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The hypothesis of path integral duality II: corrections to quantum field theoretic results

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In the path integral expression for a Feynman propagator of a spinless particle of mass m, the path integral amplitude for a path of proper length $\mathcal{R}(x,x'|g_{\mu\nu})$ connecting events x and x' in a spacetime described by the metric tensor $g_{\mu\nu}$ is $\exp -[m\,\mathcal{R}(x,x'|g_{\mu\nu})]$. In a recent paper, assuming the path integral amplitude to be invariant under the duality transformation $\mathcal{R}\to\left(L_P^2/\mathcal{R}\right)$, Padmanabhan has evaluated the modified Feynman propagator in an arbitrary curved spacetime. He finds that the essential feature of this 'principle of path integral duality' is that the Euclidean proper distance $(\Delta x)^2$ between two infinitesimally separated spacetime events is replaced by $\left[(\Delta x)^2+4L_P^2\right]$. In other words, under the duality principle the spacetime behaves as though it has a 'zero-point length' L_P, a feature that is expected to arise in a quantum theory of gravity.

In the Schwinger's proper time description of the Feynman propagator, the weightage factor for a path with a proper time s is $\exp -(m^2 s)$. Invoking Padmanabhan's 'principle of path integral duality' corresponds to modifying the weightage factor $\exp -(m^2 s)$ to $\exp -\left[m^2 s+(L_P^2/s)\right]$. In this paper, we use this modified weightage factor in Schwinger's proper time formalism to evaluate the quantum gravitational corrections to some of the standard quantum field theoretic results in flat and curved spacetimes. In flat spacetime, we evaluate the corrections to: (1) the Casimir effect, (2) the effective potential for a self-interacting scalar field theory, (3) the effective Lagrangian for a constant electromagnetic background and (4) the thermal effects in the Rindler coordinates. In an arbitrary curved spacetime, we evaluate the corrections to: (1) the effective Lagrangian for the gravitational field and (2) the trace anomaly. In all these cases, we first present the conventional result and then go on to evaluate the corrections with the modified weightage factor. We find that the extra factor $\exp -(L_P^2/s)$ acts as a regulator at the Planck scale thereby 'removing' the divergences that otherwise appear in the theory. Finally, we discuss the wider implications of our analysis.

I. THE PRINCIPLE OF PATH INTEGRAL DUALITY

All attempts to provide a quantum frame work for the gravitational field have so far proved to be unsuccessful. In the absence of a viable quantum theory of gravity, it is interesting to ask whether we can say anything at all about the effects of metric fluctuations on quantum field theory in flat and curved spacetimes.

The Planck length $L_P\equiv(G\hbar/c^3)^{1/2}$ is expected to play a vital role in the ultimate quantum theory of gravity. (We shall set $\hbar=c=1$.) Simple thought experiments clearly indicate that it is not possible to devise experimental procedures which will measure lengths with an accuracy greater than about $O(L_P)$ [1]. Also, in some simple models of quantum gravity, when the spacetime interval is averaged over the metric fluctuations, it is found to be bounded from below at $O(L_P^2)$ [2]. These results suggest that one can think of the Planck length L_P as a 'zero-point length' of spacetime. But how can such a fundamental length scale be introduced into quantum field theory in an invariant manner?

The existence of a fundamental length implies that processes involving energies higher than Planck energies will be suppressed and the ultra-violet behavior of the theory will be improved. One direct consequence of such an improved behavior will be that the Feynman propagator will acquire a damping factor for energies higher than the Planck energy. The Feynman propagator $G_{\rm F}(x,x'|g_{\mu\nu})$ in a given background metric $g_{\mu\nu}$ can be expressed in Euclidean space as

$$G_{\rm F}(x,x'|g_{\mu\nu})\equiv\sum_{\rm paths}\exp -[m\,\mathcal{R}(x,x'|g_{\mu\nu})]\,, \qquad (1)$$

