contrast. Loewenhertz and Lawrence[11] made the same analysis at finite wavelengths and zero inertia (Stokes flow), and showed that the long-wave interfacial mode grows at finite wavelength. The flow with a less viscous lower layer was again found to be always unstable, and the wave number of the fastest growing mode was inversely proportional to the thickness of the upper layer. Note that the S mode is neutrally stable in inertialess flows. In pipe and channel flows, a thin layer of low-viscosity fluid at the wall is known to stabilize the flow. This is called lubrication.[27] To borrow this terminology for interfacial flows, we would term the stabilization achieved by a more viscous wall layer[9,11] as an "anti-lubrication" one.

Chen[15] included the inertial effects in the analysis, and also surface tension and interfacial tension. Again unstable wave motion was seen to occur when the less viscous layer is in the region next to the wall for any Reynolds number and any finite interface tension and surface tension. Further, it was possible to achieve stability for the configuration with the more viscous fluid adjacent to the wall, when the Reynolds number was small enough. The analysis for a two-layer film system has been extended to three, five and multi-layer film systems and the effects of viscosity jumps have been reported.[13,14,28–30]

There is a fundamental difference between flows with viscosity jumps and flows where the viscosity is continuously stratified, and continuous stratification can bring in new physics,[31] as we shall see. However, especially considering the vast literature available about film flow on heated or cooled inclined plates, it is very surprising that variations in viscosity are seldom taken into consideration. The early work of Craik and Smith[32] is therefore notable, where an analytical solution is derived at the long-wave limit for arbitrary continuous viscosity distributions. Results were presented for the S mode in the special case of a viscosity exponentially increasing with depth, inspired by melting metal. Compared to thin films of uniform viscosity, those with a viscosity increasing away from the interface were found to be stabilizing for the S mode. Goussis and Kelly[33] obtained a similar effect with wall cooling. Observe that the viscosity gradient of Ref. 32 is of the same sign as in wall cooling. Some other efforts have been focussed on the understanding of thermocapillary instabilities.[34,35]

The above configurations, both immiscible and continuously stratified, take into account the transport of viscosity by the mean flow, but neglect the diffusion of viscosity. This assumption is similar to neglecting the thermal diffusion in a temperature equation. However, the effect of thermal diffusivity is non-negligible in the vicinity of critical layers, where the thermal convection equation becomes singular if diffusion is neglected, see, e.g., Lees and Lin[36] where this was shown first in the case of a compressible boundary layer. The same phenomenon is observed for miscible fluids, if diffusion of viscosity is neglected in the transport equation for viscosity.[37] In incompressible flow, the stability equations derived by Drazin[37] are a special case of the more general equations of Craik and Smith.[32]

Miscible two-fluid flow has been studied in other geometries such as channels and pipes.[31,38–42] In the first study of the planar Couette flow in the long-wave limit,[31] Craik realized that a continuous stratification of viscosity in the critical layer can modify stability behavior significantly. In a modified version of the Couette flow of two fluids of different viscosities considered by Yih,[16] Craik[31] showed that if the viscosity of the less viscous fluid (within which lies the critical layer for this flow) decreases away from the interface, it would lead to stabilization, while the opposite variation in its viscosity destabilizes the flow. He pointed out that this is because viscosity stratification brings about the important change that the second derivative of basic velocity with respect to the wall normal direction is no longer zero, so the perturbation cannot be neutral, even when inertia is neglected. Thus a consideration of diffusivity between two superposed liquid layers, and its consequent effect on the velocity profile, were seen to be important for stability.

Subsequent studies have revealed that stability behavior of miscible fluids can be qualitatively different from the interfacial instability of immiscible fluids. The thickness of the mixed layer was shown[39] to be an important parameter in channel flow, and the interesting result, that a thin mixed layer can result in a faster growing instability than with either a sharp viscosity jump or a continuous viscosity stratification across the entire flow, was presented. There is also a mode of channel-flow instability, termed the "overlap" (O) mode,[40] which is absent in immiscible two-fluid flow. This instability arises when the critical layer of the dominant shear mode overlaps with the viscosity gradient, similar to that found earlier by Craik[31] for Couette flow.

The problem formulation, results and discussion follow.

II. MATHEMATICAL FORMULATION

The linear stability of a gravity driven laminar two-dimensional flow of two miscible, Newtonian, incompressible fluids of equal density and different viscosities down an inclined plane tilted at an angle θ to the horizontal is considered. We work in a Cartesian coordinate system whose origin is at the unperturbed free surface. The x-axis lies on this unperturbed surface ($y = 0$) (Fig. 1), i.e., parallel to the inclined plate ($y = H$). The y-axis is normal to the plate and oriented towards it. Let $y = h(x, t)$ be the equation of the free surface at time t. Fluids of viscosity μ_1 and μ_2, respectively, occupy regions close to the inclined wall and to the free surface. There is a thin layer (called the mixed layer) where the two fluids mix and a local stratification of viscosity is created. The thickness of the mixed layer is q and $0 \leq y \leq r$ is the extent of fluid 2. The viscosity ratio is given by $m = \dfrac{\mu_2}{\mu_1}$. The downstream growth of the mixed layer thickness is neglected, and this assumption is justified when the two liquids diffuse only slowly into each other (i.e., the Peclet number, defined later in this section, is high).

The flow is governed by the continuity and Navier-Stokes equations together with a scalar-transport equation for the viscosity. The boundary conditions are the no-slip condition at the inclined plane; the balance of normal and shear stresses at the free surface; and the kinematic free surface condition at the interface. It is to be noted that, being a continuously stratified flow, the required continuities of velocity and stress components are automatically satisfied everywhere in the interior of the film.

