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Initiation of damage in a class of polymeric materials embedded with multiple localized regions of lower density

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Abstract
Fatigue and damage are the least understood phenomena in the mechanics of solids. Recently, Alagappan et al. ("On a possible methodology for identifying the initiation of damage of a class of polymeric materials", *Proc R Soc Lond A Math Phys Eng Sci* 2016; 472(2192): 20160231) hypothesized a criterion for the initiation of damage for a certain class of compressible polymeric solids, namely that damage will be initiated at the location where the derivative of the norm of the stress with respect to the stretch starts to decrease. This hypothesis led to results that were in keeping with the experimental work of Gent and Lindley("Internal rupture of bonded rubber cylinders in tension. *Proc. R. Soc. Lond. A* 1959; 249, 195–205 :10.1098) and agrees qualitatively with the results of Penn ("Volume changes accompanying the extension of rubber", *Trans Soc Rheol* 1970; 14(4): 509–517) on compressible polymeric solids. Alagappan et al. considered a body wherein there is a localized region in which the density is less than the rest of the solid. In this study, we show that the criterion articulated by Alagappan et al. is still applicable when bodies have multiple localized regions of lower density, thereby lending credence to the notion that the criterion might be reasonable for a large class of bodies with multiple inhomogeneities. As in the previous study, it is found that *damage is not initiated at the location where the stresses are the largest but instead at the location where the densities tend to the lowest value*. These locations of lower densities coincide with locations in which the deformation gradient is very large, suggesting large changes in the local volume, which is usually the precursor to phenomena such as the bursting of aneurysms.

Keywords
Damage, nonlinear polymeric solids, inhomogeneous body, failure, compressible elastic body

1. Introduction

The term "damage" is used to describe various types of failure with regard to the structural integrity of a body in that the body is unable to perform its intended purpose. Within the purely mechanical purview, damage can

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be viewed as the loss of load-carrying capacity. There are several reasons or, put differently, various different underlying physical mechanisms, that lead to a loss of load-carrying capacity. Thus identification of damage would require an understanding of the various underlying physical mechanisms that lead to loss in load-carrying capacity and the determination of which one or more of them is the cause of the damage that a particular body undergoes. As different classes of materials suffer damage in different ways one can at best hope to develop a criterion to delineate the initiation of damage only for specific classes or sub-classes of materials and also for possibly sub-classes of processes that the body is subject to. In addition to microstructural changes, entities such as the density can decrease acutely in certain locations leading to a loss in the load-carrying capacity of the body.

In order to describe damage in a meaningful manner, one needs to provide a criterion for the initiation of damage and also a rule for how the damaged material continues to respond, that is, the difference in response between the undamaged original material and the damaged material. This paper is only concerned with the development of a criterion for the initiation of damage, and that too for a special class of materials, namely compressible polymeric materials.

While the initiation of damage in most studies that have been carried out is invariably characterized with the help of damage parameters that could be scalars, vectors, tensors, or combinations of them, they usually require one to know information with regard to a reference configuration of the body, as for instance in the case of strain. Such information might be unavailable: for instance, given a body in a particular configuration one might not know the configuration in which it was free of strain. Moreover, an evolution equation is usually provided for the "damage parameter", as the material is subject to further deformation after damage has been initiated. Most often, the criterion for the initiation of damage, the choice of the "damage parameter" and the evolution equation for the "damage parameter" are ad hoc recondite specifications.

The study of damage by appealing to a damage parameter was initiated by Kachanov [1] and a discussion of "damage parameters", scalar, vector and tensor in character, are discussed in Krajcinovic [2] (see also earlier books by Kachanov [3] and Lemaitre [4] for a discussion of damage mechanics) . In an interesting study concerning the failure of rubber, Volokh [5] tries to get away from defining ad hoc internal variables as "damage parameters" and introduces the concept of "energy limiters" that according to him reflect the average "bond energy between" the "particles" that supposedly comprise the body. Volokh [5] assumes that if the failure energy, the average bond energy, is infinite, the body is hyperelastic, while if it has a finite value, the material responds as a hyperelastic body until it reaches this value and on further loading gets damaged. As bond energy usually refers to the energy in a chemical bond, it is not clear to us what he means by "bond energy" between "particles", that is, whether it refers to the bond between the chemical constituents of the body, or if he is referring to the strength of the network junctions between the long chain molecules in the polymer. Also, the failure energy associated with the "energy limiters" is calibrated by macroscopic experiments based on when the material fails (it could be, and possibly is, different in different specimens of rubber) and it is not determined by calculations based on the "bond energy" between whatever supposedly constitutes the "particles". A problem with using the value of energy from macroscopic experimental data, rather than actually computing the average bond energy, to decide on the initiation of damage, is the need to know the energy in the reference configuration in which the body is free of both stress and strain, from which the body has been deformed, the stored energy in such a stress-free and strain-free state being the datum from which the stored energy in the current configuration is determined. In a laboratory, one only has a body in the current state; it could have undergone deformation earlier and hence might not be in a strain-free or stress-free state. We reiterate that one ought to decide on the initiation of damage in terms of quantities that can be determined in the current state of the body and how they evolve when further deformed, say for instance the density, temperature, microstructure or other properties in the current state, without any recourse to knowledge concerning prior states of the body. Volokh [6] has also studied the failure of biological materials by introducing the notion of a "bulk failure energy" and an evolution equation for a parameter α. The physical underpinning of the parameter α which is referred to as the "switch parameter" is not made clear. Supposedly, when $\alpha = 0$ the body behaves as a hyperelastic body while when $\alpha < 0$ the body "is irreversibly damaged and the stored energy is dissipated".

As the literature discussing the progression of damage in a body within the context of a damage parameter is voluminous, and as our study does not concern the problem of progression of damage, we shall abstain from citing more such studies than the few mentioned above. Also, as mentioned earlier, our study does not pertain to the description of the progression of damage in a body but merely provides a methodology for identifying the initiation of damage, the constitutive relation having a different representation with the progression of damage. Our approach essentially allows one to recognize when damage is initiated.

