

Can local dynamics enhance entangling power?

Bhargavi Jonnadula,¹ Prabha Mandayam,¹ Karol Życzkowski,^{2,3} and Arul Lakshminarayan¹

¹*Department of Physics, Indian Institute of Technology Madras, Chennai, India 600036*

²*Smoluchowski Institute of Physics, Jagiellonian University, Cracow, Poland*

³*Center for Theoretical Physics, Polish Academy of Sciences, Warsaw, Poland*

(Dated: Oct. 22, 2016)

It is demonstrated here that local dynamics have the ability to strongly modify the entangling power of unitary quantum gates acting on a composite system. The scenario is common to numerous physical systems, in which the time evolution involves local operators and nonlocal interactions. To distinguish between distinct classes of gates with zero entangling power we introduce a complementary quantity called gate-typicality and study its properties. Analyzing multiple applications of any entangling operator interlaced with random local gates, we prove that both investigated quantities approach their asymptotic values in a simple exponential form. This rapid convergence to equilibrium, valid for subsystems of arbitrary size, is illustrated by studying multiple actions of diagonal unitary gates and controlled unitary gates.

Introduction: The uniquely nonclassical phenomenon of entanglement is a well-known resource for quantum information [1]. It is increasingly used to characterize complex states, from many-body ground states [2], to infinite temperature quantum phase transitions such as the ergodic to localized phase in strongly interacting many-body systems [3]. Simple coupled quantum chaotic models have been studied [4, 5] and experimentally realized [6, 7] to demonstrate the large entanglement growth wherein the subsystems are nearly maximally mixed. In general the dynamics of entanglement in a non-equilibrium context is responsible for thermalization [8]. More recently in a different setting, black-holes are conjectured to scramble quantum information in a time that is logarithmic in the entropy via entanglement [9].

While much work has centered on properties of states, quantum operators have also been studied as a physical resource for creating entanglement [10–15]. Studying directly the operators, such as propagators in time, frees us from the arbitrariness of initial states or the choice of eigenvectors. The entangling power [10, 16] while referring to an inherent property of operators on a bipartite composite system is also related to how much state entanglement can be created, on the average, using one application of the unitary on product states. Investigations on quantum transport in light harvesting complexes [17], quantum chaos [18] and thermalization [19] have made direct use of the entangling power of bipartite unitary gates.

This Letter concerns the evolution of entangling power when multiple nonlocal operators are used successively while being interlaced with local operators, a typical situation in time evolution. Local unitary invariance, in a bipartite setting, implies that $(U^A \otimes U^B)U(U^{A'} \otimes U^{B'})$ has the same entangling power as U . However, if nonlocal operations are interlaced by local dynamics, the consequences are nontrivial, as the entangling power of U and $\sqrt{U}(U^A \otimes U^B)\sqrt{U}$ are not the same.

Specifically, we are interested in the case when the non-

local operators are fixed and structured, while complexity is introduced in local gates taken randomly from a given ensemble. The resultant operators are shown to rapidly acquire properties of random operators on the composite space, in particular, large entangling powers are obtained however small the entangling power of the individual nonlocal operators may be. The resulting entangling power can be, counterintuitively, much larger than what can be achieved in the absence of the local operators. Besides the importance of such scenarios in the context of dynamical evolution, they indicate how generic bipartite gates can be prepared using local random operators [15]. It is known that such composite random operators allow for protocols such as approximate quantum encryption and data hiding that are more efficient than their deterministic counterparts [20].

We investigate the problem in detail, applying the entangling power $e_p(U)$ and introducing a complementary quantity $g_t(U)$ that, unlike $e_p(U)$, differentiates between local gates and the swap gate. Access to long-time or multiple uses of nonlocal operators is possible by analytically averaging over an ensemble of random local gates. Explicit results obtained in this way are shown to provide an excellent approximation for the time dependence of both the entangling power and g_t of multiple usage of a given bipartite unitary gate.

Entangling power and gate-typicality: Most measures of operator interaction strengths [14] are based on the Schmidt decomposition of the unitary evolution operator U acting on a bipartite space $\mathcal{H}^N \otimes \mathcal{H}^N$. The operator Schmidt decomposition and the “operator entanglement” $E(U)$ read [14]

$$U = \sum_{i=1}^{N^2} \sqrt{\lambda_i} A_i \otimes B_i, \quad E(U) = 1 - \frac{1}{N^4} \sum_{i=1}^{N^2} \lambda_i^2. \quad (1)$$

Here A_i and B_i are orthonormal operators, *i.e.* $\text{tr}(A_i A_j^\dagger) = \text{tr}(B_i B_j^\dagger) = \delta_{ij}$ and $\lambda_i \geq 0$. Unitarity implies that $\sum_{i=1}^{N^2} \lambda_i = N^2$ and therefore the set

$\{\lambda_i/N^2, 1 \leq i \leq N^2\}$ forms a discrete probability measure, and $E(U)$ is the operator linear entropy.