Using the following scaling:

$$\begin{aligned} x^* &= \frac{x}{H},\, y^* = \frac{y}{H},\, t^* = \frac{V}{H}t,\, p^* = \frac{p}{\rho V^2},\, u^* = \frac{u}{V},\, v^* = \frac{v}{V}, \\ r^* &= \frac{r}{H},\, q^* = \frac{q}{H},\, \mu^* = \frac{\mu}{\mu_1},\, m = \frac{\mu_2}{\mu_1},\, \mu_m^* = \frac{\mu_m}{\mu_1}, \end{aligned} \tag{1}$$

the governing equations and the boundary conditions are non-dimensionalized, and, dropping the superscript *, are given by

$$u_x + v_y = 0, \tag{2}$$

$$R[u_t + uu_x + vu_y] = -R\ p_x + [2\mu u_x]_x + [\mu(u_y + v_x)]_y + G, \tag{3}$$

$$R[v_t + uv_x + vv_y] = -R\ p_y + [\mu(u_y + v_x)]_x + [2\mu v_y]_y + G\cot\theta, \tag{4}$$

$$Pe[\mu_t + u\mu_x + v\mu_y] = \nabla^2\mu, \tag{5}$$

$$u = 0,\ v = 0 \quad \text{at} \quad y = 1, \tag{6}$$

$$\begin{aligned} R\ p &= \frac{2\mu}{1 + h_x^2}\left[v_y - v_x h_x + u_x h_x^2 - u_y h_x\right] \\ &\quad + \frac{S h_{xx}}{(1 + h_x^2)^{3/2}} \quad \text{at } y = h(x, t), \end{aligned} \tag{7}$$

$$(1 - h_x^2)(u_y + v_x) - 4u_x h_x = 0 \quad \text{at} \quad y = h(x, t), \tag{8}$$

$$v = h_t + uh_x \quad \text{at} \quad y = h(x, t), \tag{9}$$

Here, u and v are the components of velocity in the x and y directions, respectively, V is the basic free-surface velocity, $R = \rho VH/\mu_1$ is the Reynolds number, $G = \rho g H^2 \sin\theta/(\mu_1 V)$ and $S = \sigma/(\mu_1 V)$ are the dimensionless gravity and surface tension parameters, G being the product of the Reynolds number and the square of the Froude number, while S is the inverse of a capillary number. $Pe = VH/\chi$ is the Peclet number, χ is the mass diffusivity, $Sc = Pe/R$ is the Schmidt number, g, σ, and ρ are the acceleration due to gravity, the surface tension coefficient between fluid 2 and air and the constant density, respectively, t is time and p is pressure.

To prescribe a thin mixed layer with varying viscosity, the nondimensional base flow viscosity $\mu_B(y)$ is given by the tangent hyperbolic:

$$\mu_B(y) = \tanh(Ky + L), \tag{10}$$

where

$$K = \frac{1}{q}\left[\tanh^{-1}(1) - \tanh^{-1}(m)\right],$$

$$L = \tanh^{-1}(m) - \frac{r}{q}\left[\tanh^{-1}(1) - \tanh^{-1}(m)\right].$$

This effectively produces a three layer viscosity variation as follows:

$$\mu = \begin{cases} m, & \text{if } y = r \\ \mu_B(y), & \text{if } r \le y \le r+q \\ 1, & \text{if } y = r+q. \end{cases} \tag{11}$$

The base flow velocities can in fact be specified analytically, but are here computed numerically from Eqs. (2)–(9), dropping the time-derivative and x-derivative terms.

For the linear stability, in view of the extension of Squire's theorem to viscosity stratified flows, only two-dimensional disturbances are considered, since they are assumed to go unstable at a lower Reynolds number than corresponding three-dimensional perturbations. Perturbations may be expressed as $\hat{\phi}(x, y, t)$, $\hat{\mu}(x, y, t)$, $\hat{\eta}(x, t)$ where ϕ is the stream function perturbation such that $(\hat{u}, \hat{v}) = (\hat{\phi}_y, -\hat{\phi}_x)$. Since the flow is parallel, normal mode forms, e.g., $\hat{\phi}(x, y, t) = \tilde{\phi}(y)e^{ik(x-ct)}$ may be assumed, where $\tilde{\phi}(y)$ is the amplitude function for $\hat{\phi}$, k is the streamwise disturbance wave number, and c is the complex wave speed. A temporal stability analysis is performed and the flow is deemed linearly unstable to infinitesimal disturbances if $Im(c) > 0$. Modified Orr-Sommerfeld equations are thus derived from the non-dimensional Navier-Stokes equations, the boundary conditions and the scalar transport equation for viscosity obtained from Eqs. (2)–(9) using a standard procedure and are given by (after dropping the tilde)

$$\begin{aligned} ikR\,&\left[\phi''(U_B - c) - k^2\phi(U_B - c) - U_B''\phi\right] = \\ &\mu_B\phi'''' + 2\mu_B'\phi''' + (\mu_B'' - 2k^2\mu_B)\phi'' - 2k^2\mu_B'\phi' \\ &+ (k^2\mu_B'' + k^4\mu_B)\phi + U_B'\mu'' + 2U_B''\mu' \\ &+ (U_B''' + k^2U_B')\mu, \end{aligned} \tag{12}$$

$$ikPe\,\left[(U_B - c)\mu - \mu_B'\phi\right] = \mu'' - k^2\mu, \tag{13}$$

with the boundary conditions

$$\phi = 0 \text{ at } y = 1, \tag{14}$$

$$\phi' = 0 \text{ at } y = 1, \tag{15}$$

$$\phi'' + k^2\phi + U_B''\eta = 0 \text{ at } y = 0, \tag{16}$$

$$k\ R\ (U_B - c)\phi' + i\mu_B(\phi''' - 3k^2\phi') + i\mu_B'(\phi'' + k^2\phi) + 2iU_B''\mu - (G\cot\theta\ k + Sk^3)\eta = 0 \text{ at } y = 0, \tag{17}$$

$$\phi + (U_B - c)\eta = 0 \text{ at } y = 0, \tag{18}$$

where prime denotes differentiation with respect to y and $U_B(y)$ is the base state velocity profile. These equations, with the Peclet number set to infinity, were studied by Craik and Smith.[32]