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where $\mathcal{R}(x, x'|g_{\mu\nu})$ is the proper length of a path connecting two events x and x'. The existence of a 'zero-point length' L_P will correspond to the fact that paths with a proper length $\mathcal{R} \ll L_P$ will be suppressed in the path integral. In two recent papers [3,4], Padmanabhan has been able to capture exactly such a feature by assuming the weightage factor for a path of proper length $\mathcal{R}$ to be invariant under the duality transformation $\mathcal{R} \to \left(L_P^2/\mathcal{R}\right)$. On invoking such a 'principle of path integral duality', the Feynman propagator above takes the following form:

$$G_{\rm F}(x, x'|g_{\mu\nu}) = \sum_{\rm paths} \exp - \left[m\, \mathcal{R}(x, x'|g_{\mu\nu}) + \left(L_P^2/\mathcal{R}(x, x'|g_{\mu\nu})\right)\right]. \tag{2}$$

(It can so happen that the fundamental length scale is not actually L_P but (ηL_P), where η is a factor of order unity. Therefore, in this paper, when we say that the fundmental length is L_P, we actually mean that it is of $O(L_P)$.) It turns out that the net effect of such a modification is to add the quantity $\left(-4L_P^2\right)$ to the Lorentzian proper distance $(\Delta x)^2$ between two spacetime events x and x' in the original propagator. In other words, the postulate of the path integral duality proves to be equivalent to introducing a 'zero-point length' of $O(L_P)$ in spacetime.

Two points needs to be stressed regarding the 'principle of path integral duality'. Firstly, this duality principle is able to introduce the fundamental length scale L_P into quantum field theory in an invariant manner. Secondly, it is able to provide an ultra-violet regulator at the Planck scale thereby rendering the theory finite. These two features of the duality principle can be illustrated with the example of the Feynman propagator in flat spacetime. In flat spacetime, the Feynman propagator $G_{\rm F}(x, x'|\eta_{\mu\nu})$ can be expressed as a function of the Lorentz invariant spacetime interval $(x-x')^2$. (The metric signature we shall use in this paper is such that $(x-x')^2 > 0$ for timelike events x and x'.) As we have mentioned in the last paragraph, invoking the 'principle of path integral duality' corresponds to modifying the spacetime interval $(x-x')^2$ to $\left[(x-x')^2 - 4L_P^2\right]$ in the Feynman propagator [3,4]. Since $\left[(x-x')^2 - 4L_P^2\right]$ is again a Lorentz invariant quantity, the modified Feynman propagator proves to be Lorentz invariant. Also, in the limit of $x \to x'$ the original Feynman propagator has an ultra-violet divergence whereas the modified propagator stays finite. Thus, the 'principle of path integral duality' is not only able to introduce the fundamental length scale L_P into quantum field theory thereby rendering it finite, it is able to do so in an invariant manner.

In Schwinger's proper time formalism, the Euclidean space Feynman propagator described by Eq. (1) above can be expressed as follows [5]:

$$G_{\rm F}(x, x'|g_{\mu\nu}) = \int_0^\infty ds\, e^{-m^2 s}\, K(x, x'; s|g_{\mu\nu}), \tag{3}$$

where $K(x, x'; s|g_{\mu\nu})$ is the probability amplitude for a particle to propagate from x to x' in a proper time interval s in a given background spacetime described by the metric tensor $g_{\mu\nu}$. The path integral kernel $K(x, x'; s|g_{\mu\nu})$ is given by

$$K(x, x'; s|g_{\mu\nu}) \equiv \int \mathcal{D}x\ \exp - \left(\frac{m}{4}\int_0^s ds\, g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu\right). \tag{4}$$

In his paper [4], Padmanabhan has shown that, in Schwinger's proper time formalism, invoking the 'principle of path integral duality' corresponds to modifying the weightage given to a path of proper time s from $\exp -(m^2 s)$ to $\exp - \left[m^2 s + (L_P^2/s)\right]$. That is, the modified Feynman propagator as defined in Eq. (2) above is given by the following integral:

$$G_{\rm F}^{\rm P}(x, x'|g_{\mu\nu}) = \int_0^\infty ds\, e^{-m^2 s}\, e^{-(L_P^2 K/s)}\ K(x, x'; s|g_{\mu\nu}), \tag{5}$$

where $K(x, x'; s|g_{\mu\nu})$ is the same as in Eq. (4).

