Limits on the validity of the semiclassical theory

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Abstract

For want of a more natural proposal, it is generally assumed that the back-reaction of a quantised matter field on a classical metric is given by the expectation value of its energy-momentum tensor, evaluated in a specified state. This proposal can be expected to be quite sound only when the fluctuations in the energy-momentum tensor of the quantum field are negligible. Based on this condition, a dimensionless criterion has been suggested earlier by Kuo and Ford for drawing the limits on the validity of this semiclassical theory. In this paper, we examine this criterion for the case of a toy model, constructed with two degrees of freedom and a coupling between them that exactly mimics the behaviour of a scalar field in a Friedmann universe. To reproduce the semiclassical regime of the field theory, in the toy model, one of degrees of freedom is assumed to be classical and the other quantum mechanical. Also the backreaction is assumed to be given by the expectation values of the quantum operators involved in the equations of motion for the classical system. Motivated by the same physical reasoning as Kuo and Ford, we, here, suggest another criterion, one which will be shown to perform more reliably as we evaluate these criterions for different states of the quantum system in the toy model. Finally, from the results obtained we conclude that the semiclassical theory being considered for the toy model is reliable, during all stages of its evolution, only if the quantum system is specified to be in coherent like states. The implications of these investigations on field theory are discussed.

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1. Introduction

There exists a domain during the evolution of the universe when the energies of the on-going physical processes lie between Compton and Planck scales. In this domain, though it is sufficient to describe gravity by a classical metric the quantum nature of any matter field present has to be taken into account. In general relativity, the theory which we assume to describe gravity adequately in the regime of our interest, it is the energy-momentum tensor of the matter field, $T_{\mu\nu}$, that is reponsible for the classical geometry. The energy-momentum tensor for a quantum field being an operator, a c-number ought to be constructed out of this operator before the effect of the quantum field on a classical metric can be studied. It has been suggested earlier in literature[1], that the transition element $\langle\, out \,|\, \hat{T}_{\mu\nu} \,|\, in \rangle$ (where $|\, in \,\rangle$ and $|\, out \,\rangle$ are the asymptotic states of the quantum field), obtained by the variation of the effective action, be considered as the backreaction term. This transition element is in general a complex quantity and may lead to a complex metric which will prove rather difficult to interpret unless the imaginary part happens to be negligible or is dropped in an ad hoc manner. A more natural and plausible proposal[2, 3, 4] would be to consider the expectation value of the energy-momentum operator of the quantum field as the term that induces the non-trivial geometry. Since the theory being considered here, by itself, is incapable of providing us with a preferred state for the quantum matter field, the expectation values are to be evaluated in a state that has to be specified by hand. So the analysis of the action of a quantum field; say a massless scalar field, on the classical background metric reduces to that of solving the Einstein's equations

$$G_{\mu\nu} \;=\; R_{\mu\nu} \;-\; \frac{1}{2}\, g_{\mu\nu}\, R \;=\; 8\,\pi \left\langle\, \hat{T}_{\mu\nu} \,\right\rangle \tag{1}$$

where $\left\langle\, \hat{T}_{\mu\nu} \,\right\rangle$ is the expectation value of the energy-momentum operator (in the specified state) and the Klein-Gordon equation

$$\nabla_\mu\, \nabla^\mu\, \hat{\Phi}(x) \;=\; 0 \tag{2}$$

where $\hat{\Phi}$ is the operator corresponding to the quantised scalar field, self-consistently. (We adopt the convention $\hbar = G = c = 1$ and a metric signature of (-2) in this paper.)

Apart from the fact that the geometry has to be classical for the semiclassical theory proposed above to be reliable, *i.e* the energy scales involved should be far below the Planck scale, the fluctuations in the energy-momentum densities of the quantum field should not be too large either[3], *i.e*

$$\left\langle\, \hat{T}_{\alpha\beta}(x)\, \hat{T}_{\mu\nu}(y) \,\right\rangle \;\approx\; \left\langle\, \hat{T}_{\alpha\beta}(x) \,\right\rangle \left\langle\, \hat{T}_{\mu\nu}(y) \,\right\rangle. \tag{3}$$

So, the semiclassical theory based on (1) cannot be trusted for those states of the quantum field where the fluctuations in the energy-momentum densities are too large. The goal of this present paper is to check the validity of the semiclassical theory based on the equations (1) and (2) in time dependant background metrics like for instance, Friedmann models, for different states prescribed for the quantum field.

The calculations necessary for drawing the limits, with aid of the constraint equation (3), on the validity of the semiclassical theory proposed above will involve divergences of quantum field theory and their regularisation procedures. Since these schemes might eventually prove to sidetrack the issue of our concern, instead of analysing the validity of the semiclassical theory for the case of quantised scalar fields in time dependant metrics, we, in this paper, will study the same for the case of a toy model.

The toy model will consist of two degrees of freedom. It will be constructed such that the coupling between the two degrees of freedom exactly resembles that of a scalar field evolving in a Friedmann universe. Of the two degrees of freedom, one of them will be assumed to be classical and the other quantum mechanical so that the toy model reproduces the semiclassical nature of the field theory in the domain of our interest. The analysis of the the validity of the semiclassical theory will be carried out for the toy model with the assumption that the backreaction term is given by the expectation values (in the specified state) of the quantum operators involved in the equations of motion for the classical degree of freedom.

This paper is organised as follows. In section **2** we construct the action for the toy model taking cues from the coupling of the scalar field to a Friedmann metric, obtain the classical equations of motion and then extend the relevant equations to the semiclassical domain as described in the previous paragraph. In section **3**, the quantisation of a time dependant simple harmonic oscillator, the quantum degree of freedom in the toy model, is carried out in the Heisenberg picture. In section **4** we briefly review the motivations behind the criterion proposed by Kuo and Ford to draw the limits on the validity of the semiclassical theory, evaluate this criterion for a particular case in the toy model and point out its drawbacks. We then suggest another criterion, based on the same physical reasoning as Kuo and Ford's, but one that is more reliable, to obtain the limits on the validity of the semiclassical theory being considered for the toy model. In section **5** we calculate the criterion we have suggested and the one that has been put forward by Kuo and Ford for three different quantum states of the simple harmonic oscillator *viz* (i) vacuum, (ii) nth excited and (iii) coherent states. In the final section **6**, we comment on the conclusions that can be drawn from our investigations and also disuss the possible implications of our analysis on the study of quantum field theory in time dependant background metrics.