The operator entanglement of a unitary gate $E(U)$ is linked to its average ability to create entanglement. The *entangling power* of an operator U is defined as $e_p(U) = \overline{E_L(U|\psi_A\rangle|\psi_B)}^{\psi_A, \psi_B}$ – see [10, 16, 21]. Here $E_L(|\psi_{AB}\rangle)$ is the usual linear entropy of the state $|\psi_{AB}\rangle$, defined as $E_L = 1 - \text{tr}\rho_A^2$, where ρ_A is the reduced density matrix of A , and the average is taken over all the product states $|\psi_A\rangle|\psi_B\rangle$ distributed according to the unitarily invariant measure. Entangling power of a gate U of size N^2 is bounded as: $0 \leq e_p(U) \leq (N-1)/(N+1)$. Interestingly, both quantities are directly related as shown by Zanardi in [16], $e_p(U) = N^2[E(U) + E(US) - E(S)]/(N+1)^2$. Here S is the SWAP gate defined by $S|\psi_A\rangle|\psi_B\rangle = |\psi_B\rangle|\psi_A\rangle$, so that $E(S) = (N^2 - 1)/N^2$.

Evaluation of $E(U)$ and $E(US)$, as well as their interpretation, is facilitated by considering them as entanglement measures of four-party pure states – see Supplementary Material. Towards this end, consider two density matrices

$$\rho_R(U) = \frac{1}{N^2} U_R U_R^\dagger, \quad \rho_T(U) = \frac{1}{N^2} S U_T U_T^\dagger S, \quad (2)$$

where U_R is the reshuffling of U , while U_T is its partial transpose with respect to A . These are defined as the following, generally non-unitary, permutations of the original bi-partite unitary matrix U : $\langle ij|U_R|\alpha\beta\rangle = \langle i\alpha|U|j\beta\rangle$, and $\langle j\alpha|U_T|i\beta\rangle = \langle i\alpha|U|j\beta\rangle$.

The operator Schmidt decomposition of U is determined by the spectra of $U_R U_R^\dagger$ [22], as the eigenvalues of $\rho_R(U)$ are equal to the rescaled coefficients λ_i/N^2 from Eq. (1). As $S(SU)_R = U_T$, it is easy to relate the eigenvalues of $\rho_T(U)$ to the Schmidt decomposition of SU . It follows that $E(U) = 1 - \text{tr}(\rho_R^2(U))$, and $E(SU) = 1 - \text{tr}(\rho_T^2(U))$. The operator linear entropy $E(U)$ is thus equal to the linear entropy of the state $\rho_R(U)$. Related observations previously appeared in [23, 24].

While $E(U)$ and $E(US)$ are two independent polynomial invariants of U [25], the entangling power is a symmetric function of these quantities and hence does not differentiate local dynamics from the SWAP gate S . Therefore, it is useful to introduce a quantity complementary to the entangling power $e_p(U)$, defined by the antisymmetric combination

$$g_t(U) = \frac{N^2}{N^2 - 1} [E(U) - E(US) + E(S)]. \quad (3)$$

Observe that the maximum value, $g_t = 2$, is achieved for the SWAP gate, and locally equivalent gates, $(U^A \otimes U^B) S (U^{A'} \otimes U^{B'})$. This is consistent with the fact that, although $e_p(S) = 0$, in terms of the operator entanglement $E(U)$, the SWAP gate is a maximally nonlocal operator as all its Schmidt coefficients λ_i are equal. Op-

erationally as well, when implementing gates using teleportation and classical communication, the SWAP gate consumes maximum resources [11, 26].

The minimum value of gate-typicality is reached only for local gates, $g_t(U^A \otimes U^B) = 0$, while the mean value averaged over the Haar measure reads $\overline{g_t(U)}^U = 1$. The distribution $P(g_t)$ obtained for the Haar random unitaries is symmetric with respect to its average value. Since for large N this distribution is strongly concentrated at the mean value $\overline{g_t} = 1$, it is appropriate to call g_t as the *gate-typicality*. Note that large deviations, including gates close to S with $g_t \approx 2$, and nearly local gates with $g_t \approx 0$, are rare and atypical. Thus for any unitary gate U the quantity $|g_t(U) - 1|$ is a measure of its non-typicality.

Effect of one intermediate local operation:

Let $V = \sqrt{U}(U^A \otimes U^B)\sqrt{U}$, where one qubit local unitaries U^A and U^B are Haar random unitaries from $U(2)$. Fig. 1 shows the pairs $\{E(V), E(VS)\}$ and $\{e_p(V), g_t(V)\}$ for a fixed two-qubit gate U picked at random according to the Haar measure on $U(4)$. Here the operator \sqrt{U} has the same eigenvectors as U , and its eigenvalues are $e^{i\phi/2}$, where the eigenvalues of U are $e^{i\phi}$ and $-\pi < \phi \leq \pi$.

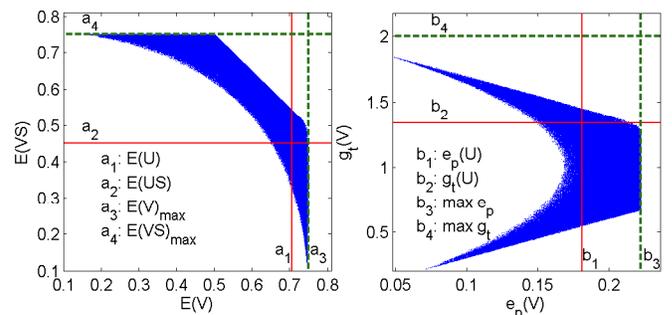


FIG. 1. Operator entanglements $E(V)$ vs $E(VS)$ (left), and entangling power vs gate-typicality (right) for two qubits. Here U is a fixed random entangling gate, $V = \sqrt{U}(U^A \otimes U^B)\sqrt{U}$ where local gates U^A, U^B are sampled according to the Haar measure on $U(2)$. Lines (with labels indicating their meaning) illustrate that local gates can increase both the entangling power and the gate-typicality of U .