Consider a D-dimensional spacetime described by the metric tensor $g_{\mu\nu}$. Let us assume that a quantized scalar field Φ of mass m satisfies the following equation of motion:

$$(\hat{H} + m^2)\Phi = 0, \tag{6}$$

where $\hat{H}$ is a differential operator in the D-dimensional spacetime. Then, in Lorentzian space, an effective Lagrangian corresponding to the operator $\hat{H}$ can be defined as follows [5]:

$$\mathcal{L}_{\rm corr} = -\frac{i}{2}\int_0^\infty \frac{ds}{s}\, e^{-im^2 s}\ K(x, x; s|g_{\mu\nu}), \tag{7}$$

where

$$K(x,x;s|g_{\mu\nu}) \equiv \langle x|e^{-i\hat{H}s}|x\rangle. \tag{8}$$

(If the quantum scalar field interacts either with itself or with an external classical field then $\mathcal{L}_{\rm corr}$ will prove to be the correction to the Lagrangian describing the classical background.) The quantity $K(x,x;s|g_{\mu\nu})$ is the path integral kernel (in the coincidence limit) of a quantum mechanical system described by time evolution operator $\hat{H}$. The integration variable s acts as the time parameter for the quantum mechanical system.

We saw above that, in Euclidean space, invoking the 'principle of path integral duality' corresponds to modifying the weightage factor $\exp -(m^2 s)$ to $\exp -\left[m^2 s + (L_P^2/s)\right]$. In Lorentzian space, this modified weightage factor is given by: $\exp -\left[im^2 s - i(L_P^2/s)\right]$. Therefore, invoking the duality principle corresponds to modifying the effective Lagrangian $\mathcal{L}_{\rm corr}$ given by Eq. (7) above to the following form:

$$\mathcal{L}^{\rm P}_{\rm corr} = -\frac{i}{2}\int_0^\infty \frac{ds}{s}\; e^{-im^2 s}\; e^{iL_P^2/s}\; K(x,x;s|g_{\mu\nu}), \tag{9}$$

where $K(x,x;s|g_{\mu\nu})$ is still given by Eq. (8). Thus, we have been able to introduce the fundamental length scale L_P into standard quantum field theory. Using this modified effective Lagrangian, we can evaluate the quantum gravitational corrections to standard quantum field theoretic results in flat and curved spacetimes.

The following remarks are in order. When we take into account the quantum nature of gravity, the metric fluctuates. Therefore, evaluating the quantum gravitational corrections to the quantum field theoretic results would mean averaging over these metric fluctuations. In some simple models of quantum gravity, it has been shown that averaging the spacetime interval $(\Delta x)^2$ over the metric fluctuations corresponds to modifying this spacetime interval to $\left[(\Delta x)^2 - L_P^2\right]$ [2]. As we have pointed out earlier, the 'principle of path integral duality' is able to achieve exactly such a feature. Also, in the last paragraph, we saw that invoking the duality principle corresponds to simply modifying the weightage factor in the effective Lagrangian. Therefore, we say that the effective Lagrangian with the modified weightage factor as given by Eq. (9) above contains the quantum gravitational corrections.

This paper is organized as follows. In the following six sections, we evaluate the quantum gravitational corrections to some of the quantum field theoretic results in flat and curved spacetimes. In sections II–V, we evaluate the corrections to: (1) the Casimir effect, (2) the effective potential for a self-interacting scalar field theory, (3) the effective Lagrangian for a constant electromagnetic background and (4) the thermal effects in the Rindler coordinates, in flat spacetime. In sections VI and VII, we evaluate the quantum gravitational corrections to: (1) the gravitational Lagrangian and (2) the trace anomaly in an arbitrary curved spacetime. In all these sections, we shall first present the conventional result and then go on to evaluate the corrections with the modified weightage factor. Finally, in section VIII, we discuss the wider implications of our analysis.