2. Constructing the action for the toy model

The action for a massless scalar field coupled to gravity is given by the equation

$$\mathcal{A} \;=\; \int d^4x\, \sqrt{-g}\, \left(\frac{1}{16\,\pi}\, R \;+\; \frac{1}{2}\, \partial_\mu \Phi\, \partial^\mu \Phi \right). \tag{4}$$

Variation of the action (4) with respect to the metric $g_{\mu\nu}$ and the field Φ would lead us to the Euler-Lagrange equations for gravity and the matter field repsectively (*viz* equations (1) and (2) with just the classical Φ and $T_{\mu\nu}$ and not their quantum operators). For a spatially flat Friedmann model described by the metric

$$ds^2 \;=\; dt^2 \;-\; a^2(t)\, \left(dx^2 \;+\; dy^2 \;+\; dz^2 \right) \tag{5}$$

where a is the scale factor, the action (4) reduces to be

$$\mathcal{A} \;=\; \int d^3x\, dt\, a^3 \left\{ -\frac{3}{8\,\pi} \left(\frac{\ddot{a}}{a} \;+\; \frac{\dot{a}^2}{a^2} \right) \;+\; \frac{1}{2} \left(\left\{ \frac{\partial \Phi}{\partial t} \right\}^2 \;-\; \left\{ \frac{|\nabla \Phi|}{a} \right\}^2 \right) \right\} \tag{6}$$

where the dots denote derivatives with respect to the comoving time t. Since any action is expected to involve derivatives of the degrees of freedom no further than the first, the terms involving the second time derivatives of a in the above action can be eliminated by integrating them by parts with respect to t. The reduced action, after the integration by parts, is obtained to be

$$\mathcal{A} \;=\; \int d^3x\, dt\, a^3 \left\{ -\frac{3}{8\pi} \left(\frac{\dot{a}^2}{a^2} \right) \;+\; \frac{1}{2} \left(\left\{ \frac{\partial \Phi}{\partial t} \right\}^2 \;-\; \left\{ \frac{|\nabla \Phi|}{a} \right\}^2 \right) \right\}. \tag{7}$$

For a time dependant metric, assumed to be homogenous and isotropic, varying the action (4) with respect to the components g_{00} and $g_{ik}(i,\, k \;=\; 1,\, 3)$ of the metric tensor ($g_{0i} \;=\; 0$ due to isotropy) yields the two Friedmann equations. Of these two, the equation obtained by varying g_{ik} involves the second time derivatives of the scale factor, whereas the one obtained by varying g_{00} is a constraint equation and depends only the on first time derivatives of a. On the other hand, the variation of the action (7) with respect to a will only yield the equation involving the second time derivatives of a and constraint equation can not be obtained from the reduced action. The reason being that in arriving at (7) from (4) we have lost the time-reparametrisation invariance of a Friedmann model having chosen g_{00} to be unity in the metric (5). To reproduce this 'lost' equation, the relevant degree of freedom has to be re-introduced in the Friedmann metric. Introducing an arbitrary function $N(t)$ into the metric (5), as follows

$$ds^2 \;=\; N^2(t)\, dt^2 \;-\; a^2(t)\, \left(dx^2 \;+\; dy^2 \;+\; dz^2 \right), \tag{8}$$

we obtain the reduced action for the above metric to be

$$\mathcal{A} \; = \; \int d^3x \, dt \, a^3 \left\{ -\frac{3}{8\,\pi\,N(t)} \left(\frac{\dot{a}^2}{a^2} \right) \; + \; \frac{1}{2} \left(\frac{1}{N(t)} \left\{ \frac{\partial \Phi}{\partial t} \right\}^2 \; - \; N(t) \left\{ \frac{|\nabla \Phi|}{a} \right\}^2 \right) \right\}. \tag{9}$$

Now, varying the above action with respect to N and setting N to be unity after the variation yields

$$\left(\frac{\dot{a}^2}{a^2} \right) \; = \; \frac{4\,\pi}{3} \left\{ \left(\frac{\partial \Phi}{\partial t} \right)^2 \; + \; \left(\frac{|\nabla \Phi|}{a} \right)^2 \right\} \tag{10}$$

which is the Friedmann equation we are interested in.

Having learned the aspects of the coupling of a scalar field to a Friedmann metric from the brief review above, it is easy to see that the following action for two coupled degrees of freedom C and q

$$\mathcal{A} \; = \; \int dt \left\{ -\frac{M}{2} \frac{\dot{C}^2}{N} \; + \; \frac{m}{2} \frac{\dot{q}^2}{N} \; - \; N \frac{m}{2} \omega^2(C) \, q^2 \right\} \tag{11}$$

exactly reproduces all the features of our interest in the reduced action (7). (The dots hereafter denote derivatives with respect to t, the time parameter for the toy model.) The variation of the action (11) with respect to N when N is set to be unity after the variation, yields the equation of constraint

$$\frac{M \, \dot{C}^2}{2} \; = \; \left(\frac{m \, \dot{q}^2}{2} \; + \; \frac{m \, \omega^2(C) \, q^2}{2} \right). \tag{12}$$

The above equation can be expressed as

$$\frac{M\dot{C}^2}{2} \; = \; H \tag{13}$$

where H, the Hamiltonian corresponding to the variable q is given by

$$H \; = \; \left(\frac{p^2}{2m} \; + \; \frac{m \, \omega^2(C) \, q^2}{2} \right). \tag{14}$$

(p in the above equation represents the momentum conjugate to the degree of freedom q), similar in structure to the Friedmann equation (10). The Euler-Lagrange equation for the variable q is

$$\ddot{q} \; + \; \omega^2(C) \, q \; = \; 0. \tag{15}$$

By comparing the actions (4) and (11), it can be easily seen that the degree of freedom C is expected to behave like the scale factor a of a Friedmann model and the the variable q is to mimic the behaviour of the scalar field.

Also, if our toy model is to reproduce the behaviour of a *quantised* scalar field in a Friedmann metric, we have to assume that the degree of freedom q behaves quantum mechanically. Then extending the classical equations of motion for C and q to the semi-classical domain in a fashion similar to what has been done in (1), we obtain that

$$\frac{M\,\dot{C}^2}{2} \;=\; \left\langle\, \hat{H}(t)\, \right\rangle \;=\; E(t) \tag{16}$$

where $\left\langle\, \hat{H}(t)\, \right\rangle$ is the expectation value (in the specified state) of the Hamiltonian operator corresponding to the classical Hamiltonian given by (14) for the degree of freedom q.

The evolution of quantum system q, a time-dependant oscillator will be discussed in the following section.

3. Quantisation of the time dependant oscillator: Heisenberg picture

The Hamiltonian operator corresponding to the the degree of freedom q is

$$\hat{H} \;=\; \frac{\hat{p}^2}{2m} \;+\; \frac{m}{2}\,\omega^2(C)\,\hat{q}^2 \tag{17}$$

where $\hat{q}$ and $\hat{p}$ are the operators corresponding to the variable q and its conjugate momentum p. In the Heisenberg picture the operators are dependant on time and the operators $\hat{q}$ and $\hat{p}$ satisfy the equations[5]

$$\frac{d\hat{q}}{dt} \;=\; i\,[\hat{H},\,\hat{q}] \;=\; \frac{\hat{p}}{m} \tag{18}$$

and

$$\frac{d\hat{p}}{dt} \;=\; i\,[\hat{H},\,\hat{p}] \;=\; -\,m\,\omega^2(C)\,\hat{q}. \tag{19}$$

Substituting (18) in (19) we obtain that

$$\frac{d^2\hat{q}}{dt^2} \;+\; \omega^2(C)\,\hat{q} \;=\; 0 \tag{20}$$

the operator equation corresponding to the classical equation (15).