It is clear from Fig. 1 that there exists a nonzero measure of local operators that can enhance the entangling power and gate-typicality of U , indicated by the solid lines. The same holds for the operator entanglements $E(V)$ and $E(VS)$ as demonstrated in Fig. 1. The mean entangling power, averaged over the Haar measure on $U(N^2)$ reads $\overline{e_p} = (N-1)^2/(N^2+1)$ [10, 16]. A sufficient condition for the existence of local operators which can increase e_p and g_t follows as a corollary to the main theorem formulated below.

Multiple iterations and averaging over local unitaries: Local gates, although they have no entangling power

themselves, catalyze the entangling power and gate-typicality when interlaced between multiple uses of U . An exact statement concerning the strength averaged over local gates is stated now as the central result of this work.

Theorem. Consider n identical nonlocal operators U interlaced by local gates $W_j = U_j^A \otimes U_j^B$:

$$U^{(n)} \equiv U W_{n-1} U \cdots W_1 U, \quad n \geq 1, \quad U^{(1)} \equiv U. \quad (4)$$

Then the mean entangling power and the mean gate-

typicality read

$$\langle e_p(U^{(n)}) \rangle_W = \bar{e}_p \left[1 - \left(1 - \frac{e_p(U)}{\bar{e}_p} \right)^n \right], \quad (5)$$

$$\text{and } \langle g_t(U^{(n)}) \rangle_W = 1 - [1 - g_t(U)]^n, \quad (6)$$

where $\langle \cdot \rangle_W$ denotes the average with respect to the Haar measure on $U(N)$.

Proof: Define the mean purities of states generated from $U^{(n)}$ of Eq. (4) and averaged over local unitaries $\{U_1^A \cdots U_{n-1}^B\}$ as

$$X_n = \langle \text{tr}[\rho_R^2(U^{(n)})] \rangle_W, \quad Y_n = \langle \text{tr}[\rho_T^2(U^{(n)})] \rangle_W, \quad (7)$$

so that $\langle E(U^{(n)}) \rangle_W = 1 - X_n$, $\langle E(SU^{(n)}) \rangle_W = 1 - Y_n$, and $X_1 = \text{tr}[\rho_R^2(U)]$, $Y_1 = \text{tr}[\rho_T^2(U)]$. Given a nonlocal operator U , and hence (X_1, Y_1) , a linear affine iterative scheme $(X_n, Y_n) \mapsto (X_{n+1}, Y_{n+1})$, follows from the independence of the local unitaries – see Supplementary Material for the details,

$$\begin{aligned} X_{n+1} &= \frac{1}{(N^2 - 1)^2} [2(N^2 + 1) - N^2(2X_1 + 2Y_1 + 2X_n + 2Y_n - X_1Y_n - Y_1X_n) + N^4(Y_1Y_n + X_1X_n)], \\ Y_{n+1} &= \frac{1}{(N^2 - 1)^2} [2(N^2 + 1) - N^2(2X_1 + 2Y_1 + 2X_n + 2Y_n - Y_1Y_n - X_1X_n) + N^4(X_1Y_n + Y_1X_n)]. \end{aligned} \quad (8)$$

It is clear that the combinations $X_n \pm Y_n$ do separate so these quantities are convenient to iterate. Thus defining

$$\xi_n = C_N \left(X_n + Y_n - \frac{4}{N^2 + 1} \right), \quad \eta_n = D_N (X_n - Y_n), \quad (9)$$

where $C_N = N^2(N^2 + 1)/(N^2 - 1)^2$, $D_N = N^2/(N^2 - 1)$ leads to simple recursion relations, $\xi_{n+1} = \xi_1 \xi_n$, $\eta_{n+1} = \eta_1 \eta_n$, with solutions $\xi_n = \xi_1^n$, $\eta_n = \eta_1^n$. It is easy to generalize this reasoning for the case in which nonlocal operators are different at each iteration in Eq. (4). Denoting different values of ξ_1 as ξ_{1k} , then $\xi_n = \prod_{k=1}^n \xi_{1k}$. Note that ξ_n is averaged over the local unitaries, while ξ_{1k} are derived from the purities of the corresponding density matrices as in Eq. (2).