II. CORRECTIONS TO THE CASIMIR EFFECT

The presence of a pair of conducting plates alters the vacuum structure of the quantum field and as a result there arises a non-zero force of attraction between the two conducting plates. This effect is called the Casimir effect. In the conventional derivation of the Casimir effect, the difference between the energy in the Minkowski vacuum and the Casimir vacuum is evaluated and a derivative of this energy difference gives the force of attraction between the Casimir plates (see, for e.g., Ref. [6], pp. 138–141). To evaluate the Casimir force in such a fashion we need to know the normal modes of the quantum field with and without the plates. Therefore, if we are to evaluate the quantum gravitational corrections to the Casimir effect by the method described above, then we need to know as to how metric fluctuations will modify the modes of the quantum field. But, we only know as to how the quantum gravitational corrections can be introduced in the effective Lagrangian. Therefore, in this section, we shall first present a derivation of the Casimir effect from the effective Lagrangian approach and then go on to evaluate the quantum gravitational corrections to this effect with the modified weightage factor. The system we shall consider in this section is a massless scalar field in flat spacetime. Also, we shall evaluate the effective Lagrangian for two cases: (i) with vanishing boundary conditions on a pair of parallel plates and (ii) with periodic boundary conditions on the quantum field.

A. Vanishing boundary conditions on parallel plates

1. Conventional result

Consider a pair of plates situated at $z = 0$ and $z = a$. Let us assume that the scalar field Φ vanishes on these plates. For such a case, the operator $\hat{H}$ corresponds to that of a free particle with the condition that its eigen functions along the z-direction vanish at $z = 0$ and $z = a$. Along the other $(D-1)$ perpendicular directions the operator $\hat{H}$ corresponds to that of a free particle without any boundary conditions. The complete quantum mechanical kernel can then be written as

$$K(x, x'; s) = K_z(z, z'; s) \times K_\perp(x_\perp, x'_\perp; s). \tag{10}$$

(We shall refer to the flat spacetime kernel $K(x, x'; s|\eta_{\mu\nu})$ simply as $K(x, x'; s)$.) In the limit $x_\perp \to x'_\perp$, $K_\perp(x_\perp, x'_\perp; s)$ is given by

$$K_\perp(x_\perp, x_\perp; s) = \left(\frac{i}{(4\pi i s)^{(D-1)/2}}\right). \tag{11}$$

The quantum mechanical kernel $K_z(z, z'; s)$ along the z-direction corresponds to that of a particle of mass $(1/2)$ in an infinite square well potential with walls at $z = 0$ and $z = a$. For such a case, the normalized eigen functions for an energy eigen value $E = (n\pi^2/a^2)$ are then given by (see, for e.g., Ref. [7], p. 65)

$$\Psi_E(z) = \sqrt{\frac{2}{a}}\ \sin(n\pi z/a), \quad \text{where} \quad n = 1, 2, 3 \ldots. \tag{12}$$

The corresponding kernel can then be written using the Feynman-Kac formula as follows (see, for instance, Ref. [8], p. 88):

$$\begin{aligned} K(z, z'; s) &= \sum_E \Psi_E(z)\ \Psi_E^*(z')\ e^{-iEs} \\ &= \left(\frac{2}{a}\right) \sum_{n=1}^{\infty} \sin(n\pi z/a)\ \sin(n\pi z'/a)\ e^{-(in^2\pi^2 s/a^2)} \\ &= \left(\frac{1}{2a}\right) \sum_{n=-\infty}^{\infty} \left\{e^{in\pi(z-z')/a} - e^{in\pi(z+z')/a}\right\}\ e^{-(in^2\pi^2 s/a^2)}. \end{aligned} \tag{13}$$