The conjugate variables q and p being observables, *viz* the position and the momentum of the harmonic oscillator, the corresponding operators are to be hermitian, *i.e* $\hat{q}^\dagger \;=\; \hat{q}$ and $\hat{p}^\dagger \;=\; \hat{p}$. So the solutions to the Heisenberg equations of motion (18) and (19) can be written down to be

$$\hat{q} \;=\; \left(\,\hat{A}\,Q \;+\; \hat{A}^\dagger\,Q^*\,\right) \tag{21}$$

and

$$\hat{p} \;=\; m\,\frac{d\hat{q}}{dt} \;=\; m\,\left(\,\hat{A}\,\dot{Q} \;+\; \hat{A}^\dagger\,\dot{Q}^*\,\right), \tag{22}$$

where $\hat{A}$ is a time-independant operator and Q satisfies the differential equation

$$\frac{d^2 Q}{dt^2} \; + \; \omega^2(C)\, Q \; = \; 0. \tag{23}$$

This equation can be solved by the ansatz[6]

$$Q \; = \; \left(\frac{\alpha(t)\, f \; + \; \beta(t)\, f^*}{\sqrt{2\, m\, \omega}} \right) \tag{24}$$

where

$$f \; = \; exp - \left\{ \int_{t_0}^{t} dt'\, \omega \,\{\, C(t')\,\} \right\} \tag{25}$$

and t_0 is an *early* time when the initial conditions for the differential equation (23) will be specified. Defining $\dot{Q} \; = \; (dQ/dt)$ to be

$$\dot{Q} \; = \; (-i\, \omega) \left(\frac{\alpha\, f \; + \; \beta\, f^*}{\sqrt{2\, m\, \omega}} \right) \tag{26}$$

(where f^* is the complex conjugate of f) and incorporating (24) and (26) in (23) we find that α and β satisfy the set of coupled equations

$$\dot{\alpha} \; = \; \left(\frac{\dot{\omega}}{2\, \omega} \right) \beta\, f^{*\,2} \quad ; \quad \dot{\beta} \; = \; \left(\frac{\dot{\omega}}{2\, \omega} \right) \alpha\, f^2. \tag{27}$$

Integrating the wronskian condition for the differential equation (23), *viz*

$$\frac{d}{dt} \left(Q\, \dot{Q}^* \; - \; Q^*\, \dot{Q} \right) \; = \; 0$$

we obtain that

$$\left(Q\, \dot{Q}^* \; - \; Q^*\, \dot{Q} \right) \; = \; \frac{i}{m} \tag{28}$$

where the constant of integration has been chosen to be (i/m). Substituting (24) and (26) into the above equation we find that the wronskian condition reduces to the relation

$$|\, \alpha \,|^2 \; - \; |\, \beta \,|^2 \; = \; 1. \tag{29}$$

Also, when (24) and (26) are substituted in the equations (18) and (19) we obtain that the evolution of the two operators $\hat{q}$ and $\hat{p}$ are described by the equations

$$\hat{q} \; = \; \frac{1}{\sqrt{2\, m\, \omega}} \left\{ \left(\alpha\, \hat{A} \; + \; \beta^*\, \hat{A}^\dagger \right) f \; + \; \left(\alpha^*\, \hat{A}^\dagger \; + \; \beta\, \hat{A} \right) f^* \right\} \tag{30}$$

and

$$\hat{p} \; = \; i \sqrt{\frac{m\, \omega}{2}} \left\{ - \left(\alpha\, \hat{A} \; + \; \beta^*\, \hat{A}^\dagger \right) f \; + \; \left(\alpha^*\, \hat{A}^\dagger \; + \; \beta\, \hat{A} \right) f^* \right\}. \tag{31}$$

When the above solutions are substituted in the following commutation relations for the operators corresponding to the canonically conjugate variables

$$[\hat{q},\, \hat{q}] \;=\; 0 \quad ; \quad [\hat{p},\, \hat{p}] \quad ; \quad [\hat{q},\, \hat{p}] \;=\; i, \tag{32}$$

we obtain the commutation relations between the operators $\hat{A}$ and $\hat{A}^{\dagger}$ to be

$$[\hat{A},\, \hat{A}] \;=\; 0 \quad ; \quad [\hat{A}^{\dagger},\, \hat{A}^{\dagger}] \;=\; 0 \quad ; \quad [\hat{A},\, \hat{A}^{\dagger}] \;=\; 1. \tag{33}$$

If the initial conditions for the equations of motion are chosen to be such that $\alpha(t_0) = 1$, $\beta(t_0) = 0$ and $\omega(t_0) = \omega_0$ (an arbitrary constant) at $t = t_0$ then the operators $\hat{q}$ and $\hat{p}$ at $t = t_0$ are given by the equations

$$\hat{q} \;=\; \frac{1}{\sqrt{2\, m\, \omega_0}} \left(\hat{A} \,+\, \hat{A}^{\dagger} \right) \quad ; \quad \hat{p} \;=\; i \sqrt{\frac{m\, \omega_0}{2}} \left(-\hat{A} \,+\, \hat{A}^{\dagger} \right). \tag{34}$$

In the Heisenberg picture the quantum states are independent of time. The state of the quantum system is usually prescribed at the same instant when the initial conditions for the equations of motion are specified. The quantum states for the matter field in which the backreaction problem in field theory is generally studied are the vacuum, n-particle and coherent states. The corresponding states for the quantum oscillator in our toy model can be defined as follows: the vacuum state $|\,0\,\rangle$ satisfies the conditon

$$\hat{A}\,|\,0\,\rangle \;=\; 0,$$

a n-particle state $|\,n\,\rangle$ is defined such that

$$\hat{A}^{\dagger}\, \hat{A}\,|\,n\,\rangle \;=\; n\,|\,n\,\rangle$$

and a coherent state $|\,\lambda\,\rangle$ follows the equation

$$\hat{A}\,|\,\lambda\,\rangle \;=\; \lambda\,|\,\lambda\,\rangle.$$

Substituting (30) and (31) in (17) we can see that the Hamiltonian operator at any time $t > t_0$ is given by

$$\hat{H}(t) \;=\; \left\{ \hat{a}^{\dagger}(t)\, \hat{a}(t) \;+\; \frac{1}{2} \right\} \omega \tag{35}$$

where

$$\hat{a}(t) \;=\; \alpha(t)\, \hat{A} \;+\; \beta^{*}(t)\, \hat{A}^{\dagger}. \tag{36}$$

The equation of motion for C, (16), then reduces to

$$\frac{M\, \dot{C}^2}{2} \;=\; \langle\, \hat{H}(t)\, \rangle \;=E(t) \;=\; \left\langle \left\{ \hat{a}^{\dagger}(t)\, \hat{a}(t) \;+\; \frac{1}{2} \right\} \omega \right\rangle \tag{37}$$

where the expectation value is evaluated in the state specified for the quantum oscillator. The expectation value of the square of the Hamiltonian operator, which will be needed later to evaluate the fluctuations in the energy, is

$$E^2(t) \;=\; \langle\, \hat{H}^2(t)\,\rangle \;=\; \left\langle \left\{ \hat{a}^{\dagger}(t)\,\hat{a}(t) \;+\; \frac{1}{2} \right\} \omega \, \left\{ \hat{a}^{\dagger}(t)\,\hat{a}(t) \;+\; \frac{1}{2} \right\} \omega \right\rangle. \tag{38}$$

4. Criterions for drawing the limits on semiclassical theory

The semiclassical theory as described by the equations (1) and (2) does not account for the fluctuations in the energy-momentum densities of the quantum field. So, as mentioned earlier, this theory can be relied upon only when the fluctuations in the energy-momentum densities are small when compared to their expectation values.