In a recent paper, Alagappan et al. [7] identified the changes (decrease) in density of a class of polymeric materials as the cause of loss of the load-carrying capacity and offered a new methodology for determining the initiation of damage that led to results that were in keeping with the experiments of Gent and Lindley [8] and Penn [9] on multi-network polymers. This methodology rests on the initiation of damage being a consequence of inhomogeneities that reflect a loss of integrity of the body at the location of the inhomogeneity due the density of the body decreasing with increasing stretch of the body. Alagappan et al. [7] articulated their rationale for why it is more meaningful to delineate the initiation of damage in terms of quantities that can be determined in the current configuration of the body and not depend on knowledge of some special state of the body such as its undeformed stress-free or strain-free state. Based on such an approach, they were able to show that for a class of inhomogeneous polymeric bodies when the density in a neighborhood in the body is initially lower than in the rest of the body, on being stretched the density in the neighborhood that was initially lower would tend to decrease much more rapidly than that in the rest of the body, and the body would fail in that it would lose its ability to carry a load and thus fail to perform as intended. In their study, Alagappan et al. [7] considered the response of a compressible elastic body, whose material properties depend on the density, wherein in a small neighborhood of a slab of polymeric material the density in the neighborhood is lower than in the rest of the body. They showed that as the body is stretched, at a certain stretch ratio, the density in the neighborhood which initially had a lower density starts to fall precipitously thereby leading to a loss of the structural integrity of the body indicating the initiation of damage of the body. Alagappan et al. [7] hypothesized that damage is initiated when the norm of the stress starts to decrease with respect to the stretch or the norm of the stress has an inflection point with respect to the density. While it is correctly stated in Alagappan et al. [7] that damage is initiated when the norm of the stress decreases with stretch, there is also an erroneous statement that the norm of the stress decreases with density. While the density in the neighborhood with the initial lower density falls drastically, the density in the rest of the domain remains relatively constant and as the slab is stretched the derivative of the norm of the stress with respect to the density increases monotonically in the far field. *Interestingly, it is not the location with the highest stress in which damage is initiated but the location of the lowest density*. This point cannot be overemphasized.

In order to ensure that it is not mere chance that the failure does not get initiated at the location of the highest stress due to the specific nature of the inhomogeneity that was considered earlier, we now consider a body that has multiple regions of lower density than the rest of the body. These regions of lower density are so chosen that they are in reasonable proximity to one another leading to extremely high stresses in the regions between these regions of lower initial density when the body is deformed. However, damage does not initiate in these locations but occurs in regions of initially lower density, and once again is initiated when the derivative of the norm of the stress with respect to the stretch becomes negative, consistent with the hypothesis for the initiation of damage advanced by Alagappan et al. [7].

In this paper we consider different patterns for the inhomogeneity. We consider a body wherein there are small neighborhoods within which the density is lower to start with than in the rest of the body. We consider two forms of distribution of inhomogeneity around the central region, one in which the inhomogeneities are located along the x- and y-axes and the other in which they are located at an angle of 45° from the x- and y-axes. We find that the criterion for damage hypothesized by Alagappan et al. [7] once again predicts failure in regions wherein the density is lowest.

The organization of the paper is as follows. In the next section we provide a brief review of the kinematics and introduce the compressible elastic models that we study. In order to be in a position to compare the results of our work with the previous study by Alagappan et al. [7] we consider the same two classes of bodies considered by them, the generalization of the compressible body due to Gent [10] and the compressible neo-Hookean model. In Section 3 we introduce the precise structure of the inhomogeneity in the body and derive the equations that govern the deformation of the body for both classes of models studied. This is followed by the solutions of the relevant boundary value problems and a discussion of the results obtained in Section 4.

2. Preliminaries

Let $\mathbf{X}$ denote a typical material point in the reference configuration (κ_R) of a body and $\mathbf{x}$ denote the same point at time 't' in the current configuration denoted by (κ_t). Let $\boldsymbol{\chi}_{\kappa_R}$, the motion of the body, be a one-to-one mapping of κ_R onto κ_t, that is,

$$\mathbf{x} = \boldsymbol{\chi}_{\kappa_R}(\mathbf{X}, t). \tag{1}$$

The deformation gradient $\mathbf{F}_{\kappa_R}$ is defined as $\mathbf{F}_{\kappa_R} := \frac{\partial \chi_{\kappa_R}}{\partial \mathbf{X}}$. The displacement $\mathbf{u}$ is defined as $\mathbf{u} := \mathbf{x} - \mathbf{X}$ and the relations between the displacement gradient and deformation gradient are given by

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \nabla_{\mathbf{X}} \mathbf{u} = \mathbf{F} - \mathbf{I}, \quad \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \nabla_{\mathbf{x}} \mathbf{u} = \mathbf{I} - \mathbf{F}^{-1}. \tag{2}$$

The right and left Cauchy–Green tensors and unimodular tensors corresponding to them are defined by

$$\mathbf{B} := \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{C} := \mathbf{F}^{\mathrm{t}}\mathbf{F}, \quad \bar{\bar{\mathbf{C}}} := \det \mathbf{F}^{-2/3}\mathbf{C} \quad \text{and} \quad \bar{\bar{\mathbf{B}}} := \det \mathbf{F}^{-2/3}\mathbf{B}. \tag{3}$$

The above kinematical definitions are sufficient for our purposes.

Let $\mathbf{T}$ denote the Cauchy stress tensor. We consider here an inhomogeneous generalization of the Gent model (see [7, 10]) whose Cauchy stress is related to the Cauchy–Green tensor $\mathbf{B}$ through

$$\begin{aligned} \mathbf{T} =& \frac{\mu_0(\mathbf{X})}{(\det \mathbf{F})^{n(\mathbf{X})+1}} \frac{1}{\left(1 - \left(\frac{I_1 - 3}{I_m(\mathbf{X}) - 3}\right)\right)} \left[\bar{\bar{\mathbf{B}}} - \frac{1}{3}\mathrm{tr}(\bar{\bar{\mathbf{B}}})\mathbf{I}\right] \\ &+ \frac{n(\mathbf{X})\mu_0(\mathbf{X})}{2(\det \mathbf{F})^{n(\mathbf{X})+1}} (I_m(\mathbf{X}) - 3) \log\left(1 - \left(\frac{I_1 - 3}{I_m(\mathbf{X}) - 3}\right)\right)\mathbf{I} \\ &+ \frac{K_0(\mathbf{X})\nu(\mathbf{X})}{(\det \mathbf{F})^{n(\mathbf{X})}} \left[(\det \mathbf{F})^{\nu(\mathbf{X})-1} - \frac{1}{(\det \mathbf{F})^{\nu(\mathbf{X})+1}}\right]\mathbf{I} \\ &- \frac{n(\mathbf{X})K_0(\mathbf{X})}{(\det \mathbf{F})^{n(\mathbf{X})+1}} \left[(\det \mathbf{F})^{\nu(\mathbf{X})} + \frac{1}{(\det \mathbf{F})^{\nu(\mathbf{X})}} - 2\right]\mathbf{I}. \end{aligned} \tag{4}$$

In the above equation, I_1 is the first principal invariant of the unimodular Cauchy–Green tensor $\bar{\bar{\mathbf{B}}}$. The parameters $\mu_0(\mathbf{X})$ and $K_0(\mathbf{X})$ are the shear and bulk moduli, I_m is the stretch limit, ν is the volumetric exponent, and n is the exponent related to the extent of degradation of the material modulus. The constitutive expression (4) is in terms of the deformation gradient from a special reference configuration and this might seem to contradict the requirement that one not use quantities which require special configuration from which measurements are to be made. This is not a problem with regard to what is considered in this paper as one can cast elasticity within a purely Eulerian framework (see Rajagopal and Srinivasa [11]). We shall not use the Eulerian approach to elasticity here.