The quantity ξ_1 is related to the entangling power of U and, remarkably, it follows from the definition that $\xi_1 = 1 - e_p(U)/\bar{e}_p$. This in turn, along with $\xi_n = \xi_1^n$, results in Eq. (5). The complementary quantity η_n distinguishes between the SWAP and local gates, and provides additional motivation to introduce the gate-typicality (3). Hence using the definition (9) of η_1 one can show that $\eta_1 = 1 - g_t(U)$ and the advertised exponential convergence (6) follows. \square

Corollary. If $e_p(U) < e_p(\sqrt{U})(2 - e_p(\sqrt{U})/\bar{e}_p)$, then there exist local operators U^A and U^B such that $e_p(U) <$

$$e_p(\sqrt{U}(U^A \otimes U^B)\sqrt{U}).$$

The corollary follows as the theorem implies that if $\langle e_p(U^{(2)}) \rangle_W > e_p(U^2)$ then $e_p(U^2) < e_p(U)(2 - e_p(U)/\bar{e}_p)$. The entangling power of UWU averaged over local unitaries is larger than the entangling power of U^2 and therefore there exist members of the ensemble of local unitary operators such that $e_p(U(U^A \otimes U^B)U) > e_p(U^2)$. If the nonlocal gate U is chosen at random from $U(4)$ numerical results indicate that about 28% of them satisfy the condition in the corollary, one such realization is shown in Fig. 1. Thus as long as $e_p(U) \neq 0$, $\langle e_p(U^{(n)}) \rangle_W \rightarrow \bar{e}_p$ as $n \rightarrow \infty$, and the convergence is exponential with the rate which depends on the entangling power $e_p(U)$. A similar statement about the gate-typicality g_t follows. Thus if $g_t(U) < g_t(\sqrt{U})(2 - g_t(\sqrt{U}))$, then there exist local operators such that $g_t(U) < g_t(\sqrt{U}(U^A \otimes U^B)\sqrt{U})$.

Note that the above statements solve completely and exactly the problem of finding the entangling power and gate-typicality of an iterated nonlocal operation averaged over random local gates which interlace the dynamics. Two convergence rates follow, $\log|1/\xi_1|$ for the entangling power and $\log|1/\eta_1|$ for the gate-typicality. Any entangling gate U iterated with interlaced random local unitaries will lead to typical entangling power and mean typicality at these rates. Also the purities tend to their

mean values:

$$X_\infty = Y_\infty = \frac{2}{N^2 + 1}, \quad (10)$$

is the fixed point *independent* of unitary U , as long as it is not itself a local operator or the swap gate.

The ranges of ξ_1 and η_1 are

$$-\frac{2}{(N^2 - 1)} \leq \xi_1 \leq 1, \quad -1 \leq \eta_1 \leq 1. \quad (11)$$

The upper-bounds are reached by unitary operators that are local. In addition $\xi_1 = 1$ also for the SWAP gate, while for η_1 the lower-bound of -1 is reached *only* in this case. The lower-bound of ξ_1 follows from fact that X_1 and Y_1 are density matrix purities and hence cannot be less than $1/N^2$ each. However, this is not a tight bound for $N = 2$ and this is related to the nonexistence of absolutely maximally entangled states of four qubits [27] and of multiunitary matrices of order four [28]. The bound is tight in all dimensions except 2 and possibly 6, which follows also from a previous study of the entangling power of permutations [29]. As shown below for $N = 2$ the minimum value of ξ_1 is achieved if U is the CNOT gate.

Examples: To present our Theorem in action we now discuss three paradigmatic unitary gates: (a) the two qubit CNOT gate, (b) unitaries $U \equiv U_d$ consisting of only diagonal elements, and (c) higher dimensional controlled gates.

(a) Two qubit CNOT gate reads $|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x$, and simple calculations yield $X_1 = 1/2$, $Y_1 = 1/4$, $e_p(\text{CNOT}) = 2/9$ and $g_t(\text{CNOT}) = 2/3$. Using Eqs. (5,6) results in $\langle e_p(\text{CNOT}^{(n)}) \rangle_W =$

$$\frac{1}{5} \left(1 - \frac{(-1)^n}{9^n} \right), \quad \langle g_t(\text{CNOT}^{(n)}) \rangle_W = 1 - \frac{1}{3^n}. \quad (12)$$

The entangling power decreases from the maximum possible $2/9$ to the average $1/5$ at the rate of $2 \log 3$, while the gate-typicality increases from $2/3$ to 1 at the rate of $\log 3$.

(b) Random diagonal unitaries U_d studied in [30, 31] arise as interactions in many Floquet models. Note first that the square of a diagonal unitary matrix remains diagonal, so the entangling power and the gate-typicality of U_d^n remains approximately the same during the time evolution. However, if the evolution is interlaced by local dynamics the situation changes dramatically. Applying Eq. (7) we obtain

$$\overline{X_1} = \overline{\text{tr}(\rho_R^2(U_d))}^{U_d} = \frac{2N - 1}{N^2}, \quad \text{var}(X_1) = 2 \frac{(N - 1)^2}{N^6}. \quad (13)$$

where the bar indicates additional averaging over the diagonal elements, which are uniform random phases – the average $\overline{X_1}$ over the unimodular ensemble is derived in

[31]. As the gate U_d is diagonal it is invariant with respect to partial transposition, so $\rho_T(U_d) = I_{N^2}/N^2$ and $Y_1 = 1/N^2$. Thus for generic diagonal unitaries one has $X_1 \sim 2/N$ and $Y_1 = 1/N^2$. These values imply the following behavior for $n \ll N$,

$$\Delta X_n = \frac{2^n}{N^n} \left[1 + \mathcal{O}\left(\frac{n}{N}\right) \right], \quad \Delta Y_n = \mathcal{O}\left(\frac{2^n n}{N^{n+1}}\right), \quad (14)$$

where $\Delta X_n = X_n - X_\infty$ and $\Delta Y_n = Y_n - Y_\infty$. Thus Y_n almost reaches its typical value only after two generic diagonal gates as $Y_2 \sim 2/N^2$. The value of $X_2 \sim 6/N^2$ and reflects significant deviations from stationarity after two iterations, while $X_3 \sim 2/N^2$, the same leading order as X_∞ .