Motivated by this fact, Kuo and Ford[7] have suggested that the dimensionless quantity

$$\Delta_{\alpha\beta\mu\nu}(x,\; y) \;\equiv\; \left| \frac{\langle\, : \hat{T}_{\alpha\beta}(x)\,\hat{T}_{\mu\nu}(y) :\, \rangle \;-\; \langle\, : \hat{T}_{\alpha\beta}(x) :\, \rangle \langle\, : \hat{T}_{\mu\nu}(x) :\, \rangle}{\langle\, : \hat{T}_{\alpha\beta(x)}\,\hat{T}_{\mu\nu}(y) :\, \rangle} \right| \tag{39}$$

(where the semicolons represent renormalisation of the expectation values) be considered as a measure of the fluctuations in the energy-momentum densities of the quantum field. When the fluctuations in the energy-momentum densities are negligible, this quantity will be far less than unity and the semiclassical theory being considered here will prove to be quite sound. But when the fluctuations are large the above quantity is expected to be of order unity reflecting a complete breakdown of the proposed semiclassical theory.

The numerous components and the dependance on the two spacetime points make the quantity $\Delta_{\alpha\beta\mu\nu}(x,\, y)$ an extremely cumbersome object to handle. For the sake of simplicty, as Kuo and Ford themselves suggest, we can confine our attention to either the evaluation of the purely temporal component of this quantity in the coincidence limit (*i.e* when $x \,\to\, y$)

$$\Delta_{KF2}(x) \;\equiv\; \left| \frac{\langle\, : \hat{T}_{00}^2(x) :\, \rangle \;-\; \langle\, : \hat{T}_{00}(x) :\, \rangle^2}{\langle\, : \hat{T}_{00}^2(x) :\, \rangle} \right| \tag{40}$$

(subscript KF standing for Kuo and Ford) or the criterion

$$\Delta_{KF1}(x) \;\equiv\; \left| \frac{\langle\, : \hat{T}_{00}^2(x) :\, \rangle \;-\; \langle\, : \hat{T}_{00}(x) :\, \rangle^2}{\langle\, : \hat{T}_{00}(x) :\, \rangle^2} \right|. \tag{41}$$

The quantities Δ_{KF1} and Δ_{KF2} are related to each other by the equation

$$\Delta_{KF2} \;=\; \left(\frac{\Delta_{KF1}}{\Delta_{KF1} \;+\; 1} \right). \tag{42}$$

In the next section, when similar quantities are evaluated for the case of our toy model it will be shown that the criterions Δ_{KF1} and Δ_{KF2} yield equivalent results. The evaluation of Δ_{KF2} was carried out by Kuo and Ford for different states of a quantised massless scalar field in flat space. The limits on the validity of the semiclassical theory can not be drawn from the evaluation of these criterions in flat space but has to be carried out for the case of quantum fields in curved backgrounds.

For the Friedmann model, the time dependant metric for which we are trying to obtain the limits on the validity of the semiclassical theory as described by equations (1) and (2), the adiabatic limit corresponds to the case when the scale factor a is a slowly varying function of time, *i.e* when $(\dot{a}/a) \rightarrow 0$, In this limit the Friedmann metric is almost Minkoskian and in flat space the fluctuations in the energy-momentum densities *are* neglible. Extending this example, in the adiabatic limit, the fluctuations in the energy-momentum densities of a quantum field in an arbitrary spacetime can be, in general, expected to be small. And a dimensionless criterion, supposed to reflect the magnitude of these fluctuations, proposed for drawing the limits on the validity of the semiclassical theory, should identically vanish in this limit.

The adiabatic limit for our toy model corresponds to the case when the coupling term $\omega(C(t))$ is a slowly evolving function in time, *i.e* when $(d\omega/dt) \rightarrow 0$. The equation (27) implies that, for the initial conditions that have been specified (*viz* $\alpha(t_0) = 1$ and $\beta(t_0) = 0$), in the adiabatic limit $\alpha \rightarrow 1$ and $\beta \rightarrow 0$ for $t > t_0$. And in this limit, since the semiclassical theory that has been put forward for the toy model is known to be reliable, any criterion suggested for drawing the limits on its validity should reduce to zero when $\beta \rightarrow 0$.

The back-reaction term (refer to (16)) for our toy mechanical model is the expectation value of the Hamiltonian corresponding to the quantum degree of freedom. So, for the toy model, it is the magnitude of the fluctuations in the energy of the quantum oscillator that will decide the validity of the semiclassical theory. So the quantities equivalent to Δ_{KF1} and Δ_{KF2} for the case of our toy model are

$$\Delta_{KF1}(t) \equiv \left| \frac{\left\langle :\hat{H}^2(t): \right\rangle - \left\langle :\hat{H}(t): \right\rangle^2}{\left\langle :\hat{H}(t): \right\rangle^2} \right|, \tag{43}$$

and

$$\Delta_{KF2}(t) \equiv \left| \frac{\left\langle :\hat{H}^2(t): \right\rangle - \left\langle :\hat{H}: \right\rangle^2}{\left\langle :\hat{H}^2(t): \right\rangle} \right|. \tag{44}$$

The semi-colons in the above quantities represent regularisation, performed either by normal ordering of the operators or by vacuum subtraction. Renormalisation of the expectation values can be achieved through normal ordering by moving all the $\hat{a}^{\dagger}$'s to the left of $\hat{a}$'s in the various operators involved. And vacuum subtraction implies regularisation carried out by deducting the vacuum term that prevails at $t = t_0$ (when $\alpha = 1$ and $\beta = 0$) from the expectation values.

Though a detailed analysis will be carried out in the next section, to illustrate the drawbacks of the criterions Δ_{KF1} and Δ_{KF2}, we, in this section, evaluate these quantities for a particular case of the toy model. When the the quantum system in our toy model is specified to be in a vacuum state and renormalisation is achieved by normal ordering, the square of the fluctuations in energy of the quantum oscillator is given by the equation

$$\left\langle\, :\hat{H}^2:\, \right\rangle \;-\; \left\langle\, :\hat{H}:\, \right\rangle^2 \;=\; |\,\beta\,|^2 \;+\; 2\,|\,\beta\,|^4. \tag{45}$$

and the criterions Δ_{KF1} and Δ_{KF2} follow the expressions

$$\Delta_{KF1(NO)} \;=\; \left\{\frac{1\;+\;2\,|\,\beta\,|^2}{|\,\beta\,|^2}\right\} \quad ; \quad \Delta_{KF2(NO)} \;=\; \left\{\frac{1\;+\;2\,|\,\beta\,|^2}{1\;+\;3\,|\,\beta\,|^2}\right\}.$$

(The subscript (NO) represents normal ordering.) The above equations clearly show that in the adiabatic limit, *i.e* when $\beta\;\rightarrow\;0$, though the fluctuations in the energy of the oscillator do vanish completely, the criterions Δ_{KF1} and Δ_{KF2} rather than reducing to zero, as they should, they take on the values infinity and unity respectively suggesting a breakdown of the semiclassical theory. But in the adiabatic limit, the semiclassical theory *is* reliable and the criterions *should* vanish.