As in the previous study by Alagappan et al. [7] we shall consider the deformation of a rectangular slab with regions wherein the density is lower. We non-dimensionalize the equation (**??**) by using a characteristic length scale 'L' (the length of the layer under consideration) and modulus K_1: $u = L\bar{u}$, $v = L\bar{v}$, $w = L\bar{w}$ and $\mathbf{T} = K_1\bar{\mathbf{T}}$. The final non-dimensionalized equation is

$$\bar{\mathbf{T}} = \alpha_1 \left[\bar{\bar{\mathbf{B}}} - \frac{1}{3}\mathrm{tr}(\bar{\bar{\mathbf{B}}})\mathbf{I}\right] + [\alpha_2 + \alpha_3 - \alpha_4]\,\mathbf{I}, \tag{5}$$

where

$$\begin{aligned} \alpha_1 &= \frac{\mu_0(\bar{\mathbf{X}})}{K_1(\det \bar{\mathbf{F}})^{n(\mathbf{X})+1}} \frac{1}{\left(1 - \left(\frac{I_1 - 3}{I_m(\bar{\mathbf{X}}) - 3}\right)\right)}, \\ \alpha_2 &= \frac{n(\bar{\mathbf{X}})\mu_0(\bar{\mathbf{X}})}{2K_1(\det \bar{\mathbf{F}})^{n(\bar{\mathbf{X}})+1}} (I_m(\bar{\mathbf{X}}) - 3)\log\left(1 - \left(\frac{I_1 - 3}{I_m(\bar{\mathbf{X}}) - 3}\right)\right), \\ \alpha_3 &= \frac{K_0(\bar{\mathbf{X}})\nu(\bar{\mathbf{X}})}{K_1(\det \bar{\mathbf{F}})^{n(\bar{\mathbf{X}})}} \left[(\det \bar{\mathbf{F}})^{\nu(\bar{\mathbf{X}})-1} - \frac{1}{(\det \bar{\mathbf{F}})^{\nu(\bar{\mathbf{X}})+1}}\right], \\ \text{and} \quad \alpha_4 &= \frac{K_0(\bar{\mathbf{X}})n(\bar{\mathbf{X}})}{K_1(\det \bar{\mathbf{F}})^{n(\bar{\mathbf{X}})+1}} \left[(\det \bar{\mathbf{F}})^{\nu(\bar{\mathbf{X}})} + \frac{1}{(\det \bar{\mathbf{F}})^{\nu(\bar{\mathbf{X}})}} - 2\right]. \end{aligned}$$

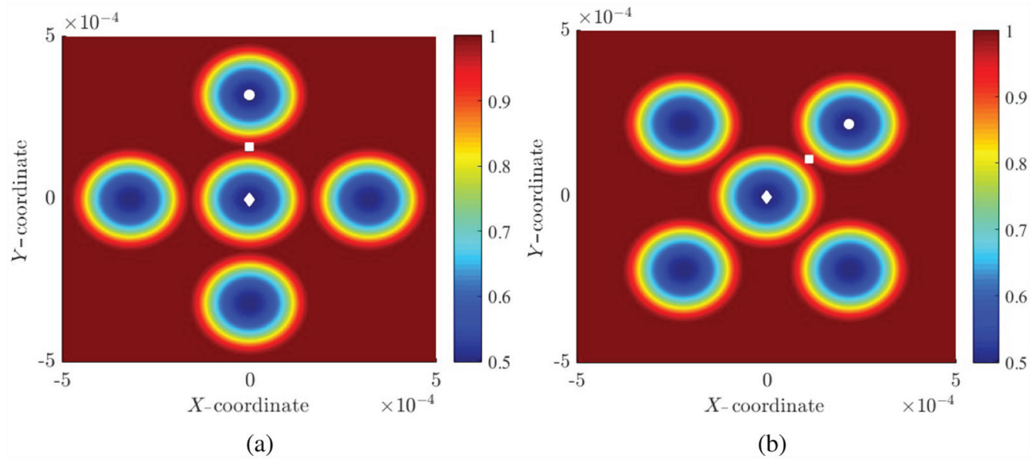


Figure 1. Initial density profile in the small localized region around the central inhomogeneity. The filled diamond, square and circle shapes represents critical points A, B and C respectively. Further discussions are provided in Section **??**. (a) Case 1. (b) Case 2.

The other model that we consider here is a generalization of the compressible neo-Hookean body whose Cauchy stress is related to the Cauchy–Green tensor **B** through

$$\mathbf{T} = K_0(\mathbf{X})\frac{(\det \mathbf{F} - 1)}{(\det \mathbf{F})^3}\mathbf{I} + \mu_0(\mathbf{X})\frac{1}{(\det \mathbf{F})}\text{dev } \bar{\bar{\mathbf{B}}}, \tag{6}$$

where $\frac{\mu_0(\mathbf{X})}{\det \mathbf{F}}$ and $\frac{K_0(\mathbf{X})}{(\det \mathbf{F})^3}$ are related to the shear and bulk moduli which decrease with increase in det **F**. We non-dimensionalize this Cauchy stress in the same manner as for the generalized Gent model. The final non-dimensionalized equation is given by

$$\bar{\mathbf{T}} = \frac{K_0(\bar{\mathbf{X}})}{K_1}\frac{(\det \bar{\mathbf{F}} - 1)}{(\det \bar{\mathbf{F}})^3}\mathbf{I} + \frac{\mu_0(\bar{\mathbf{X}})}{K_1}\frac{1}{(\det \bar{\mathbf{F}})}\text{dev } \bar{\bar{\mathbf{B}}}. \tag{7}$$

Hereafter for the ease of notation we will ignore the bar over **X** and denote the non-dimensional quantity as **X**.

3. Geometry of the body and the governing equations

We consider a thin square plate of length 'L'. In order to understand the response of an inhomogeneous body wherein there are multiple regions of lower density with regard to the initiation of damage as well as to the location of the initiation of damage, we consider two different cases of initial reference densities as shown in Figure **??**. Due to the symmetry of the problem, we consider only a quarter of the configuration of the body. In this case we have three locations where the initial reference density is at a minimum surrounded by a small neighborhood in which a steep fall takes place from the density in the rest of the body: one at the center of the domain (point A) and the remaining two depending on the orientation of the inhomogeneity around the central one. We study two patterns for the inhomogeneities: one where they are located along the abscissa and the ordinate, and the other where they lie on lines making 45° with the abscissa and the ordinate. We subject this body to an equal bi-axial state of strain.