Using the Poisson sum formula (see, for instance, Ref. [9], Part I, p. 483) this kernel can be rewritten as (see, for e.g., Ref. [10], p. 46)

$$\begin{aligned} &K_z(z, z'; s) \\ &\quad = \left(\frac{1}{(4\pi i s)^{1/2}}\right) \sum_{n=-\infty}^{\infty} \left\{\exp\left[i(z - z' + 2na)^2/4s\right] - \exp\left[i(z + z' + 2na)^2/4s\right]\right\} \end{aligned} \tag{14}$$

so that in the coincidence limit (i.e. as $z = z'$), this kernel reduces to

$$\begin{aligned} K_z(z, z; s) &= \left(\frac{1}{(4\pi i s)^{1/2}}\right) \sum_{n=-\infty}^{\infty} \left\{\exp(in^2a^2/s) - \exp\left[i(z + na)^2/s\right]\right\} \\ &= \left(\frac{1}{(4\pi i s)^{1/2}}\right) \left\{1 + 2\sum_{n=1}^{\infty} e^{(in^2a^2/s)} - \sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}. \end{aligned} \tag{15}$$

Therefore, the complete kernel (in the coincidence limit) corresponding to the operator $\hat{H}$ with the conditions that its eigen functions vanish at $z = 0$ and $z = a$ is given by

$$\begin{aligned} K(x, x; s) &= K_z(z, z; s) \times K_\perp(x_\perp, x_\perp; s) \\ &= \left(\frac{i}{(4\pi i s)^{D/2}}\right) \left\{1 + 2\sum_{n=1}^{\infty} e^{(in^2a^2/s)} - \sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}. \end{aligned} \tag{16}$$

Substituting this kernel in Eq. (7) and setting $m = 0$, we obtain that

$$\mathcal{L}_{\rm corr} = \left(\frac{1}{2(4\pi i)^{D/2}}\right) \int_0^\infty \frac{ds}{s^{(D/2)+1}} \left\{1 + 2\sum_{n=1}^{\infty} e^{in^2a^2/s} - \sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}. \tag{17}$$

In flat spacetime, in the absence of any boundary conditions on the quantum scalar field Φ, the quantum mechanical kernel for the operator $\hat{H}$ corresponds to that of a free particle. The kernel for such a case is given by

$$K^0(x,x;s) = \left(\frac{i}{(4\pi i s)^{D/2}}\right). \tag{18}$$

That is there exists a non-zero $\mathcal{L}_{\rm corr}$ even for a free field in flat spacetime. Therefore, the flat spacetime contribution as given by

$$\mathcal{L}^0_{\rm corr} = \left(\frac{1}{2(4\pi i)^{D/2}}\right)\int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{-im^2 s} \tag{19}$$

has to be subtracted from all $\mathcal{L}_{\rm corr}$.

On subtracting the quantity $\mathcal{L}^0_{\rm corr}$ (given by Eq. (19) above with m set to zero) from the expression for $\mathcal{L}_{\rm corr}$, we obtain that

$$\begin{aligned}
\bar{\mathcal{L}}_{\rm corr} &= \left(\mathcal{L}_{\rm corr} - \mathcal{L}^0_{\rm corr}\right)\\
&= \left(\frac{1}{2(4\pi i)^{D/2}}\right)\int_0^\infty \frac{ds}{s^{(D/2)+1}}\left\{2\sum_{n=1}^{\infty} e^{(in^2a^2/s)} - \sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}\\
&= \left(\frac{1}{2(4\pi i)^{D/2}}\right)\left\{2\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{(in^2a^2/s)}\right.\\
&\qquad\qquad \left. - \sum_{n=-\infty}^{\infty}\int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{i(z+na)^2/s}\right\}.
\end{aligned} \tag{20}$$

The integrals in the above result can be expressed in terms of the Gamma function (see, for e.g., Ref. [11], p. 934). Therefore

$$\begin{aligned}
\bar{\mathcal{L}}_{\rm corr} &= \left(\frac{\Gamma(D/2)}{2(4\pi)^{D/2}}\right)\left\{2\sum_{n=1}^{\infty}(n^2a^2)^{-D/2} - \sum_{n=-\infty}^{\infty}(z+na)^{2.(-D/2)}\right\}\\
&= \left(\frac{\Gamma(D/2)}{2(4\pi a^2)^{D/2}}\right)\left\{2\sum_{n=1}^{\infty} n^{-D} - \sum_{n=-\infty}^{\infty}\left[n+(z/a)\right]^{-D}\right\}.
\end{aligned} \tag{21}$$