This 'bad' behaviour on the part of these criterions can be corrected for, if the expectation values of the operators in the quantities Δ_{KF1} and Δ_{KF2} are not renormalised. Or in other words, if the vacuum terms are re-introduced in the expectation values that these criterions are made up of. Performing this, we obtain the criterions necessary for drawing the limits on the validity of the semiclassical theory for the toy model to be either the quantity

$$\Delta_{SC1}(t) \;\equiv\; \left|\,\frac{\left\langle\,\hat{H}^2(t)\,\right\rangle \;-\; \left\langle\,\hat{H}(t)\,\right\rangle^2}{\left\langle\,\hat{H}(t)\,\right\rangle^2}\,\right|, \tag{46}$$

(subscript SC stands for semiclassical) or the one

$$\Delta_{SC2}(t) \;\equiv\; \left|\,\frac{\left\langle\,\hat{H}^2(t)\,\right\rangle \;-\; \left\langle\,\hat{H}(t)\,\right\rangle^2}{\left\langle\,\hat{H}^2(t)\,\right\rangle}\,\right|. \tag{47}$$

The two quantities Δ_{SC1} and Δ_{SC2} are related to each other by the equation

$$\Delta_{SC2} \;=\; \left(\frac{\Delta_{SC1}}{\Delta_{SC1}\;+\;1}\right). \tag{48}$$

When no renormalisation is carried out the square of the fluctuations in the energy of the oscillator is

$$\left\langle\,\hat{H}^2\,\right\rangle \;-\; \left\langle\,\hat{H}\,\right\rangle^2 \;=\; 2\,|\,\beta\,|^2 \;+\; 2\,|\,\beta\,|^4$$

so that the quantities Δ_{SC1} and Δ_{SC2} are given by the expressions

$$\Delta_{SC1} \;=\; \left\{ \frac{2\,|\,\beta\,|^2 \;+\; 2\,|\,\beta\,|^4}{|\,\beta\,|^2 \;+\; |\,\beta\,|^4 \;+ \frac{1}{4}} \right\} \quad ; \quad \Delta_{SC2} \;=\; \left\{ \frac{2\,|\,\beta\,|^2 \;+\; 2\,|\,\beta\,|^4}{3\,|\,\beta\,|^2 \;+\; 3\,|\,\beta\,|^4 \;+\; \frac{1}{4}} \right\}.$$

The two quantities Δ_{SC1} and Δ_{SC2} do vanish in the adiabatic limit ($i.e$ when $\beta \rightarrow 0$), clearly illustrating that the criterions Δ_{SC1} and Δ_{SC2} perform more reliably than either Δ_{KF1} or Δ_{KF2}.

5. Δ_{KF} and Δ_{SC} for different quantum states

In the following three sub-sections we evaluate the Δ's for different states of the quantum oscillator in our toy model.

(i). In a vacuum state

If the state of the quantum oscillator is defined to be a vacuum state at the early time t_0 (when the other initial conditions have been specified) then the expectation values of the operators $\hat{H}$ and $\hat{H}^2$ without any renormalisation are

$$\begin{aligned} E \;=\; \langle\, \hat{H} \,\rangle = \langle\, 0\,| \left\{ \hat{a}^\dagger\, \hat{a} \;+\; \frac{1}{2} \right\} \omega |\, 0 \,\rangle \\ = \left\{ |\,\beta\,|^2 \;+\; \frac{1}{2} \right\} \omega \end{aligned} \tag{49}$$

and

$$\begin{aligned} E^2 \;=\; \langle\, \hat{H}^2 \,\rangle \;=\; \langle\, 0\,| \left\{ \hat{a}^\dagger\, \hat{a} \;+\; \frac{1}{2} \right\} \omega \left\{ \hat{a}^\dagger\, \hat{a} \;+\; \frac{1}{2} \right\} \omega |\, 0 \,\rangle \\ = \left\{ 3\,|\,\beta\,|^2 \;+\; 3\,|\,\beta\,|^4 \;+\; \frac{1}{4} \right\} \omega^2. \end{aligned} \tag{50}$$

When the operators involved in the above expressions are normal ordered, the expectation values are

$$\begin{aligned} E_{(NO)} \;=\; \langle\; :\hat{H}:\; \rangle = \langle\, 0\,| \left\{ \hat{a}^\dagger\, \hat{a} \right\} \omega |\, 0 \,\rangle \\ = |\,\beta\,|^2\, \omega \end{aligned} \tag{51}$$

and

$$\begin{aligned} E^2_{(NO)} \;=\; \langle\; :\hat{H}^2:\; \rangle \;=\; \langle\, 0\,| \left\{ a^\dagger\, \hat{a}^\dagger\, \hat{a}\, \hat{a} \right\} \omega^2 |\, 0 \,\rangle \\ = \left\{ |\,\beta\,|^2 \;+\; 3\,|\,\beta\,|^4 \right\} \omega^2. \end{aligned} \tag{52}$$

For the case, when renormalisation is achieved by the vacuum subtraction, $i.e$

$$\begin{aligned} E_{(VS)} \;&=\; \langle\; :\hat{H}:\; \rangle \\ &= \langle\, 0\,| \left\{ \hat{a}^\dagger\, \hat{a} \;+\; \frac{1}{2} \right\} \omega |\, 0 \,\rangle \;-\; \langle\, 0\,| \left\{ \hat{A}^\dagger\, \hat{A} \;+\; \frac{1}{2} \right\} \omega |\, 0 \,\rangle \end{aligned} \tag{53}$$

and

$$
\begin{aligned}
E^2_{(VS)} &= \langle\, : \hat{H}^2 : \,\rangle \\
&= \langle\, 0 \,| \left\{ \hat{a}^\dagger \, \hat{a} \, + \, \frac{1}{2} \right\} \omega \, \left\{ \hat{a}^\dagger \, \hat{a} \, + \, \frac{1}{2} \right\} \omega | \, 0 \,\rangle \\
&- \langle\, 0 \,| \left\{ \hat{A}^\dagger \, \hat{A} \, + \, \frac{1}{2} \right\} \omega \, \left\{ \hat{A}^\dagger \, \hat{A} \, + \, \frac{1}{2} \right\} \omega \, | \, 0 \,\rangle,
\end{aligned} \tag{54}
$$

the expressions for $E_{(VS)}$ and $E_{(VS)}$ are the same as the quantities E and E^2 but without the $(\omega/2)$ and the $(\omega^2/4)$ terms respectively. Substituting the above results in the equations (43) and (44), we obtain that

$$
\Delta_{KF1(NO)} \;=\; \left\{ \frac{1 \,+\, 2\,|\,\beta\,|^2}{|\,\beta\,|^2} \right\} \quad ; \quad \Delta_{KF2(NO)} \;=\; \left\{ \frac{1 \,+\, 2\,|\,\beta\,|^2}{1 \,+\, 3\,|\,\beta\,|^2} \right\} \tag{55}
$$

and

$$
\Delta_{KF1(VS)} \;=\; \left\{ \frac{3 \,+\, 2\,|\,\beta\,|^2}{|\,\beta\,|^2} \right\} \quad ; \quad \Delta_{KF2(VS)} \;=\; \left\{ \frac{3 \,+\, 2\,|\,\beta\,|^2}{3 \,+\, 3\,|\,\beta\,|^2} \right\}, \tag{56}
$$

where the subscripts (NO) and (VS) represent renormalisation by normal ordering and vacuum subtraction respectively.