3.1. Governing equations

As in the previous study by Alagappan et al. [7], we shall assume displacement components in the X-, Y- and Z-directions to be $\bar{u}(X, Y)$, $\bar{v}(X, Y)$ and $\bar{w}(X, Y, Z) = Z\phi(X, Y)$. That such an approximation is appropriate was borne out by the previous study by Alagappan et al. [7]. The form for '$\bar{w}$' is also chosen based on the study done by Alagappan et al. [7]. Under the above assumption, the deformation gradient is given by, as a consequence of (**??**),

$$\mathbf{F} = \begin{bmatrix} 1 + \frac{\partial \bar{u}}{\partial X} & \frac{\partial \bar{u}}{\partial Y} & 0 \\ \frac{\partial \bar{v}}{\partial X} & 1 + \frac{\partial \bar{v}}{\partial Y} & 0 \\ Z\frac{\partial \phi}{\partial X} & Z\frac{\partial \phi}{\partial Y} & 1 + \phi \end{bmatrix}. \tag{8}$$

The linear dependence of the displacement in the Z-direction results in deformation gradient components ($Z\frac{\partial \phi}{\partial X}$ and $Z\frac{\partial \phi}{\partial Y}$) and Cauchy stress components ($\bar{T}_{XZ}$ and $\bar{T}_{YZ}$) being negligible (see [7] for a detailed discussion).

We shall assume that the boundary conditions are

$$\bar{u}(1, Y) = u_1 \quad \forall\, 0 \le Y \le 1,$$
$$\bar{v}(X, 1) = u_1 \quad \forall\, 0 \le X \le 1,$$

and the symmetry conditions are

$$\bar{u}(0, Y) = 0 \quad \forall\, 0 \le Y \le 1,$$
$$\bar{v}(X, 0) = 0 \quad \forall\, 0 \le X \le 1.$$

One could assume boundary conditions for the traction but this makes the problem very cumbersome. The above is akin to a semi-inverse approach or that of using separation of variables; the validity of the assumption is to be verified a posteriori and if found to be inappropriate the assumption has to be changed. We expect the assumption to be valid in view of the previous work by Alagappan et al. [7].

We also have $\bar{P}_{ZZ} = 0$, since the thin sheet is traction-free on the top and bottom surfaces. We shall express the equation of equilibrium in terms of the Piola–Kirchhoff tensor $\mathbf{P}$ as

$$\text{Div}\, \bar{\mathbf{P}} = \mathbf{0}.$$

The relation between the Cauchy stress and first Piola–Kirchhoff tensor is given by

$$\bar{\mathbf{P}} = (\det \bar{\mathbf{F}})\bar{\mathbf{T}}\bar{\mathbf{F}}^{-\mathrm{T}}.$$

In virtue of the above relationship between the Cauchy and Piola–Kirchhoff stress tensors, the Piola–Kirchhoff stress components for the generalized Gent model are as follows:

$$\bar{P}_{XX} = \beta_1 + \beta_1 \frac{\partial \bar{u}}{\partial X} + \left(\frac{\partial \bar{v}}{\partial Y} + 1\right)(\phi + 1)(\beta_3 - \beta_1\beta_2),$$
$$\bar{P}_{YY} = \beta_1 + \beta_1 \frac{\partial \bar{v}}{\partial Y} + \left(\frac{\partial \bar{u}}{\partial X} + 1\right)(\phi + 1)(\beta_3 - \beta_1\beta_2),$$
$$\bar{P}_{XY} = \beta_1 \frac{\partial \bar{u}}{\partial Y} - \frac{\partial \bar{v}}{\partial X}(\phi + 1)(\beta_3 - \beta_1\beta_2),$$
$$\bar{P}_{YX} = \beta_1 \frac{\partial \bar{v}}{\partial X} - \frac{\partial \bar{u}}{\partial Y}(\phi + 1)(\beta_3 - \beta_1\beta_2),$$
$$\bar{P}_{ZZ} = (\beta_3 - \beta_1\beta_2)\left(\frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{v}}{\partial Y} + \frac{\partial \bar{u}}{\partial X}\frac{\partial \bar{v}}{\partial Y} - \frac{\partial \bar{u}}{\partial Y}\frac{\partial \bar{v}}{\partial X} + 1\right)$$
$$+\beta_1(\phi + 1),$$

Table 1. Material parameters for the generalized Gent model.

Parameters	Functions
ρ_0	Smoothed Heaviside function (see Figure **??**)
μ_0	$10^2 \rho_0$
K_0	$10^4 \rho_0$
ν	Smoothed piecewise linear function (see Figure 2(a))
I_m	Smoothed piecewise linear function (see Figure 2(b))
K_1	10^4
n	0.5

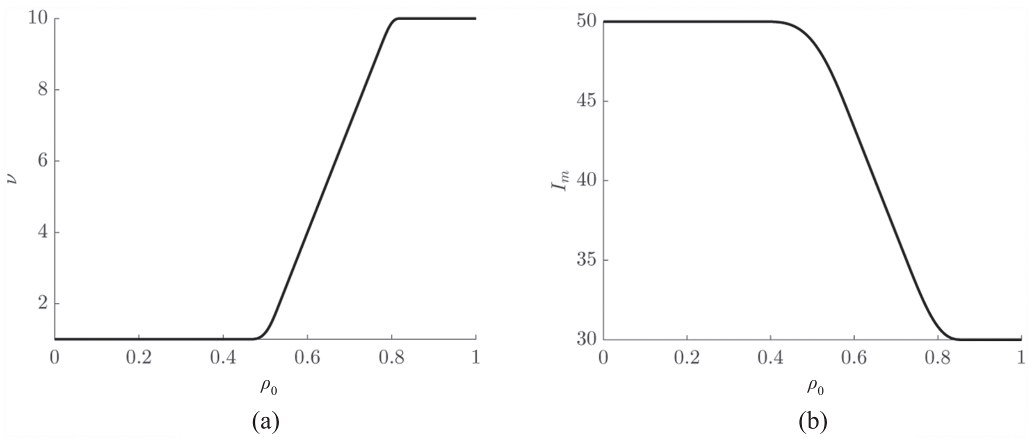


Figure 2. Material parameters variation. (a) Variation of ν. (b) Variation of I_m.

where

$$\beta_1 = \frac{\mu_0(\bar{\mathbf{X}})}{K_1(\det \bar{\mathbf{F}})^{(n(\bar{\mathbf{X}})+2/3)}} \frac{1}{\left(1 - \left(\frac{\mathrm{tr}(\bar{\bar{\mathbf{B}}})-3}{I_m(\bar{\mathbf{X}})-3}\right)\right)},$$

$$\beta_2 = \frac{1}{3} \frac{\mathrm{tr}(\bar{\bar{\mathbf{B}}})}{\det \bar{\mathbf{F}}},$$

$$\beta_3 = \frac{n\mu_0(\bar{\mathbf{X}})(I_m(\bar{\mathbf{X}}) - 3)}{2K_1(\det \bar{\mathbf{F}})^{n(\bar{\mathbf{X}})}} \log\left(1 - \left(\frac{\mathrm{tr}(\bar{\bar{\mathbf{B}}}) - 3}{I_m(\bar{\mathbf{X}}) - 3}\right)\right) + \frac{K_0(\bar{\mathbf{X}})\nu(\bar{\mathbf{X}})}{K_1(\det \bar{\mathbf{F}})^{n(\bar{\mathbf{X}})}} \left[(\det \bar{\mathbf{F}})^{\nu(\bar{\mathbf{X}})} - \frac{1}{(\det \bar{\mathbf{F}})^{\nu(\bar{\mathbf{X}})}}\right] - \frac{K_0(\bar{\mathbf{X}})n(\bar{\mathbf{X}})}{K_1(\det \bar{\mathbf{F}})^{n(\bar{\mathbf{X}})}} \left[(\det \bar{\mathbf{F}})^{\nu(\bar{\mathbf{X}})} + \frac{1}{(\det \bar{\mathbf{F}})^{\nu(\bar{\mathbf{X}})}} - 2\right].$$

The material parameters that have been assumed in this study for the generalized Gent model are shown in Table **??**. The smoothed piecewise linear functions ν and I_m, as a function of the density, are depicted in Figure 2a and Figure 2b, respectively.