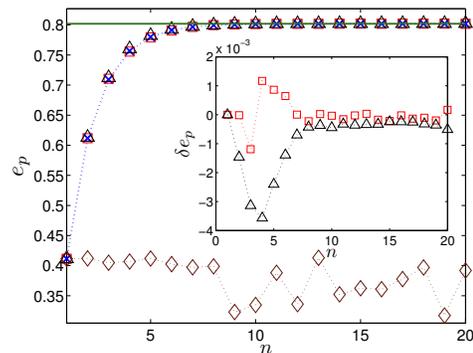


FIG. 2. Entangling power e_p after n actions of a controlled unitary gate $U \in U(N^2)$ plotted for $N = 10$. Data from one realization of $U^{(n)}$ (\square) and one realization of $[(U^A \otimes U^B) U]^n$ (\triangle) are shown. Values averaged over local gates according to Eq. (5) are indicated by (\times), while the insets show deviations from this average. Horizontal line denotes the average over the unitary group. The case U^n with no interlacing gates, for which entangling power do not converge to typical values, is marked by (\diamond).

These results imply that, up to the lowest order in $1/N$, the entangling power increases from the value $1 - (4/N)$, typical for diagonal unitaries to the mean value over the Haar measure, $\overline{e_p} = 1 - (2/N)$. On the other hand, the gate-typicality g_t increases from the value $1 - (2/N)$ characteristic to diagonal unitaries to 1 under the influence of random local gates.

(c) Controlled unitaries can be implemented using a simple nonlocal protocol [32] with prior entanglement. Consider a controlled gate $U = P_1^A \otimes \mathbb{1}^B + P_2^A \otimes V^B$, where $P_1^A + P_2^A = \mathbb{1}_N$ and $P_i^A P_j^A = \delta_{ij} P_i^A$, and V^B is some N dimensional unitary operator. In this case $X_1 = (N^2 + |\text{tr} V^B|^2)/2N^2$, $Y_1 = 1/N^2$. For $n \ll N$ the iteration results in

$$\begin{aligned} \Delta X_n &= \frac{1}{2^n} \left[1 + \mathcal{O}\left(\frac{n}{N^2}\right) \right], \\ \Delta Y_n &= \frac{1}{2^n N^2} \left[-(n+1) + \mathcal{O}\left(\frac{n}{N^2}\right) \right]. \end{aligned} \quad (15)$$

The details of the gate V^B are relevant by its trace only up to higher orders represented by the symbol \mathcal{O} . The

contrast with Eq. (14) is apparent as this indicates a much slower convergence to the asymptotic values. It is also clear that the quantity Y_n approaches its limiting value faster than X_n . For instance, for $n = 2$ one has $X_2 \sim 1/4$, while $Y_2 \sim 5/(4N^2)$. The Haar average is reached within precision $\mathcal{O}(1/N^4)$ by X_n for time $n \sim 4 \log_2 N$.

Under the action of random local gates the entangling power increases from the initial value close to $1/2 - 1/N$ to the stationary value of $1 - 2/N$, while the gate-typicality increases from $1/2$ to 1. If no local operators were used, observe first that U^n is also a controlled unitary. Assuming now that V^B is taken as a Haar random unitary from $U(N)$ then the average form factor $\langle |\text{tr}(V^B)^n|^2 \rangle$ equal to n for $n \leq N$ and to N for $n > N$ – see [33] – implies that the entangling power of U^n decreases to $1/2 - (3/2N)$ for times long enough, while the gate-typicality decreases to $1/2 - (1/2N)$.

Simulations in Fig. 2 illustrates that the formulae derived for entangling power averaged over an ensemble of local interlacing gates form excellent approximations to time evolution even for a *single* realization of local gates. More remarkably, formulae derived also work if the *same* local gates are applied at every iteration, so that $U^{(n)} = (U^A \otimes U^B U)^n$ – see inset for the smallness of the deviations. Thus these results are of direct relevance to the study of iterated coupled quantum Floquet systems such as in [4, 18]. Similar qualitative behavior is observed for gate-typicality as well. Thus correlations introduced by the repeated action of local gates are not significant, and the entangling power and gate-typicality continue to reach their asymptotic values exponentially fast.

Summary and outlook: Iterating nonlocal unitary operators with interlaced local dynamics is a typical scenario in both time evolution and simple quantum circuits. We have shown here that two quantities characterizing the interaction strength, namely the entangling power and gate-typicality are significantly modified by subsequent application of local gates. We have shown that both quantities converge exponentially to their asymptotic values and computed the mean convergence rates under the assumption that local gates are distributed randomly according to the Haar measure on $U(N)$.