III. DETERMINATION OF THE THICKNESS OF THE VISCOSITY-STRATIFIED LAYER

A physical value of the thickness of the mixed layer, q, can be decided as follows. Consider that parallel streams of two miscible fluids come into contact at a streamwise location $x = 0$, at a constant y. This could be achieved, for example, by a splitter plate placed at $x < 0$, with the two fluids on either side of it. For $x > 0$, the two fluids will mix into each other giving a stratified layer, whose thickness q grows downstream. Hence, q will be a function of x. The streamwise growth of q is dictated by the Peclet number as described below.

The viscosity is directly proportional to the concentration of each fluid in the mixed layer, and the steady mean concentration c at any location satisfies

$$U\frac{\partial c}{\partial x} + V\frac{\partial c}{\partial y} = \frac{1}{Pe}\left[\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right]. \tag{19}$$

To obtain a strictly correct value of q, this equation would need to be solved in two dimensions, coupled with the momentum equation, and boundary conditions. To estimate instead the order of q given the physical parameters, we will now proceed by assuming similarity in x, due to slow diffusion. Making a boundary layer approximation, we have $V \ll U$ and $\partial^2/\partial x^2 \ll \partial^2/\partial y^2$. We are left with $U\partial c/\partial x \simeq Pe^{-1}\partial^2 c/\partial y^2$, which gives the scaling $Uc/x \simeq Pe^{-1}c/q^2$, or $q \simeq \sqrt{x/(UPe)} = \sqrt{x/(UScRe)}$. We now briefly address the value of $q = 0.2$ chosen for most of this study, at $r = 0.4$. Figure 2 shows that $U \approx 0.7$ at $y = 0.4$ (the exact value depends on m). For a representative case of $Re = 500$ and $Sc = 20$, we get that $q = 0.2$ is found around $x \approx 280$. In an experiment with a film thickness of 0.5 cm, this equals a streamwise distance of 140 cm. If we take the liquid density and viscosity to resemble those of water, the Reynolds number indicated would require velocities of O(10) cm/s, which is realistic. Shorter distances for the realization of this value of q will be obtained at lower r, Re or Sc. Furthermore, in Appendix A, we show that the instabilities are even stronger for a somewhat thinner mixed layer ($q = 0.15$), which would also lie further upstream.

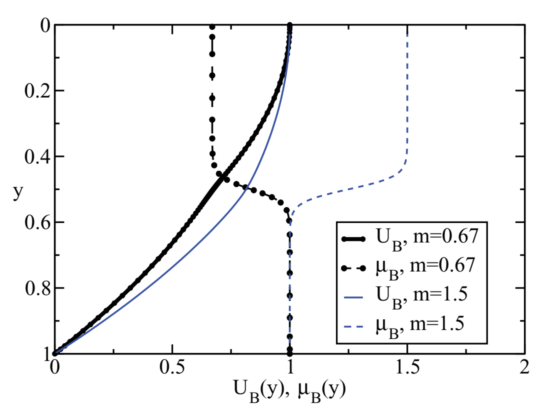

FIG. 2. Typical basic velocity and viscosity profiles. The interface is denoted by $y = 0$ and the y coordinate increases to 1 at the wall. r and q are chosen to be 0.4 and 0.2, respectively, to place the mixed layer in the middle of the film.

The scaling argument can also be used to validate the parallel flow assumption — when using the (one-dimensional) stability equation, we assume that the steady flow and the mixed layer thickness q vary at a much larger length scale than the disturbance wavelength. At $Re < 1000$, the most unstable wavelength is typically O(1), which gives to the requirement $dq/dx \ll 1$. At $Re = 10$, $r = 0.4$ and $q = 0.2$, this gives the requirement $Sc \gg 0.7$, and for the same at $Re = 500$, we obtain $Sc \gg 0.004$. We may thus conclude that the parameter range used here is well inside the regime where the parallel flow assumption is valid.

IV. RESULTS

Equations (12)–(18) determine the linear stability to infinitesimal two-dimensional disturbances of the miscible two-layer flow down an inclined plane. A Chebyshev spectral collocation discretization is used to solve the eigenvalue problem, with the n collocation points ($y_{c,1}$, ..., $y_{c,n}$) defined as

$$y_{c,j} = 0.5\cos\{[\pi(j-1)/(n-1)] + 1\}.$$

Since the gradients are large in the mixed layer, a large number of grid points are required in this region. This is achieved by using the stretched collocation point distribution[40] (y_1, ..., y_n), given by

$$y_j = \frac{a}{\sin h(by_0)}\left[\sin h\{(y_c - y_0)b\} + \sin h(by_0)\right],$$

where a is the midpoint of the mixed layer and

$$y_0 = \frac{0.5}{b} ln\left[\frac{1+(e^b-1)a}{1+(e^{-b}-1)a}\right].$$

The stretching parameter b represents the degree of clustering of the points around $y = a$. In the computations that follow, we have taken $b = 8$. We have checked the resolution independency of our results as follows: when the number of collocation points was doubled from $n = 81$ to $n = 161$, the eigenvalues were unchanged up to five decimal places, for the range of parameters considered.