Table 2. Material parameters for the generalized Neo-Hookean model.

Parameters	Functions
ρ_0	Smoothed Heaviside function (see Figure **??**)
μ_0	$10^2\rho_0$
K_0	$100(1-\exp(-8\rho_0))\mu_0$
K_1	10^4

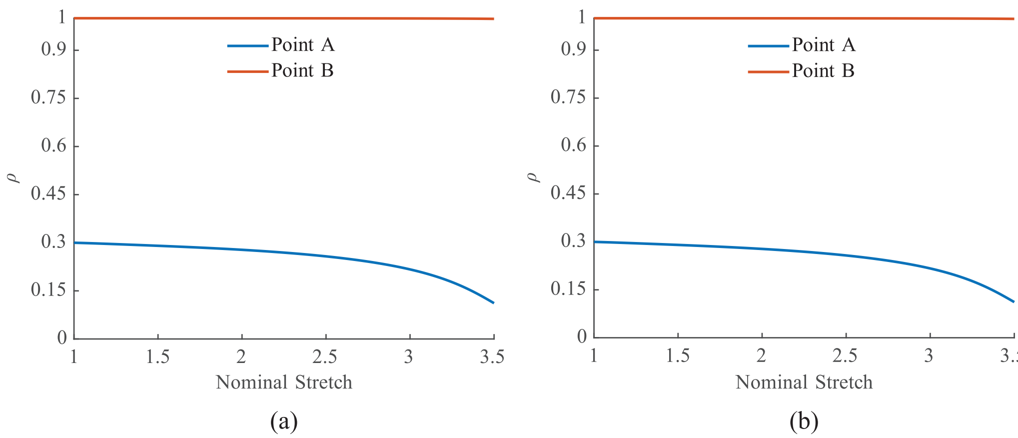


Figure 3. Density variation with nominal stretch for the generalized Gent model. (a) Case 1. (b) Case 2.

The Piola–Kirchhoff stress components for the generalized compressible neo-Hookean model are found to be as follows:

$$\bar{P}_{XX} = \alpha_2 + \alpha_2\frac{\partial \bar{u}}{\partial X} + \alpha_1\left(\frac{\partial \bar{v}}{\partial Y}+1\right)(\phi+1) \tag{9}$$

$$\bar{P}_{YY} = \alpha_2 + \alpha_2\frac{\partial \bar{v}}{\partial Y} + \alpha_1\left(\frac{\partial \bar{u}}{\partial X}+1\right)(\phi+1) \tag{10}$$

$$\bar{P}_{XY} = \alpha_2\frac{\partial \bar{u}}{\partial Y} - \alpha_1\frac{\partial \bar{v}}{\partial X} - \alpha_1\phi\frac{\partial \bar{v}}{\partial X} \tag{11}$$

$$\bar{P}_{YX} = \alpha_2\frac{\partial \bar{v}}{\partial X} - \alpha_1\frac{\partial \bar{u}}{\partial Y} - \alpha_1\phi\frac{\partial \bar{u}}{\partial Y} \tag{12}$$

$$\bar{P}_{ZZ} = \alpha_2(\phi+1) + \alpha_1\left(\frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{v}}{\partial Y} + \frac{\partial \bar{u}}{\partial X}\frac{\partial \bar{v}}{\partial Y} - \frac{\partial \bar{u}}{\partial Y}\frac{\partial \bar{v}}{\partial X} + 1\right) \tag{13}$$

where

$$\alpha_1 = \frac{K_0(\det \bar{\mathbf{F}} - 1)}{K_1(\det \bar{\mathbf{F}})^3} - \frac{\mu_0}{3K_1 \det \bar{\mathbf{F}}}\mathrm{tr}\bar{\bar{\mathbf{B}}},$$

$$\alpha_2 = \frac{\mu_0}{K_1}\det \bar{\bar{\mathbf{F}}}^{-2/3}.$$

The material parameters used for this model are shown in Table **??**.

The equilibrium equations, in the absence of body forces, are

$$\frac{\partial \bar{P}_{XX}}{\partial X} + \frac{\partial \bar{P}_{XY}}{\partial Y} = 0 \tag{14}$$

$$\frac{\partial \bar{P}_{YX}}{\partial X} + \frac{\partial \bar{P}_{YY}}{\partial Y} = 0 \tag{15}$$

$$\bar{P}_{ZZ} = 0 \tag{16}$$

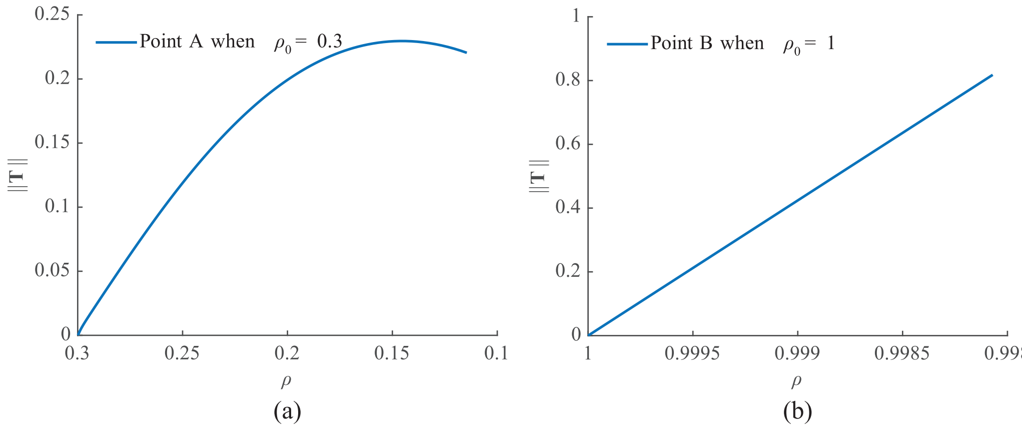


Figure 4. Variation of the norm of the stress tensor with density for the generalized Gent model (case 1). (a) At point A. (b) At point B.