As typical for ergodic problems, a generic realization is shown to closely follow the average behavior. Our analytic predictions hold even when the same local unitary gate is applied several times. Additional numerical investigations show that other moments $\text{tr}(\rho_R^k(U^{(n)}))$ and $\text{tr}(\rho_T^k(U^{(n)}))$, with $k \neq 2$, and the von Neumann entropies also exponentially approach their limiting values, as the density of the rescaled eigenvalues of $\rho_R(U^{(n)})$ and $\rho_T(U^{(n)})$ for large N approaches the Marcenko-Pastur distribution [34].

A detailed study and further interpretation of the gate-

typicality is called for. It is important to investigate the extent to which nonrandom local operators influence the approach to equilibrium of a periodically interlaced unitary dynamics. Applications to Floquet models of condensed matter physics and quantum chaos would be interesting. Generalizations to multipartite settings as well as to generalized quantum operations are worth studying.

We are grateful to Som Bandyopadhyay for discussions on the ancilla interpretation and Steven Tomsovic for comments. This work was supported by the Polish National Science Center under the project number DEC-2015/18/A/ST2/00274, by the John Templeton Foundation under the project No. 56033, and the Indian DST (INSPIRE) project PHY1415305DSTXPRAN.

-
- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
 - [2] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, *Rev. Mod. Phys.*, **80**, 517 (2008).
 - [3] J. A. Kjäll, J. H. Bardarson, and F. Pollmann, *Phys. Rev. Lett.*, **113**, 107204 (2014).
 - [4] A. Lakshminarayan, *Phys. Rev. E*, **64**, 036207 (2001).
 - [5] C. M. Trail, V. Madhok, and I. H. Deutsch, *Phys. Rev. E*, **78**, 046211 (2008).
 - [6] S. Chaudhury, A. Smith, B. E. Anderson, S. Ghose, and P. S. Jessen, *Nature*, **461**, 768 (2009).
 - [7] C. Neill and et al., *Nat. Phys.*, 10.1038/nphys3830 (2016).
 - [8] N. Linden, S. Popescu, A. J. Short, and A. Winter, *Phys. Rev. E*, **79**, 061103 (2009).
 - [9] N. Lashkari, D. Stanford, M. Hastings, T. Osborne, and P. Hayden, *J. of High Energ. Phys.*, **2013**, 22 (2013).
 - [10] P. Zanardi, C. Zalka, and L. Faoro, *Phys. Rev. A*, **62**, 030301 (2000).
 - [11] D. Collins, N. Linden, and S. Popescu, *Phys. Rev. A*, **64**, 032302 (2001).
 - [12] G. Vidal and J. I. Cirac, *Phys. Rev. Lett.*, **88**, 167903 (2002).
 - [13] K. Hammerer, G. Vidal, and J. I. Cirac, *Phys. Rev. A*, **66**, 062321 (2002).
 - [14] M. A. Nielsen, C. M. Dawson, J. L. Dodd, A. Gilchrist, D. Mortimer, T. J. Osborne, M. J. Bremner, A. W. Harrow, and A. Hines, *Phys. Rev. A*, **67**, 052301 (2003).
 - [15] J. Emerson, Y. S. Weinstein, M. Saraceno, S. Lloyd, and D. G. Cory, *Science*, **302**, 2098 (2003).
 - [16] P. Zanardi, *Phys. Rev. A*, **63**, 040304 (2001).
 - [17] F. Caruso, A. W. Chin, A. Datta, S. F. Huelga, and M. B. Plenio, *Phys. Rev. A*, **81**, 062346 (2010).
 - [18] R. Demkowicz-Dobrzański and M. Kuś, *Phys. Rev. E*, **70**, 066216 (2004).
 - [19] V. Scarani, M. Ziman, P. Štelmachovič, N. Gisin, and V. Bužek, *Phys. Rev. Lett.*, **88**, 097905 (2002).
 - [20] P. Hayden, D. Leung, P. W. Shor, and A. Winter, *Communications in Mathematical Physics*, **250**, 371 (2004), ISSN 1432-0916.
 - [21] X. Wang and P. Zanardi, *Phys. Rev. A*, **66**, 044303 (2002).
 - [22] K. Życzkowski and I. Bengtsson, *Open Systems & Infor-*

mation Dynamics, **11**, 3 (2004), ISSN 1573-1324.

- [23] X. Wang, B. C. Sanders, and D. W. Berry, Phys. Rev. A, **67**, 042323 (2003).
- [24] Z. Ma and X. Wang, Phys. Rev. A, **75**, 014304 (2007).
- [25] M. Grassl, M. Rötteler, and T. Beth, Phys. Rev. A, **58**, 1833 (1998).
- [26] J. Eisert, K. Jacobs, P. Papadopoulos, and M. B. Plenio, Phys. Rev. A, **62**, 052317 (2000).
- [27] A. Higuchi and A. Sudbery, Physics Letters A, **273**, 213 (2000), ISSN 0375-9601.
- [28] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Życzkowski, Phys. Rev. A, **92**, 032316 (2015).
- [29] L. Clarisse, S. Ghosh, S. Severini, and A. Sudbery, Phys. Rev. A, **72**, 012314 (2005).
- [30] Y. Nakata, P. S. Turner, and M. Muraio, Phys. Rev. A, **86**, 012301 (2012).
- [31] A. Lakshminarayan, Z. Puchała, and K. Życzkowski, Phys. Rev. A, **90**, 032303 (2014).
- [32] L. Yu, R. B. Griffiths, and S. M. Cohen, Phys. Rev. A, **81**, 062315 (2010).
- [33] F. Haake, M. Kus, H.-J. Sommers, H. Schomerus, and K. Życzkowski, J. Phys., A **29**, 3641 (1996).
- [34] V. A. Marcenko and L. A. Pastur, Mathematics of the USSR-Sbornik, **1**, 457 (1967).
- [35] P. A. Mello, J. Phys., A **23**, 4061 (1990).
- [36] B. Collins and P. Śniady, Commun.Math. Phys., **264**, 773 (2006), ISSN 1432-0916.
- [37] Z. Puchała and J. A. Miszczak, arXiv:1109.4244 (2011).