Figure 2 presents the typical base state velocity $U_B(y)$ and viscosity $\mu_B(y)$ profiles when $r = 0.4$ for the viscosity ratios $m = 1.5$ (less viscous fluid adjacent to the inclined plane, i.e., the lubrication case) and $m = 0.67$ (more viscous fluid adjacent to the inclined plane, i.e., the anti-lubrication case). The thickness q of the mixed layer has been fixed at 0.2 here, and in all the results presented in the main body of the paper. In Appendix A, we show how the results depend on small changes in q. The base state velocity $U_B(y)$ vanishes at $y = 1$, while $\partial U_B/\partial y = 0$ at the free surface, $y = 0$. In this figure, the velocity and viscosity scales used for non-dimensionalization correspond to the maximum velocity at the free surface and the viscosity of the fluid layer closer to the inclined plane.

However, to present stability results, in order to make a fairer comparison across different viscosity ratios, we use a Reynolds number based on average quantities, i.e.,

$$Re \equiv \frac{\rho H \bar{V}}{\bar{\mu}},$$

where the overbar stands for average across the film.

As mentioned before, the long wave limit has been studied in detail by Craik and Smith[32] in the zero diffusivity limit for a viscosity profile increasing exponentially away from the interface. Since there is no clear "wall location" in that paper, it is not possible to define a clear ratio between wall and interface viscosities, so we are unable to make quantitative comparisons. However, we made computations setting $Sc = 0$ and $m < 1$ for several locations and thicknesses of the mixed layer at very small wave numbers, and found that the instability of Reynolds number increases with decreasing m, consistent with Ref. 32.

We now discuss the lubrication case, $m = 1.5$. Neutral boundaries of the surface (S) mode for different Schmidt numbers are presented in Fig. 3 for $r = 0.4$, $q = 0.2$, and angle of inclination $\theta = 10^\circ$. The figure also shows the single fluid ($m = 1$) neutral boundary of Yih. Stabilization of the

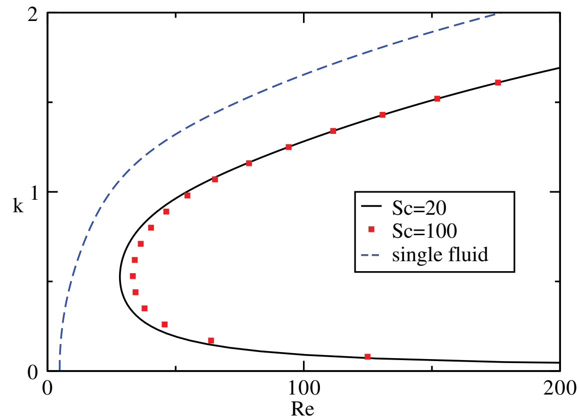

FIG. 3. Neutral stability boundaries of the surface mode for different Schmidt numbers when $r = 0.4$, $q = 0.2$, $m = 1.5$, and $\theta = 10°$. The case of a single fluid[21] is also shown for comparison.

S mode is observed at low k for $m = 1.5$. This stabilization will be important for applications. There is no low Reynolds number long-wave ($Re \to 0$, $k \to 0$) S-mode instability of miscible two-layer flow with the less viscous fluid near the wall, and this is in contrast to the single fluid case. Stability is not very sensitive to variations in diffusivity, although a mild stabilization is observed with decreased diffusivity (between $Sc = 20$ and $Sc = 100$). This result is qualitatively different from that predicted for immiscible two-fluid flow with sharp jump in viscosity,[9] in which the surface mode is unstable at low wave numbers and small Reynolds numbers.

Simultaneously, there is another mode which goes unstable in this flow, and whose neutral boundaries are shown in Fig. 4. Whereas the S mode has phase speed larger than the free-surface velocity, the phase speed of this other mode is 0.76–0.79, which is close to the base flow velocity (0.81) at the centre of the mixing layer ($y = 0.5$, since $r = 0.4, q = 0.2$). As first described in miscible Couette flow,[31] the latter instability arises entirely due to overlap of the critical layer with the viscosity gradient, and is therefore called an overlap mode. This mode of instability is absent in immiscible two-fluid flow. To distinguish it from another overlap mode discussed below, we term it O1. This mode is very sensitive to diffusivity. When $Sc = 100$, the flow is unstable to the O1 mode at very low Reynolds number and over a wide range of wave numbers. The range of unstable wave numbers decreases with increase in Reynolds number at this diffusivity level. When the diffusivity level of the fluids is increased (to $Sc = 20$), the unstable region is confined within a small blob in $Re - k$ space.

The neutral boundaries of the S and O1 modes we discussed at $Sc = 20$ are compared in Fig. 5 against the corresponding neutral boundaries when the two fluids are interchanged, but all

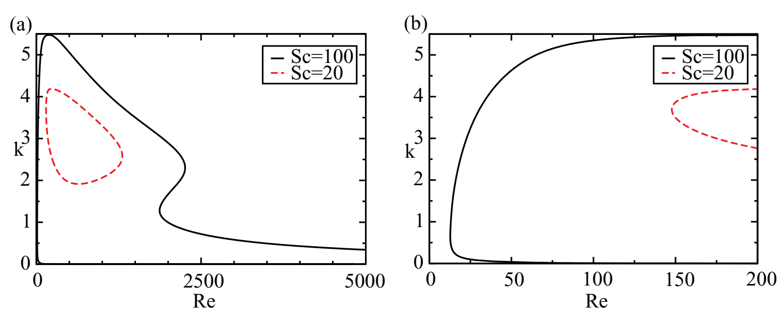

FIG. 4. (a) Neutral stability boundaries of the overlap mode O1 for different Schmidt numbers when $r = 0.4$, $q = 0.2$, $m = 1.5$. (b) The same, zoomed to low Re, to provide a comparison with the surface mode of Fig. 3.