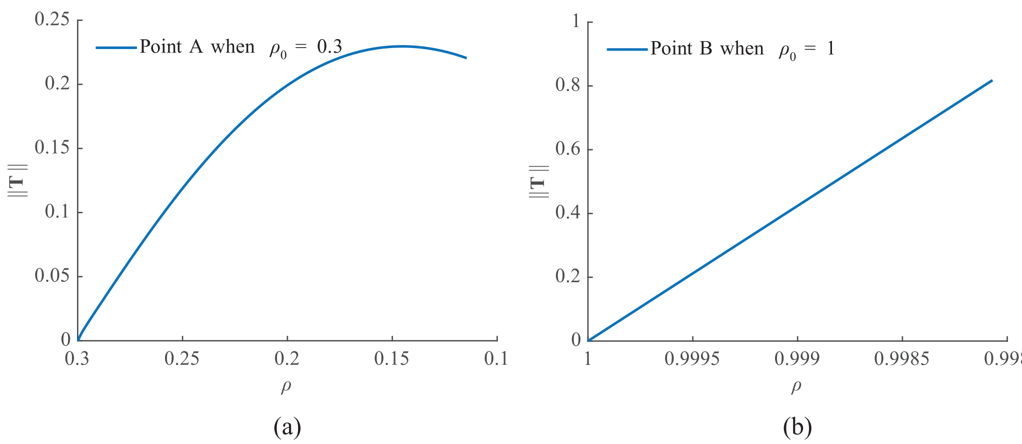


Figure 5. Variation of norm of the stress tensor with density for the generalized Gent model (case 2). (a) At point A. (b) At point B.

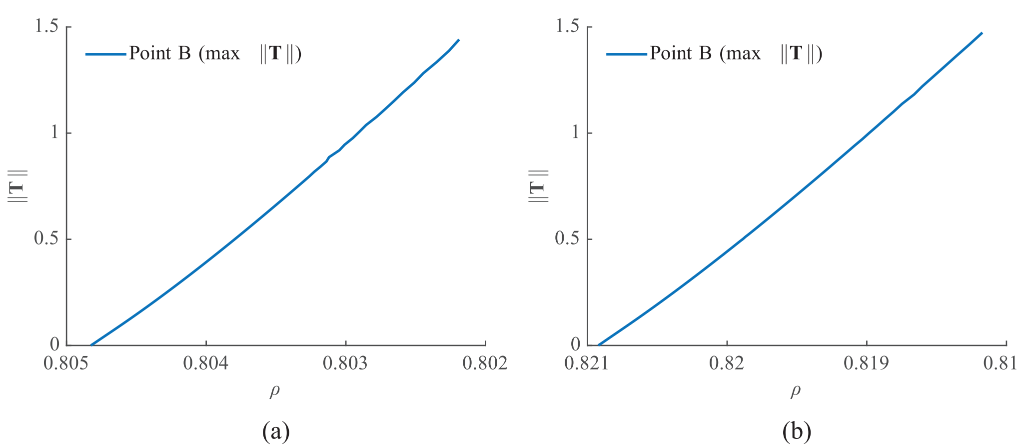
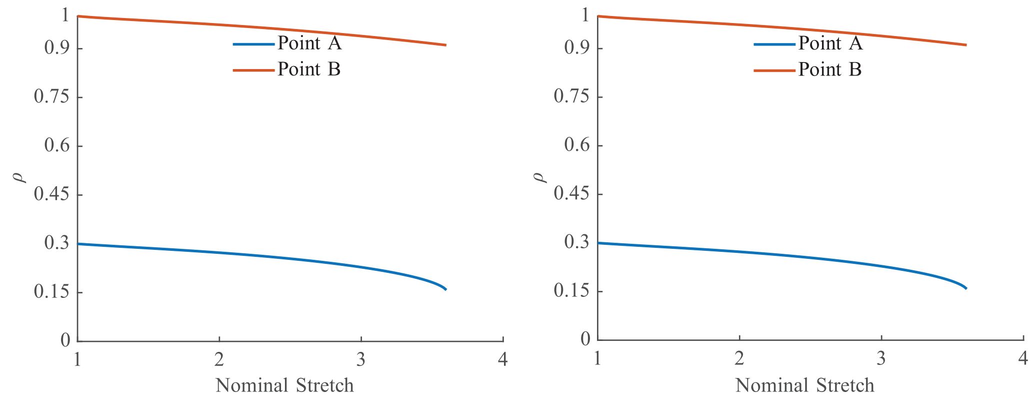


Figure 6. Variation of the maximum of the norm of the stress tensor with density for the generalized Gent model. (a) Case 1. (b) Case 2.

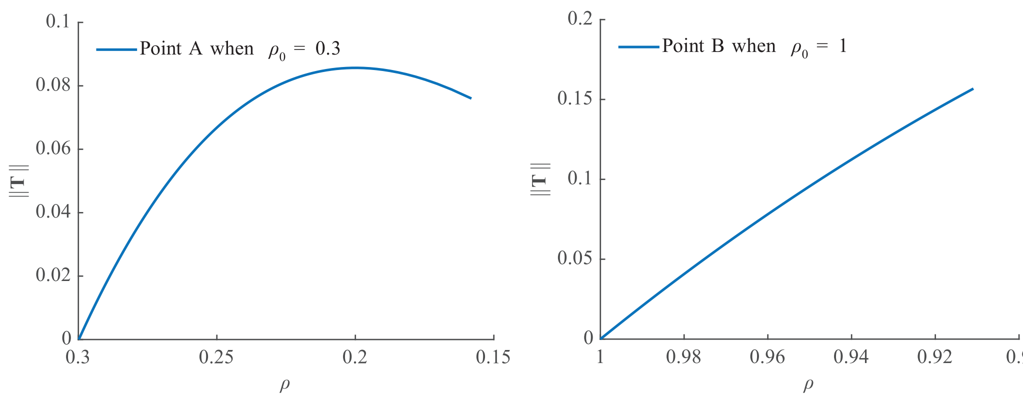
and the balance of mass is given by

$$\rho = \rho_0 / \det \bar{\mathbf{F}}. \tag{17}$$



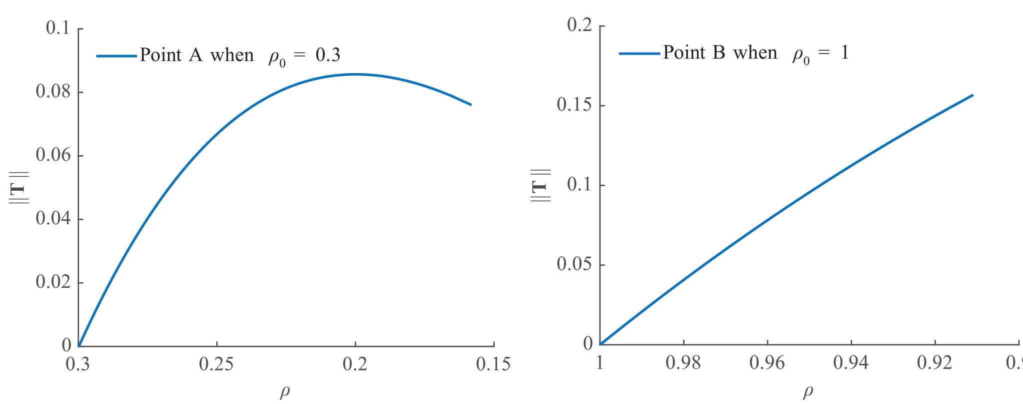
(a) (b)

Figure 7. Density variation with nominal stretch for the generalized neo-Hookean model. (a) Case 1. (b) Case 2.