Supplementary Material

$$X_{UV} \equiv \langle \text{tr}[\rho_R^2(\mathcal{N})] \rangle_{U^A, U^B} = \frac{1}{N^4} U_{m_1 \gamma_1}^{k\alpha} V_{m_2 \gamma_2}^{l\beta} U_{m_3 \gamma_3}^{\alpha' k'} V_{m_4 \gamma_4}^{\beta' l'} \bar{U}_{m_5 \gamma_5}^{\alpha' \alpha} \bar{V}_{m_6 \gamma_6}^{\beta' \beta} \bar{U}_{m_7 \gamma_7}^{kk'} \bar{V}_{m_8 \gamma_8}^{ll'} \left\langle \left(\mathcal{U}^A \bar{\mathcal{U}}^A \right)_{[m]} \right\rangle \left\langle \left(\mathcal{U}^B \bar{\mathcal{U}}^B \right)_{[\gamma]} \right\rangle,$$

where $\left(\mathcal{U}^A \bar{\mathcal{U}}^A \right)_{[m]} = U_{m_1 m_2}^A U_{m_3 m_4}^A \bar{U}_{m_5 m_6}^A \bar{U}_{m_7 m_8}^A$; $\left(\mathcal{U}^B \bar{\mathcal{U}}^B \right)_{[\gamma]} = U_{\gamma_1 \gamma_2}^B U_{\gamma_3 \gamma_4}^B \bar{U}_{\gamma_5 \gamma_6}^B \bar{U}_{\gamma_7 \gamma_8}^B$,

(16)

and the bar indicates the complex conjugate. A similar expression holds for $Y_{UV} = \langle \text{tr}(\rho_T^2(\mathcal{N})) \rangle_{U^A, U^B}$. The average over the local unitaries U^A, U^B are independent and such averages over the unitary group have long been

A. Operator entanglement and the ancilla interpretation:

It is useful to view $E(U)$ as entanglement in a pure state between A and B along with a bi-partite ancilla $A'B'$ – see Fig. 3. Let AA' be in the standard maximally entangled state $|\phi_{AA'}^+\rangle = \sum_{j=1}^N |jj\rangle/\sqrt{N}$, with A' being an ancilla with A , also of dimension N , and let BB' be in a similar state. If U acts between A and B subsystems, the reduced density matrices of AA' and $A'B$ are respectively, $\rho_R(U) = U_R U_R^\dagger/N^2$, and $\rho_T(U) = S U_T U_T^\dagger S/N^2$. Here S is the SWAP operator, U_R is the reshuffling of U , while U_T is its partial transpose with respect to A and are defined in the main text. The operator linear entropy $E(U)$ is thus the linear entropy of the state $\rho_R(U)$ and measures the entanglement in this quadripartite state with respect to the partition $AA'|BB'$. Since $E(SU) = E(US)$ it represents the entanglement of the same state with respect to the partition $AB'|A'B$ partition – see Fig. 3.

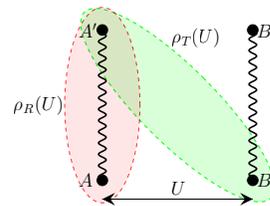


FIG. 3. Operational significance of $E(U)$. It captures the entanglement generated by the action of U on a pure state of systems A, B along with ancillas A' and B' , across the splitting $AA'|BB'$. Similarly $E(US)$ is the entanglement with respect to the partition $AB'|A'B$.

B. Proof for the iterative scheme:

Consider the operator $\mathcal{N} = U (U^A \otimes U^B) V$. Denoting the matrix elements $\langle ij|U|pq \rangle \equiv U_{ij}^{pq}$ and summing over repeated indices one arrives at

known [35, 36] (see [37] for a Mathematica function to calculate the averages). While they are in general expressed in terms of the so-called Weingarten functions, this particular 4-term average is simple enough:

$$\begin{aligned} \langle U_{i_1 j_1} U_{i_2 j_2} \bar{U}_{i'_1 j'_1} \bar{U}_{i'_2 j'_2} \rangle &\equiv \int_{U(N)} U_{i_1 j_1} U_{i_2 j_2} \bar{U}_{i'_1 j'_1} \bar{U}_{i'_2 j'_2} dU = \frac{1}{N^2 - 1} (\delta_{i_1 i'_1} \delta_{i_2 i'_2} \delta_{j_1 j'_1} \delta_{j_2 j'_2} + \delta_{i_1 i'_2} \delta_{i_2 i'_1} \delta_{j_1 j'_2} \delta_{j_2 j'_1}) \\ &- \frac{1}{N(N^2 - 1)} (\delta_{i_1 i'_1} \delta_{i_2 i'_2} \delta_{j_1 j'_2} \delta_{j_2 j'_1} + \delta_{i_1 i'_2} \delta_{i_2 i'_1} \delta_{j_1 j'_1} \delta_{j_2 j'_2}) \end{aligned} \quad (17)$$