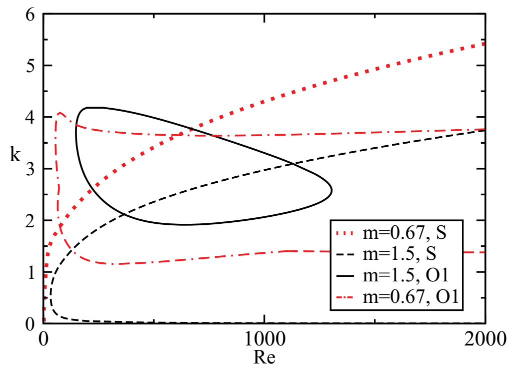

FIG. 5. Neutral stability boundaries for O1 and S modes at $Sc = 20$, $r = 0.4$, $q = 0.2$, $\theta = 10^{\circ}$ for two viscosity ratios.

other parameters are kept the same, i.e., when m is set at 0.67. At this Sc, it is seen that placing the lower viscosity closer to the wall stabilizes both the S mode and the O1 mode.

This shear flow is also always unstable to the traditional (without overlap) Tollmien-Schlichting (TS) modes developing in the wall boundary layer, but at Reynolds numbers higher than the range shown in this figure (see coming figures). The Tollmien-Schlichting modes on an inclined plane were studied in detail by Lin,[43] who also indicated that the TS waves can sometimes have a higher growth rate and dominate the S mode at infinite times, if the wavelength of surface perturbations present in the system is small enough.

How does varying the relative volumetric flow rates of the two fluids change stability behavior? To answer this, neutral boundaries are presented in Figs. 6 and 7 for different locations of the mixed layer when $\theta = 10^{\circ}$ for $Sc = 20, m = 1.5$. To retain clarity, the S mode is not shown in this figure. At $r = 0.315$ (Fig. 6) there are two distinct regimes of instability, both of which may be attributed to overlap conditions, the O1 blob at high wave numbers, and the finger-like region at significantly lower wave numbers, termed here the O2 mode. (In Appendix A we show that the O1 and O2 modes, being critical layer modes, lie within the same range of phase speeds, but are distinct and have vastly different wave numbers. These modes emerge out of the overlap mechanism, and are therefore both of the Craik type.) When the mixed layer is moved towards the wall a little, to $r = 0.4$, the O instability regions are much larger, and besides, the TS mode instability makes an appearance at high Reynolds numbers ($Re \sim 3000$) and moderate wave numbers ($k \sim 1$). The phase speeds of the O1 and O2 modes are close to each other, and lie within the mixed layer, whereas the TS mode is of low frequency, and hence has a low-lying critical layer, at $y \sim 0.9$ for $m = 1.5$. As r increases to 0.5 (Fig. 7), the O1 and O2 instability regions merge together. The TS mode is now unstable even at

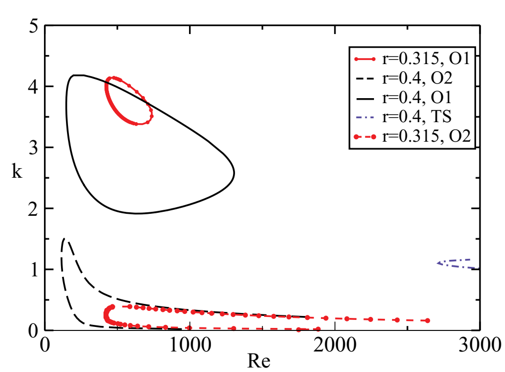

FIG. 6. Neutral boundaries of the O1, O2, and TS modes for different locations of the mixed layer when $\theta = 10^{\circ}$, $Sc = 20$, and $m = 1.5$. Two distinct overlap modes are seen for smaller values of r. The Tollmien-Schlichting mode is seen at high Reynolds numbers and $k \sim O(1)$ for $r = 0.4$. The surface mode is not shown in this figure.

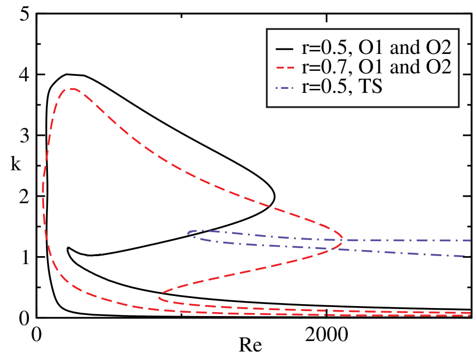

FIG. 7. Neutral boundaries of the overlap modes O1, O2, and TS modes for different locations of the mixed layer when $\theta = 10^\circ$, $Sc = 20$, and $m = 1.5$. The two overlap modes merge together. The Tollmien-Schlichting mode is seen at high Reynolds numbers and $k \sim O(1)$ for $r = 0.5$. The surface mode is not shown in this figure.

$Re \sim 1000$. At higher r $(= 0.7)$, O1 and O2 are still conjoined, but the neutral TS mode has shifted to the right to extremely high Reynolds numbers (not shown). We may conclude that stability is influenced by the relative mass fluxes in the two liquid layers.