Now, consider the case $D=4$. The first series in the above expression for $\bar{\mathcal{L}}_{\rm corr}$ can be expressed in terms of the Riemann zeta function (see, for e.g., Ref. [12], p. 334) and, the second series can be summed using the following relation (cf. Ref. [13], Vol. I, p. 652):

$$\sum_{n=-\infty}^{\infty}(k+\alpha)^{-p} = \left(\frac{\pi(-1)^{p-1}}{(p-1)!}\right)\frac{d^{p-1}}{d\alpha^{p-1}}\left[\cot(\pi\alpha)\right], \tag{22}$$

with the result

$$\bar{\mathcal{L}}_{\rm corr} = \left(\frac{\pi^2}{1440a^4}\right) - \left(\frac{\pi^2}{96a^4\sin^4(\pi z/a)}\right)\left(3 - 2\sin^2(\pi z/a)\right). \tag{23}$$

Let us now consider the case $D=2$. For such a case, from Eqs. (21) and (22), we obtain that

$$\bar{\mathcal{L}}_{\rm corr} = \left(\frac{\pi}{24a^2}\right)\left(1 - 3{\rm cosec}^2(\pi z/a)\right). \tag{24}$$

Note that there for both the cases $D=4$ as well as $D=2$, there arises a term that is dependent on z. (Compare these results with Eqs. (5.11) and (5.15) that appear in Ref. [14], p. 105.)

2. Results with the modified weightage factor

Let us now evaluate the effective Lagrangian with the modified weightage factor. Substituting the kernel (16) in Eq. (9) and setting $m = 0$, we obtain that

$$\mathcal{L}_{\rm corr}^{\rm P} = \left(\frac{1}{2(4\pi i)^{D/2}}\right) \int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{iL_P^2/s} \left\{1 + 2\sum_{n=1}^{\infty} e^{in^2a^2/s} - \sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}. \tag{25}$$

Just as in the case of $\mathcal{L}_{\rm corr}$, there exists a non-zero $\mathcal{L}_{\rm corr}^{\rm P}$ even for a free field in flat spacetime. This flat spacetime contribution has to be subtracted from all $\mathcal{L}_{\rm corr}^{\rm P}$. It is given by

$$\mathcal{L}_{\rm corr}^{\rm P0} = \left(\frac{1}{2(4\pi i)^{D/2}}\right) \int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{-im^2 s}\, e^{iL_P^2/s}. \tag{26}$$

On subtracting this quantity (with m set to zero) from the expression for $\mathcal{L}_{\rm corr}^{\rm P}$, we get

$$\begin{aligned}
\bar{\mathcal{L}}_{\rm corr}^{\rm P} &= \left(\mathcal{L}_{\rm corr}^{\rm P} - \mathcal{L}_{\rm corr}^{\rm P0}\right) \\
&= \left(\frac{1}{2(4\pi i)^{D/2}}\right) \int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{iL_P^2/s} \left\{2\sum_{n=1}^{\infty} e^{in^2a^2/s} - \sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\} \\
&= \left(\frac{1}{2(4\pi i)^{D/2}}\right) \left\{2\sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{i\left[n^2a^2+L_P^2\right]/s}\right. \\
&\qquad \left. - \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{i\left[(z+na)^2+L_P^2\right]/s}\right\}.
\end{aligned} \tag{27}$$