Thus there are 16 terms that should be computed for finding X_2 , and a few are calculated below. Let

$$\begin{aligned} \left\langle \left(\mathcal{U}^A \bar{\mathcal{U}}^A \right)_{[m]} \right\rangle &\equiv \frac{1}{N^2 - 1} [D_1 + D_2 - \frac{1}{N} (D_3 + D_4)], \\ \left\langle \left(\mathcal{U}^B \bar{\mathcal{U}}^B \right)_{[\gamma]} \right\rangle &\equiv \frac{1}{N^2 - 1} [D'_1 + D'_2 - \frac{1}{N} (D'_3 + D'_4)], \end{aligned} \quad (18)$$

where D_i and D'_j , $i, j = 1, \dots, 4$, are products of Kronecker deltas which can be read from Eq. (17). For example, the term corresponding to $D_4 D'_1$ in X_2 is

$$\begin{aligned} &\frac{-1}{N^5 (N^2 - 1)^2} \delta_{m_1 m_7} \delta_{m_3 m_5} \delta_{m_2 m_6} \delta_{m_4 m_8} \delta_{\gamma_1 \gamma_5} \delta_{\gamma_2 \gamma_6} \delta_{\gamma_3 \gamma_7} \delta_{\gamma_4 \gamma_8} \\ &\quad U_{m_1 \gamma_1}^{k\alpha} V_{l\beta}^{m_2 \gamma_2} U_{m_3 \gamma_3}^{\alpha' k'} V_{\beta' l'}^{m_4 \gamma_4} \bar{U}_{m_5 \gamma_5}^{\alpha' \alpha} \bar{V}_{\beta' \beta}^{m_6 \gamma_6} \bar{U}_{m_7 \gamma_7}^{kk'} \bar{V}_{l' l}^{m_8 \gamma_8} \\ &= \frac{-1}{N^2 (N^2 - 1)^2} \langle k\alpha\alpha'k' | U \otimes U | m_1 \gamma_1 m_3 \gamma_3 \rangle \\ &\quad \langle m_3 \gamma_1 m_1 \gamma_3 | U^\dagger \otimes U^\dagger | \alpha' \alpha k k' \rangle \\ &= \frac{-1}{N^2 (N^2 - 1)^2} \text{tr}[(U \otimes U) S_{AA'} (U^\dagger \otimes U^\dagger) S_{AA'}] \\ &= \frac{-N^2}{(N^2 - 1)^2} \text{tr}(\rho_R^2(U)) \equiv \frac{-N^2}{(N^2 - 1)^2} X_1^U \end{aligned} \quad (19)$$

As another example, the term corresponding to $D_3 D'_4$ is

found to be

$$\begin{aligned} &\frac{1}{N^6 (N^2 - 1)^2} \text{tr}[(U \otimes U) S_{BB'} (U^\dagger \otimes U^\dagger) S_{AA'}] \\ &\quad \text{tr}[(V \otimes V) S_{AA'} (V^\dagger \otimes V^\dagger) S_{AA'}] \\ &= \frac{N^2}{(N^2 - 1)^2} \text{tr}(\rho_T^2(U)) \text{tr}(\rho_R^2(V)) \\ &\equiv \frac{N^2}{(N^2 - 1)^2} Y_1^U X_1^V. \end{aligned} \quad (20)$$

Evaluating all such terms including those for Y_{UV} results in

$$\begin{aligned} X_{UV} &= \frac{1}{(N^2 - 1)^2} [2(N^2 + 1) - 2N^2 (X_1^U + Y_1^U + X_1^V + Y_1^V) \\ &\quad + N^4 (X_1^U X_1^V + Y_1^U Y_1^V) + N^2 (Y_1^U X_1^V + X_1^U Y_1^V)], \\ Y_{UV} &= \frac{1}{(N^2 - 1)^2} [2(N^2 + 1) - 2N^2 (X_1^U + Y_1^U + X_1^V + Y_1^V) \\ &\quad + N^4 (X_1^U Y_1^V + Y_1^U X_1^V) + N^2 (Y_1^U Y_1^V + X_1^U X_1^V)]. \end{aligned} \quad (21)$$

Now let $V = U (U_2^A \otimes U_2^B) U \dots (U_{n-1}^A \otimes U_{n-1}^B) U$, and take the average of both sides of Eq. (21) over all the local operators U_2^A, \dots, U_{n-1}^B . By definition then $\langle X_{UV} \rangle_{U_2^A, \dots, U_{n-1}^B} \equiv X_n$, $X_1^U \equiv X_1$ is independent of the local operators and $\langle X_1^V \rangle_{U_2^A, \dots, U_{n-1}^B} \equiv X_{n-1}$, with identical expressions for Y . Hence the recursion in Eq. (8) follows.