(a) (b)

Figure 8. Variation of the norm of the stress tensor with density for generalized neo-Hookean model (case 1). (a) At point A. (b) At point B.



(a) (b)

Figure 9. Variation of the norm of the stress tensor with density for generalized neo-Hookean model (case 2). (a) At point A. (b) At point B.

4. Results and discussions

The effect of multiple regions of lower density was studied using the two different cases for the initial density variation as discussed earlier. In the cases considered, the only variation is the placement of the inhomogeneity around the central inhomogeneity. In both the cases, the region which has the lowest initial reference density

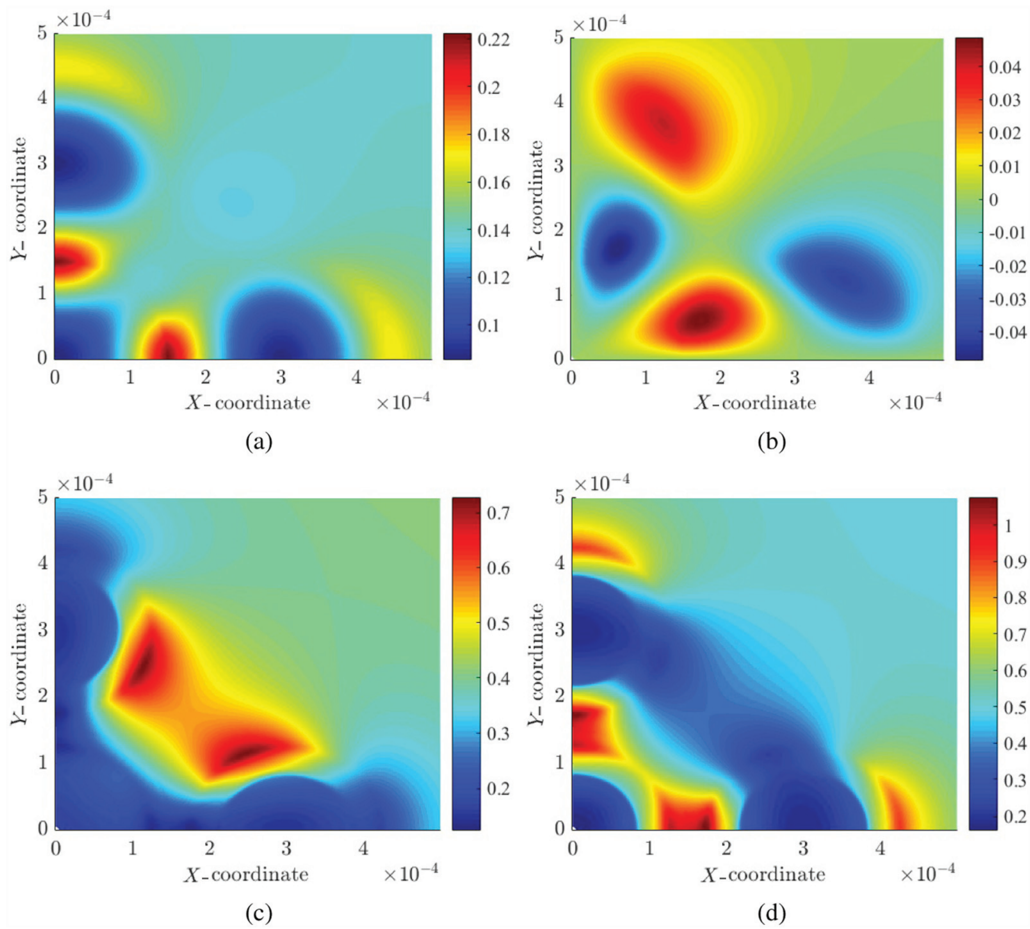


Figure 10. Case 1 for generalized neo-Hookean model. (a) ||**T**|| (b) Shear stress component. (c) Radial stress. (d) Circumferential stress.

(point A; see Figure **??**) deteriorates faster in comparison with the region of highest initial reference density (point B; see Figure **??**). Even though the placements of the inhomogeneity were different, both showed similar variation of density with stretch, which can be observed in Figure **??**. The percentage reduction of the density at point A is 62.83 while at point B it is 0.199, for a stretch of 3.5. Figures **??** and **??** show the variation of the norm of the stress with stretch. As the sheet is stretched the norm of the stress increases to a certain limit after which it starts decreasing. Thus, the initiation of damage occurs when the derivative of the norm of the stress decreases with stretch as hypothesized by Alagappan et al. [7]. The critical value of the density at which it happens is same in both the cases discussed, that is, 0.14, which is a decrease of around 53%. The stretch corresponding to this density is 3.4 (see Figure **??**). Even though the density is lowest at point A damage initiates at this location. The stress is at a maximum in the region between the inhomogeneities (point B; see Figure **??**) and it does not show any sign of initiation of damage at point B (see Figure **??**). Moreover, the density also does not reduce significantly at B as observed in point A. It is important to bear in mind that once damage is initiated the constitutive relation that has been assumed for the material is valid no more. The material ceases to respond to further stretch in an elastic (that is, non-dissipative) manner and some other constitutive relation that captures the inelastic response of the material has to be put into place. Here calculations were carried out past the point where damage is initiated just to show that the norm of the stress decreases with increasing stretch.

The results of the biaxial stretching for a generalized compressible neo-Hookean body are shown in Figures **??** through **??**. The results are similar to the generalized Gent model except for the magnitude of the quantities involved. The critical value of the density at which the derivative of the norm of the stress decreases with stretch is 0.2, which is same in both the cases discussed earlier. The percentage reductions in density at point A and point B are 33.33% and 6.78% respectively which corresponds to a stretch of 3.4.