Figure 8 presents the effects of diffusivity on the critical Reynolds number (Re_{cr}) for two viscosity ratios $m = 1.5$ and $m = 0.67$. Flow is unstable in the region above a given curve. The surface mode always becomes unstable at a rather low Reynolds number and is not sensitive to Schmidt number. For this mode, it is stabilizing to place the more viscous fluid near the interface and the less viscous fluid near the inclined plane. On the other hand, the overlap mode is very sensitive to the Schmidt number. At high Schmidt number, the overlap mode O2 overtakes the surface mode as being unstable at a lower Reynolds number. At extremely low diffusivity (Sc large), the O2 mode begins to resemble the inertialess immiscible ($Re \to 0, Sc \to \infty$) mode. For an immiscible mode in planar Couette flow at low inertia Craik[31] showed that diffusion was stabilizing, and our results are in line with this. At high Schmidt number, flow is unstable whether we are in the lubrication or anti-lubrication geometry, i.e., irrespective of which fluid is nearer the wall. We note that at low Peclet number, the parallel flow approximation is no longer a good one, so the results at the extreme left of this figure will need re-examination with this approximation released.

We summarize stability at small Reynolds number by displaying contours of maximum growth rate ω_i, including all modes, and across a range of r and m, at two Reynolds numbers (Fig. 9, (a) $Re = 15$ and (b) $Re = 95$). When viewed across this range, we see that r and even the Reynolds number

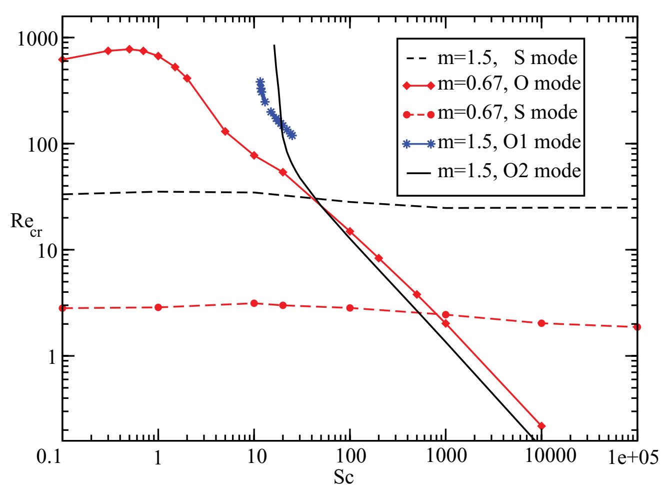

FIG. 8. Critical Reynolds number as a function of Schmidt number for $r = 0.4$, $q = 0.2$, and $\theta = 10^\circ$.

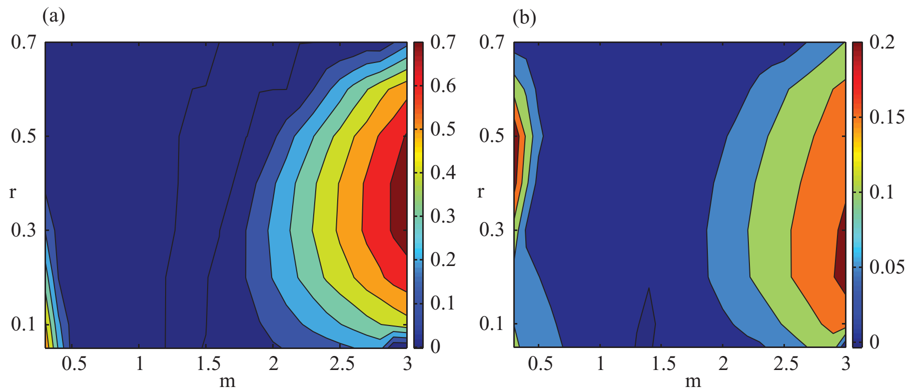

FIG. 9. Contours of maximum ω_i among all modes, when $Sc = 20$. (a) $Re = 15$, (b) $Re = 95$. Red contour represents high maximum growth rate, blue contour represents low maximum growth rates.

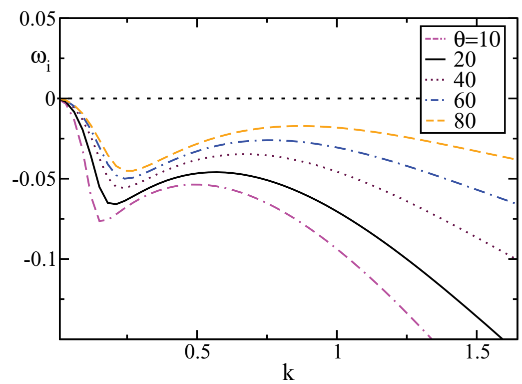

FIG. 10. Growth rate of the most unstable mode at a viscosity ratio $m = 1.5$, when $Re = 15, Sc = 1, r = 0.4, q = 0.2$.

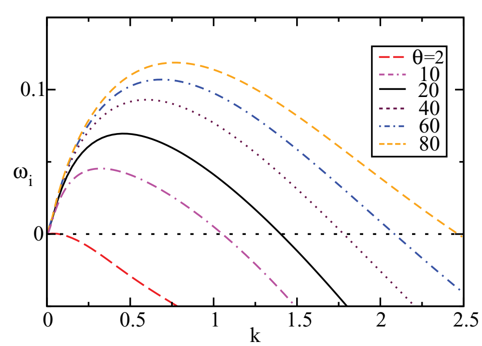

FIG. 11. Growth rate of the most unstable mode at a viscosity ratio $m = 0.67$, when $Re = 15, Sc = 1, r = 0.4, q = 0.2$.

are in fact relatively minor players, and that it is the viscosity ratio m which decides stability at a given Sc. A region close to $m = 1.5$ is most stable, and flow is destabilized on either side. Overall, the lubrication configuration is more unstable at high m.