From the expectation values evaluated above, we find that the criterions we have suggested for drawing the limits on the semiclassical theory for the toy model, *viz* $\Delta_{SC1\,\&\,2}$ are given by the expressions

$$
\Delta_{SC1} \;=\; \left\{ \frac{2\,|\,\beta\,|^2 \,+\, 2\,|\,\beta\,|^4}{|\,\beta\,|^2 \,+\, |\,\beta\,|^4 \,+\, \frac{1}{4}} \right\} \quad ; \quad \Delta_{SC2} \;=\; \left\{ \frac{2\,|\,\beta\,|^2 \,+\, 2\,|\,\beta\,|^4}{3\,|\,\beta\,|^2 \,+\, 3\,|\,\beta\,|^4 \,+\, \frac{1}{4}} \right\}. \tag{57}
$$

(ii). In a nth excited state

For the case when the quantum state of the oscillator is specified to be a nth excited state, the expectation values when no renormalisation has been carried out are

$$
\begin{aligned}
E \;=\; \langle\, \hat{H} \,\rangle = \;&= \langle\, n \,| \left\{ \hat{a}^\dagger \, \hat{a} \, + \, \frac{1}{2} \right\} \omega | \, n \,\rangle \\
&= \left\{ |\,\beta\,|^2 \, (2\,n \,+\, 1) \,+\, n \,+\, \frac{1}{2} \right\} \omega
\end{aligned} \tag{58}
$$

and

$$
\begin{aligned}
E^2 \;&=\; \langle\, :\, \hat{H}^2\, :\, \rangle \;=\; \langle\, n\,|\left\{\hat{a}^\dagger\,\hat{a} \;+\; \frac{1}{2}\right\}\,\omega\,\left\{\hat{a}^\dagger\,\hat{a} \;+\; \frac{1}{2}\right\}\,\omega|\,n\,\rangle \\
&= \left\{\left(n^2 \;+\; n\right)\left(1 \;+\; 6\,|\,\beta\,|^2 \;+\; 6\,|\,\beta\,|^4\right)\right. \\
&\qquad\qquad \left. +\; \left(3\,|\,\beta\,|^2 + \;3\,|\,\beta\,|^4 \;+\; \frac{1}{4}\right)\;\right\}\omega^2.
\end{aligned} \tag{59}
$$

When renormalisation is achieved by normal ordering, the expecation values are given by the expressions

$$
\begin{aligned}
E_{(NO)} \;&=\; \langle\, :\hat{H}:\,\rangle = \;\langle\, n\,|\,\{\,\hat{a}^\dagger\,\hat{a}\,\}\;\omega|\,n\,\rangle \\
&= \left\{\,|\,\beta\,|^2\;(2\,n \;+\; 1\,) \;+\; n\right\}\,\omega
\end{aligned} \tag{60}
$$

and

$$
\begin{aligned}
E^2_{(NO)} \;&=\; \langle\, :\hat{H}^2:\,\rangle \;=\; \langle\, n\,|\,\{\,\hat{a}^\dagger\,\hat{a}^\dagger\,\hat{a}\,\hat{a}\,\}\;\omega^2|\,n\,\rangle \\
&= \left\{n^2\left(1 \;+\; 6\,|\,\beta\,|^2 \;+\; 6\,|\,\beta\,|^4\right) \;+\; n\,\left(-1 \;+\; 2\,|\,\beta\,|^2 \;+\; 6\,|\,\beta\,|^4\right)\right. \\
&\qquad\qquad\qquad \left. +\;\left(\,|\,\beta\,|^2 \;+\; 3\,|\,\beta\,|^4\right)\;\right\}\omega^2.
\end{aligned} \tag{61}
$$

For regularisation of the expectation values by vacuum subtraction, *i.e*

$$
\begin{aligned}
E_{(VS)} \;&=\; \langle\; :\hat{H}:\;\rangle \\
&= \langle\, n\,|\left\{\hat{a}^\dagger\,\hat{a} \;+\; \frac{1}{2}\right\}\,\omega|\,n\,\rangle - \;\langle\, 0\,|\left\{\hat{A}^\dagger\,\hat{A} \;+\; \frac{1}{2}\right\}\,\omega|\,0\,\rangle
\end{aligned} \tag{62}
$$

and

$$
\begin{aligned}
E^2_{(VS)} \;&=\; \langle\; :\hat{H}^2:\;\rangle \\
&= \langle\, n\,|\left\{\hat{a}^\dagger\,\hat{a} \;+\; \frac{1}{2}\right\}\,\omega\,\left\{\hat{a}^\dagger\,\hat{a} \;+\; \frac{1}{2}\right\}\,\omega|\,n\,\rangle \\
&- \;\langle\, 0\,|\left\{\hat{A}^\dagger\,\hat{A} \;+\; \frac{1}{2}\right\}\,\omega\,\left\{\hat{A}^\dagger\,\hat{A} \;+\; \frac{1}{2}\right\}\,\omega|\,0\,\rangle,
\end{aligned} \tag{63}
$$

the expressions for $E_{(VS)}$ and $E^2_{(VS)}$ are the same as the quantities E and E^2 but without the $(\omega/2)$ and $(\omega^2/4)$ terms respectively. Substituting the quantities evaluated above in the equations (43) and (44), we find that the criterions suggested by Kuo and Ford are given by the following expressions:

$$
\Delta_{KF1(NO)} \;=\; \left\{\left|\,\frac{|\,\beta\,|^4\,\left(2\,n^2 \;+\; 2\,n \;+\; 2\right) \;+\; |\,\beta\,|^2\;(\,2\,n \;+\; 1\,) \;-\; n}{|\,\beta\,|^4\;(\,4\,n^2 \;+\; 4\,n \;+\; 1\,) \;+\; |\,\beta\,|^2\;(\,4\,n^2 \;+\; 2\,n\,) \;+\; n^2}\,\right|\right\} \tag{64}
$$

$$\Delta_{KF2(NO)} = \left\{ \left| \frac{|\beta|^4 \left(2n^2 + 2n + 2\right) + |\beta|^2 (2n + 1) - n}{|\beta|^4 (6n^2 + 6n + 3) + |\beta|^2 (6n^2 + 2n + 1) + (n^2 - n)} \right| \right\} \tag{65}$$

$$\Delta_{KF1(VS)} = \left\{ \frac{|\beta|^4 \left(2n^2 + 2n + 2\right) + |\beta|^2 \left(2n^2 + 4n + 3\right) + n}{|\beta|^4 (4n^2 + 4n + 1) + |\beta|^2 (4n^2 + 2n) + n^2} \right\} \tag{66}$$

and

$$\Delta_{KF2(VS)} = \left\{ \frac{|\beta|^4 \left(2n^2 + 2n + 2\right) + |\beta|^2 \left(2n^2 + 4n + 3\right) + n}{\left(|\beta|^4 + |\beta|^2\right) (6n^2 + 6n + 3) + (n^2 + n)} \right\}. \tag{67}$$