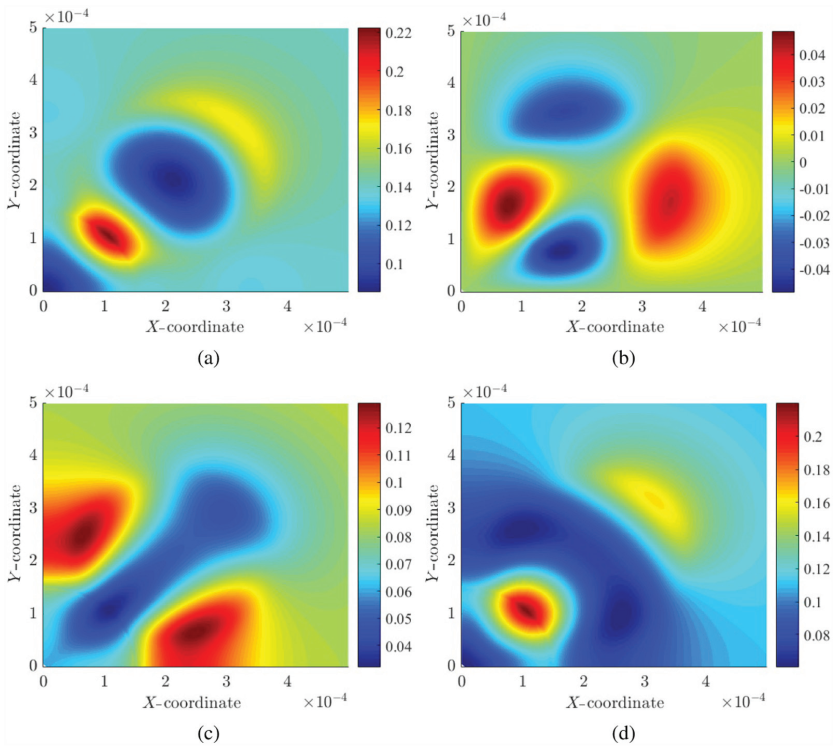


Figure 11. Case 2 for generalized neo-Hookean model. (a) $||\mathbf{T}||$ (b) Shear stress component, $T_{r\theta}$. (c) Radial stress. (d) Circumferential stress.

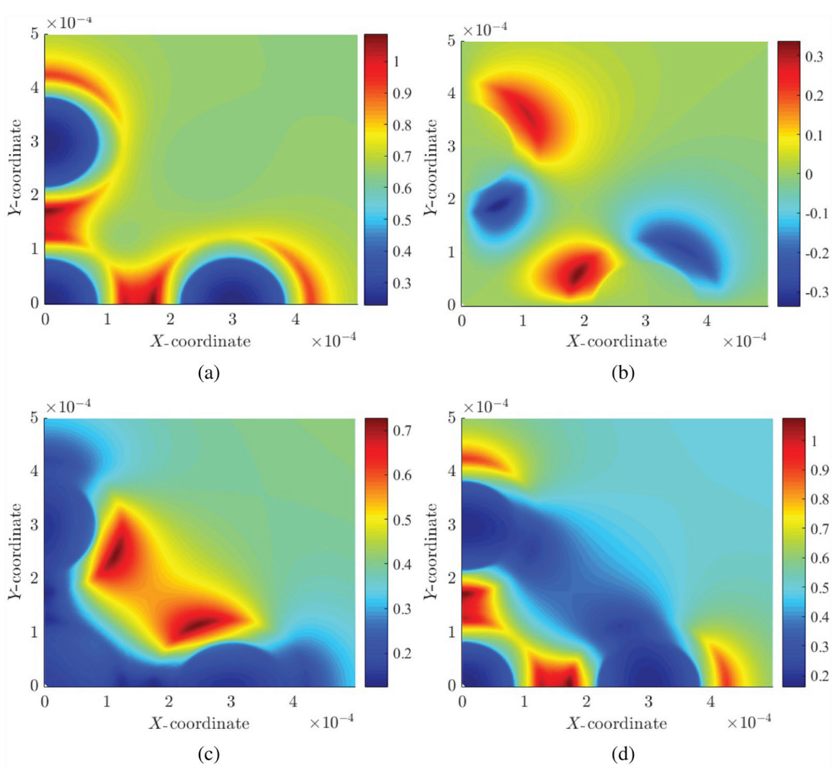


Figure 12. Case 1 for generalized Gent model. (a) $||\mathbf{T}||$ (b) Shear stress component. (c) Radial stress. (d) Circumferential stress.

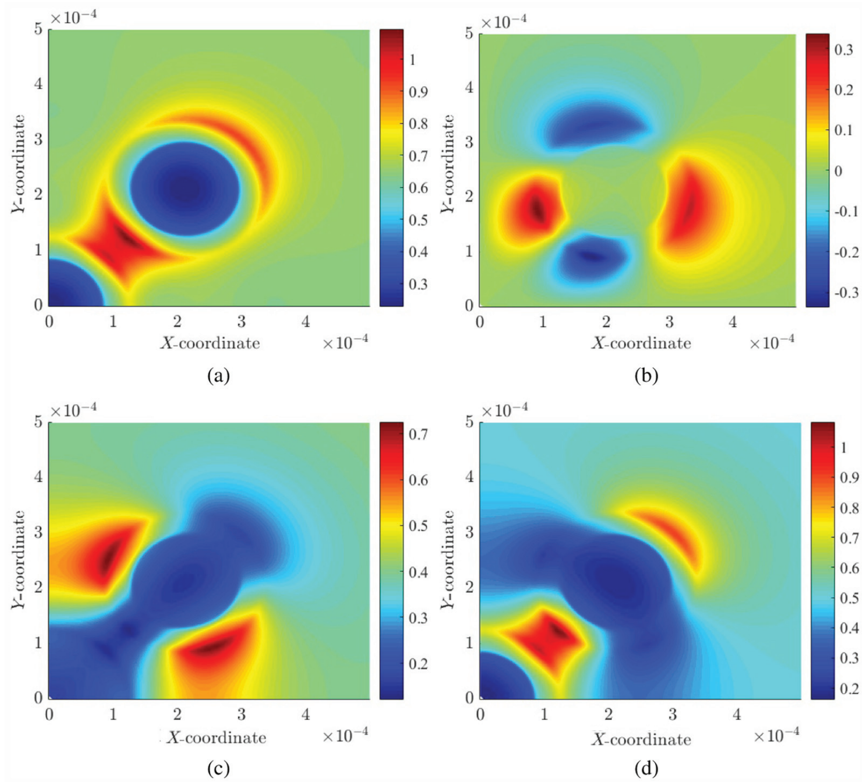


Figure 13. Case 2 for generalized Gent model. (a) $||\mathbf{T}||$ (b) Shear stress component, $T_{r\theta}$. (c) Radial stress. (d) Circumferential stress.

Figures **??** and **??** and Figures **??** and **??** show the norm of the stress, shear, radial and circumferential stress at a stretch of 3.4 for the generalized Gent and neo-Hookean models for cases 1 and 2 respectively. The norm of the stress and the circumferential stress give a clear picture of the interaction between the inhomogeneities in the two cases considered here. If not for these additional inhomogeneities surrounding the central region, the norm of the stress, as well as value of the the circumferential stress, will increase monotonically in the annular region wherein the transition occurs from the maximum value to the minimum value for the density. It is very important to recognize that though the maximum of the norm of the stress occurs in the region between the two inhomogeneities, the loss of the structural integrity of the material with regard to its load-carrying capacity occurs at the location where the density is lowest, which coincides with the location where the derivative of the norm of the stress with respect to the stretch starts to decrease, which is also the inflection point for the norm of the stress versus the density.

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