The growth rate of the most unstable mode is presented as a function of inclination angle θ for $m = 1.5$ in Fig. 10, and for $m = 0.67$ in Fig. 11, when $Re = 15, Sc = 1, r = 0.4$. The flow is progressively more unstable as the solid plate is made more vertical. The results further confirm that $m = 1.5$ stabilizes in comparison to $m = 0.67$ at small Re. It is to be noted that at this Re, the flow of a single fluid layer down an inclined plane can never be stable for any angle of inclination considered, whereas having a layer of fluid with the same viscosity near the inclined wall with $r = 0.4$ and

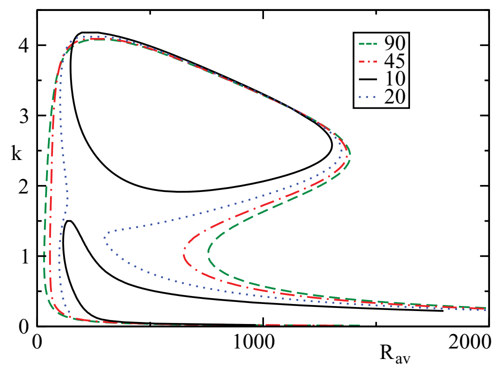

FIG. 12. Effect of θ on the neutral stability boundaries, at $m = 0.67$, when $Re = 15$, $Sc = 1$, $r = 0.4$, $q = 0.2$.

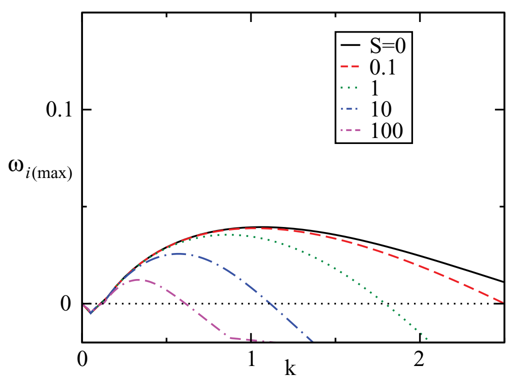

FIG. 13. Effect of surface tension, for different values of inverse capillary number S, for $Re = 75$, $m = 1.5$. Here $Sc = 10$, $r = 0.4$, $q = 0.2$, and $\theta = 10°$.

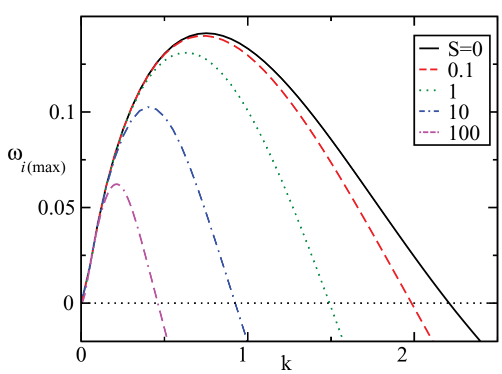

FIG. 14. Effect of surface tension, for different values of inverse capillary number S, for $Re = 10$, $m = 0.67$. Here $Sc = 10$, $r = 0.4$, $q = 0.2$, and $\theta = 10°$.

$m = 1.5$, a viscosity stratified flow is stabilized even for a high angle of inclination. The neutral stability boundaries as functions of θ are also shown in Fig. 12. This figure confirms that increasing inclination destabilizes. It is also seen that the instability regions for the two overlap modes O1 and O2 are separate for $\theta = 10$, but merge at higher inclination angles.

The influence of surface tension S, which has been set to zero so far, is examined. A higher Reynolds is chosen for $m = 1.5$ (Fig. 13) than for $m = 0.67$ (Fig. 14), so that the base cases are both unstable. Surface tension plays its expected role, having little effect at low wave numbers while damping high wave number disturbances. Re-examining a portion of Fig. 8 by including surface tension (Fig. 15), we see that, when $m = 0.67$, even with $S = 100$, it is not possible to stabilize the

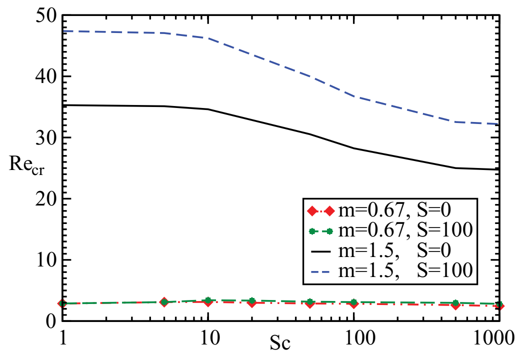

FIG. 15. Effects of surface tension on critical Reynolds number for two different viscosity ratios when $r = 0.4$, $q = 0.2$, and $\theta = 10°$.

system much. This is because the longest wave is the most unstable. On the other hand, when the fluid close to the inclined plane is less viscous ($m > 1$), the fact that finite wavelengths are the least stable means that surface tension can enhance the stabilizing effect at all diffusivity levels.

V. CONCLUSIONS

We have shown that the stability of two-fluid flows with a miscible interface of finite thickness is qualitatively different from that of two-fluid flows with a sharp interface between the two liquids. In particular, an overlap mode of instability, resulting when the critical layer of the dominant mode is located within the viscosity gradient, is found, which does not exist for immiscible flow. This instability mechanism was first identified by Craik[31] in the long-wave limit for two-layer miscible Couette flow, and was also seen to generate new instability modes in channel flow.[40] This mechanism is here identified to be operating in interfacial flow as well. At moderate miscibility levels, a lubricating configuration of $m = 1.5$ is the most stable with respect to both overlap and surface modes, and this could be useful in applications. This is in contrast to immiscible flows as well as flows with viscosity variations (of similar magnitude) spanning the entire flow, both of which are more unstable in the lubrication configuration and stabler in the anti-lubrication. As miscibility levels are lowered ($Sc \sim 10^4$), the trend reverses, and the overlap instability begins to resemble the interfacial mode in the immiscible case. In particular, the critical Reynolds number approaches 0, and the anti-lubrication configuration is now less unstable. Flow is unstable at large viscosity contrasts irrespective of the direction of viscosity stratification. The role of the overlap mode in triggering nonlinearities will be an interesting question for the future. It is hoped that the present work will motivate experiments on miscible two-fluid free-surface flows.