Whereas the criterions $\Delta_{SC1\,\&\,2}$, when the expectation values E and E^2 are substituted in the equations (46) and (47) are given by the expressions

$$\Delta_{SC1} = \left\{ \left(\frac{2|\beta|^2 + 2|\beta|^4}{\left(1 + 2|\beta|^2\right)^2} \right) \left(\frac{n^2 + n + 1}{n^2 + n + \frac{1}{4}} \right) \right\} \tag{68}$$

and

$$\Delta_{SC2} = \left\{ \left(\frac{2|\beta|^2 + 2|\beta|^4}{1 + 6|\beta|^2 + 6|\beta|^4} \right) \left(\frac{n^2 + n + 1}{n^2 + n + \frac{1}{2}} \right) \right\}. \tag{69}$$

(iii). In a coherent state

When the quantum oscillator is specified to be in a coherent state the expectation values when no renormalisation is carried out are given by the equations

$$\begin{aligned} E &= \langle \hat{H} \rangle = \langle \lambda | \left\{ \hat{a}^\dagger \hat{a} + \frac{1}{2} \right\} \omega | \lambda \rangle \\ &= \left\{ |\lambda|^2 \left(1 + 2|\beta|^2\right) + \lambda^2 \alpha \beta + \lambda^{*2} \alpha^* \beta^* + |\beta|^2 + \frac{1}{2} \right\} \omega \end{aligned} \tag{70}$$

and

$$
\begin{aligned}
E^2 \;=\; \left\langle \hat{H}^2 \right\rangle \;&=\; \left\langle \lambda \right| \left\{ \hat{a}^\dagger \hat{a} \;+\; \frac{1}{2} \right\} \omega \left\{ \hat{a}^\dagger \hat{a} \;+\; \frac{1}{2} \right\} \omega |\lambda\rangle \\
&= \left\{ \left(|\lambda|^4 \;+\; 2\,|\lambda|^2 \right) \left(1 \;+\; 6\,|\beta|^2 \;+\; 6\,|\beta|^4 \right) \right. \\
&+ \left(2\,|\lambda|^2 \;+\; 3 \right) \left(\lambda^2 \alpha \beta \;+\; \lambda^{*2} \alpha^* \beta^* \right) \left(1 \;+\; 2\,|\beta|^2 \right) \\
&+ \left(\lambda^4 \alpha^2 \beta^2 \;+\; \lambda^{*4} \alpha^{*2} \beta^{*2} \right) \\
&+ \left. \left(3|\beta|^2 \;+\; 3\,|\beta|^4 \right) \;+\; \frac{1}{4} \right\} \omega^2.
\end{aligned} \tag{71}
$$

When the operators are normal ordered the expectation values are

$$
\begin{aligned}
E_{NO} \;&=\; \left\langle \;:\hat{H}:\; \right\rangle = \left\langle \lambda \right| \left\{ \hat{a}^\dagger \hat{a}(t) \right\} \omega \left| \lambda \right\rangle \\
&= \left\{ |\lambda|^2 \left(1 \;+\; 2\,|\beta|^2 \right) \;+\; \lambda^2 \alpha \beta \;+\; \lambda^{*2} \alpha^* \beta^* \;+\; |\beta|^2 \right\} \omega
\end{aligned} \tag{72}
$$

and

$$
\begin{aligned}
E^2_{NO} \;&=\; \left\langle \;:\hat{H}^2:\; \right\rangle \;=\; \left\langle \lambda \right| \left\{ \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \right\} \omega^2 |\lambda\rangle \\
&= \left\{ |\lambda|^4 \left(1 \;+\; 6\,|\beta|^2 \;+\; 6\,|\beta|^4 \right) \right. \\
&+ |\lambda|^2 \left(8\,|\beta|^2 \;+\; 12\,|\beta|^4 \right) \\
&+ \left(\lambda^2 \alpha \beta \;+\; \lambda^{*2} \alpha^* \beta^* \right) \left\{ 1 \;+\; 6\,|\beta|^2 \;+\; |\lambda|^2 \left(2 \;+\; 4\,|\beta|^2 \right) \right\} \\
&+ \left. \left(\lambda^4 \alpha^2 \beta^2 \;+\; \lambda^{*4} \alpha^{*2} \beta^{*2} \right) + \; 3\,|\beta|^4 \;+\; |\beta|^2 \right\} \omega^2.
\end{aligned} \tag{73}
$$

For the case, when renormalisation is achieved by vacuum subtraction, *i.e*

$$
\begin{aligned}
E_{(VS)} \;&=\; \left\langle \;:\hat{H}:\; \right\rangle \\
&= \left\langle \lambda \right| \left\{ \hat{a}^\dagger \hat{a} \;+\; \frac{1}{2} \right\} \omega \left| \lambda \right\rangle - \left\langle 0 \right| \left\{ \hat{A}^\dagger \hat{A} \;+\; \frac{1}{2} \right\} \omega \left| 0 \right\rangle
\end{aligned} \tag{74}
$$

and

$$
\begin{aligned}
E^2_{(VS)} \;&=\; \left\langle \;:\hat{H}^2:\; \right\rangle \\
&= \left\langle \lambda \right| \left\{ \hat{a}^\dagger \hat{a} \;+\; \frac{1}{2} \right\} \omega \left\{ \hat{a}^\dagger \hat{a} \;+\; \frac{1}{2} \right\} \omega \left| \lambda \right\rangle \\
&- \left\langle 0 \right| \left\{ \hat{A}^\dagger \hat{A} \;+\; \frac{1}{2} \right\} \omega \left\{ \hat{A}^\dagger \hat{A} \;+\; \frac{1}{2} \right\} \omega \left| 0 \right\rangle,
\end{aligned} \tag{75}
$$

the expectation values, $E_{(VS)}$ and $E^2_{(VS)}$ are given by the same expressions as the quantities E and E^2 but without the $(\omega/2)$ and $(\omega^2/4)$ terms respectively. For the coherent state being considered, the expressions for the Δ's prove to be rather lenghthy. Due to this reason, we do not write them down here explicitly but just quote their values in the different limits of interest in the tables below.

The numerators of the various Δ's evaluated earlier in this section contain either the quantity $\left(\langle\, : \hat{H}^2 : \,\rangle \,-\, \langle\, : \hat{H} : \,\rangle^2 \right)$ or the one $\left(\langle\, \hat{H}^2 \,\rangle \,-\, \langle\, \hat{H} \,\rangle^2 \right)$, both of them being the square of the fluctuations in the energy of the quantum oscillator in the toy model. These quantities, as can be seen from the expressions for the Δ's, are proportional to at least the second power of $|\,\beta\,|$ and hence a large value for β would imply large fluctuations and hence a breakdown of the semiclassical theory.

The expressions for the different Δ's in the two limits of interest, *viz* $\beta \,\rightarrow\, 0$ and $\beta \,\rightarrow\, \infty$ are summarised in the tables I and II respectively.

Table I ($\beta \,\rightarrow\, 0$)

	Vacuum	nth excited	Coherent
Δ_{SC1}	0	0	$\left(\frac{\vert\lambda\vert^2}{\vert\lambda\vert^4 + \vert\lambda\vert^2 + \frac{1}{4}} \right)$
Δ_{SC2}	0	0	$\left(\frac{\vert\lambda\vert^2}{\vert\lambda\vert^4 + 2\,\vert\lambda\vert^2 + \frac{1}{4}} \right)$
$\Delta_{KF1(NO)}$	∞	$\left(\frac{1}{n} \right)$	0
$\Delta_{KF2(NO)}$	1	$\left(\frac{1}{n-1} \right)$	0
$\Delta_{KF1(VS)}$	∞	$\left(\frac{1}{n} \right)$	$\left(\frac{2}{\vert\lambda\vert^2} \right)$
$\Delta_{KF2(VS)}$	1	$\left(\frac{1}{n-1} \right)$	$\left(\frac{2}{2 + \vert\lambda\vert^2} \right)$

Table II ($\beta \to \infty$)

	Vacuum	nth excited	Coherent
Δ_{SC1}	2	$\left(\frac{n^2 + n + 1}{2n^2 + 2n + \frac{1}{2}} \right)$	$\left(\frac{\lvert\lambda\rvert^2 (8 + 4c_1) + 2}{\left(\lvert\lambda\rvert^2 (2 + c_1) + 1 \right)^2} \right)$
Δ_{SC2}	$\left(\frac{2}{3}\right)$	$\left(\frac{n^2 + n + 1}{3n^2 + 3n + \frac{3}{2}} \right)$	$\left(\frac{\lvert\lambda\rvert^2 (8 + 4c_1) + 2}{\lvert\lambda\rvert^4 (6 + 4c_1 + c_2) + \lvert\lambda\rvert^2 (12 + 6c_1) + 3} \right)$
$\Delta_{KF1(NO)}$	2	$\left(\frac{n^2 + n + 1}{2n^2 + 2n + \frac{1}{2}} \right)$	$\left(\frac{\lvert\lambda\rvert^2 (8 + 4c_1) + 2}{\left(\lvert\lambda\rvert^2 (2 + c_1) + 1 \right)^2} \right)$
$\Delta_{KF2(NO)}$	$\left(\frac{2}{3}\right)$	$\left(\frac{n^2 + n + 1}{3n^2 + 3n + \frac{3}{2}} \right)$	$\left(\frac{\lvert\lambda\rvert^2 (8 + 4c_1) + 2}{\lvert\lambda\rvert^4 (6 + 4c_1 + c_2) + \lvert\lambda\rvert^2 (12 + 6c_1) + 3} \right)$
$\Delta_{KF1(VS)}$	2	$\left(\frac{n^2 + n + 1}{2n^2 + 2n + \frac{1}{2}} \right)$	$\left(\frac{\lvert\lambda\rvert^2 (8 + 4c_1) + 2}{\left(\lvert\lambda\rvert^2 (2 + c_1) + 1 \right)^2} \right)$
$\Delta_{KF2(VS)}$	$\left(\frac{2}{3}\right)$	$\left(\frac{n^2 + n + 1}{3n^2 + 3n + \frac{3}{2}} \right)$	$\left(\frac{\lvert\lambda\rvert^2 (8 + 4c_1) + 2}{\lvert\lambda\rvert^4 (6 + 4c_1 + c_2) + \lvert\lambda\rvert^2 (12 + 6c_1) + 3} \right)$

The quantities c_1 and c_2 in the table II are given to be

$$c_1 = 2\cos(a + b + 2l) \quad ; \quad c_1 = 2\cos(2a + 2b + 4l) \tag{76}$$

where a, b and l are the arguments of the complex quantities α, β and λ respectively.

As listed in Table I, in the adiabatic limit, the quantities Δ_{SC1} or Δ_{SC2} do not vanish for the coherent state because these states are not energy eigen states and hence they do possess fluctuations in the energy. Whereas, when the operators are normal ordered these fluctuations in the energy for the coherent state do vanish in the adiabatic limit, as is reflected by the values of Δ_{KF1} and Δ_{KF2} in Table I. For the other vacuum and the nth excited states the quantities Δ_{KF1} and Δ_{KF2} do not vanish in the adiabtic limit whereas the criterions Δ_{SC1} or Δ_{SC2} do and hence, as mentioned in the previous section, perform more reliably. So, though all the criterions listed in tables above yield equivalent results in the limit $\beta \to \infty$, to draw the limits on the validity of the semiclassical theory for the toy model we have to concentrate on the quantities Δ_{SC1} or Δ_{SC2}.

In the limit when β is large, it can be seen from Table II, that the criterions Δ_{SC1} and Δ_{SC2} are of order unity for the vacuum and the nth excited state. Hence the semiclassical theory will not prove to be reliable if the quantum oscillator is specified to be in one of these two states. Excited states with a large value for n are generally assumed to be reliable states to study the semiclassical theory. In the limit $\beta \to \infty$ the criterions Δ_{SC1} and Δ_{SC2} are of order unity, in nth excited states for large n, suggesting the breakdown of the theory.

Whereas, if the quantum oscillator is specified to be in a coherent state, the results tabulated for the criterions Δ_{SC1} and Δ_{SC2}, clearly show that, for a large value for the parameter λ of the coherent state, thses dimensionless criterions die down as $\left(1 / |\, \lambda \,|^2 \right)$ irrespective of the value of β. So the semiclassical theory being considered for the toy model will prove to be absolutely reliable if such states are specified for the quantum oscillator.

6. Conclusions

The results of the above section quite clearly prove that the semiclassical theory being considered for the toy model, can be relied upon, during all stages of the evolution, only if the quantum system is specified to be in coherent like states. It is quite possible, due to the nature of the coupling between the degrees of feedom chosen for the toy model, that this conclusion might prove to be valid even for quantised scalar fields in time dependant metrics. If the backreaction problem *has* to be in studied in states, for the quantum system, which do not possess a coherent nature, the semiclassical theory being considered in this paper is bound to prove rather inadequate and the fluctuations will have to be accounted for in the backreaction term. When done so, the semiclassical theory can be expected to be described by an equation similar in form to the Langevin equation.

It has been claimed, in section **4** and the discussions following the tables in section **5**, that the criterions Δ_{SC1} and Δ_{SC2} perform more reliably than either Δ_{KF1} or Δ_{KF2}. This was achieved by introducing the vacuum terms in the criterions Δ_{KF1} and Δ_{KF2} to yield the quantities Δ_{SC1} and Δ_{SC2} so that they provide reliable results in the adiabatic limit. For the case of quantum field theory, the quantities Δ_{SC1} and Δ_{SC2} which involve non-renormalised expectation values, will prove to be a ratio of divergences and hence will make no sense. So, only quantities involving renormalised expectation values can be evaluated. Since, it has been illustrated, for the case of the toy model, that the quantities Δ_{KF1} and Δ_{KF2} prove to be unreliable in the adiabatic limit, for the field theoretic case, it would be advisable to concentrate on just the fluctuations in the energy-momentum density of the quantum field. Negligible fluctuations can then be considered to be a positive result for the validity of the semiclassical theory. And, when there is a prolific production of particles taking place, either Δ_{KF1} or Δ_{KF2} can be relied upon to reflect the validity of the semiclassical theory.

Though, in this paper, the unreliability of the criterions Δ_{KF1} and Δ_{KF2} in the adiabatic limit and means of improving upon these quantities was pointed out for the case of the toy model, its main objective was to illustrate the limited validity of the semiclassical theory.

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