APPENDIX A: ROLE OF THE MIXED LAYER THICKNESS

As discussed above, Ern *et al.*,[39] have shown that increasing the mixed layer thickness stabilizes the flow in a channel. This same behavior is displayed by the overlap and the TS modes in the present geometry as well, see Fig. 16. While the critical Reynolds number of the overlap mode is not very sensitive to the thickness q, it is seen that the range of wave numbers and Reynolds numbers over which each mode is displayed decreases with increasing q. At a q of 0.15 the two overlap modes O1 and O2 are merged into a large region of instability, whereas for $q = 0.25$, the two modes display distinct and small regions of instability. On the surface mode (Fig. 17), the effect is reversed, with increasing q implying increasing unstable region. The critical Reynolds number is also affected for this mode. In order to demonstrate that the production layers of different modes are well separated, we present in Fig. 18 the phase speeds $c = \omega/k$ for the different modes for the case $q = 0.15$ where all four modes occupy distinct regions. The TS mode propagates with the smallest phase speed as is

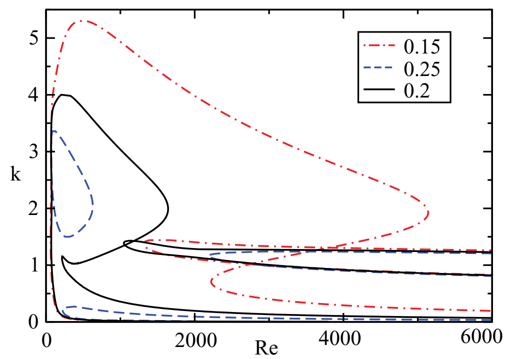

FIG. 16. Effect of changing the mixed-layer thickness q (in the legend) on the neutral boundaries of the overlap and TS modes. The other parameters are $r = 0.5$, $m = 1.5$, $\theta = 10°$, $Sc = 20$.

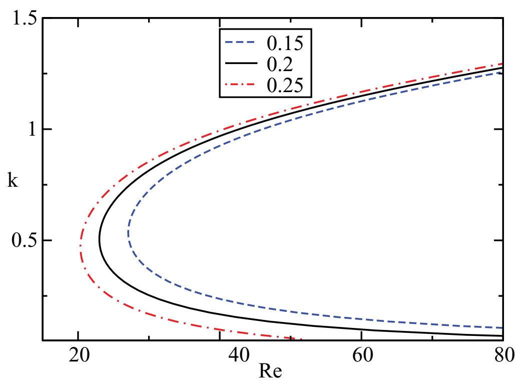

FIG. 17. Effect of changing the mixed-layer thickness q (in the legend) on the surface mode. Neutral boundaries are shown. The other parameters are $r = 0.5$, $m = 1.5$, $\theta = 10$, $Sc = 20$.

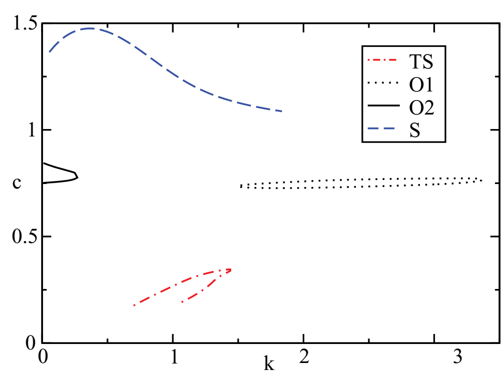

FIG. 18. Phase speed and a function of the wave number for neutral perturbations, when r =0.5, $q = 0.15$, $m = 1.5$, $\theta = 10°$, $Sc = 20$.

traditionally observed. The overlap modes are focused in a layer where $c \sim U$, whereas the surface modes show $c > 1$ exactly as in the single fluid case.

APPENDIX B: RELATION TO THE INTERFACIAL PERTURBATION IN THE IMMISCIBLE LIMIT

Since the whole viscosity field is perturbed (by the introduction of viscosity perturbation μ), a separate perturbation variable h for the interface between fluids 1, 2 and the mixing layer is not required. However, the equation for h in the immiscible case would state that the viscosity along a particle path (the perturbed interface) is constant. Equivalently, the interface follows the particles. Below, we will argue how our analysis leads to the same result in the absence of diffusion ($Pe^{-1} \to 0$).

In Eq. (13), we impose the linearized version of

$$D/Dt(\mu_B + \mu) = Pe^{-1}\nabla^2(\mu_B + \mu) \tag{B1}$$

(where $D/Dt = \frac{\partial}{\partial t} + (U+u)\frac{\partial}{\partial x} + (V+v)\frac{\partial}{\partial y}$), that is, the viscosity along a particle path changes only by viscosity diffusion. In the immiscible limit ($Pe^{-1} \to 0$), we would recover

$$D/Dt(\mu_B + \mu) = 0, \tag{B2}$$

that is, the lines of constant viscosity are equal to the particle paths (in the absence of diffusion). The perturbed interface between fluid i (i = 1,2) and the mixed layer is per definition a line of constant viscosity, and hence follows a particle path by Eq. (B2). This recovers the immiscible result.

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