As before, the integrals can be expressed in terms of the Gamma function with the result

$$\begin{aligned}
\bar{\mathcal{L}}_{\rm corr}^{\rm P} &= \left(\frac{\Gamma(D/2)}{2(4\pi)^{D/2}}\right) \left\{2\sum_{n=1}^{\infty} \left[n^2a^2 + L_P^2\right]^{-D/2} - \sum_{n=-\infty}^{\infty} \left[(x+na)^2 + L_P^2\right]^{-D/2}\right\} \\
&= \left(\frac{\Gamma(D/2)}{2(4\pi a^2)^{D/2}}\right) \left\{2\sum_{n=1}^{\infty} \left[n^2 + (L_P^2/a^2)\right]^{-D/2}\right. \\
&\qquad \left. - \sum_{n=-\infty}^{\infty} \left[(n+(z/a))^2 + (L_P^2/a^2)\right]^{-D/2}\right\}.
\end{aligned} \tag{28}$$

The series in the above result cannot be written in closed form for the case $D = 4$. Consider the case $D = 2$. For such a case the series in $\bar{\mathcal{L}}_{\rm corr}^{\rm P}$ above can be expressed in a closed form. Making use of the following two relations (cf. Ref. [13], Vol. 1, p. 685):

$$\sum_{n=0}^{\infty} (n^2 + \alpha^2)^{-1} = \left(\frac{1}{2\alpha^2}\right) + \left(\frac{\pi}{2\alpha}\right) \coth(\pi\alpha) \tag{29}$$

and

$$\sum_{n=-\infty}^{\infty} \left[(n+\alpha)^2 + \beta^2\right]^{-1} = \left(\frac{\pi}{\beta}\right)\, \sinh(2\pi\beta)\left[\cosh(2\pi\beta) - \cos(2\pi\alpha)\right]^{-1}, \tag{30}$$

we find that $\bar{\mathcal{L}}_{\rm corr}^{\rm P}$ can be expressed as follows:

$$\begin{aligned}
\bar{\mathcal{L}}_{\rm corr}^{\rm P} = \Bigg\{ &-\frac{1}{8\pi L_P^2} + \left(\frac{1}{8aL_P}\right) \coth(\pi L_P/a) \\
&+ \left(\frac{1}{8aL_P}\right) \sinh(2\pi L_P/a)\left[\cos(2\pi z/a) - \cosh(2\pi L_P/a)\right]^{-1} \Bigg\}.
\end{aligned} \tag{31}$$

In the limit of $L_P \to 0$, we find that

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} \to \left\{ \left(\frac{\pi}{24a^2}\right) \left(1 - 3{\rm cosec}^2(\pi z/a)\right) \right.$$
$$\left. - L_P^2 \left(\frac{\pi^3}{360a^4}\right) \left(1 + 30{\rm cosec}^2(\pi z/a) - 45{\rm cosec}^4(\pi z/a)\right) \right\}. \tag{32}$$

That is, the lowest order quantum gravitational corrections appear at $O\left(L_P^2/a^2\right)$.

B. Periodic boundary conditions

1. Conventional result

Let us now consider the case wherein we impose periodic boundary conditions on the quantum field along the z-direction. That is, let us assume that the scalar field Φ takes on the same value at the points z and $(z+a)$. For such a case, the operator $\hat{H}$ corresponds to that of a free particle with the condition that its eigen functions along the z-direction take on the same value at z and $(z+a)$. Along the other $(D-1)$ perpendicular directions the operator $\hat{H}$ corresponds to that of a free particle without any boundary conditions. The complete kernel can then be written as in Eq. (10). where $K_\perp(x_\perp, x_\perp; s)$ is given by Eq. (11).

The normalized eigen functions of the operator $\hat{H}$ along the z-direction corresponding to an energy eigen value $E = (4n^2\pi^2/a^2)$ are then given by

$$\Psi_E(z) = \sqrt{\frac{1}{a}}\, e^{(2in\pi z/a)} \quad \text{where} \quad n = 0, \pm 1, \pm 2, \ldots. \tag{33}$$

The corresponding kernel can then be written using the Feynman-Kac formula as follows (see, for instance, Ref. [8], p. 88):

$$K(z, z'; s) = \sum_E \Psi_E(z)\, \Psi_E^*(z')\, e^{-iEs} = \left(\frac{1}{a}\right) \sum_{n=-\infty}^{\infty} \exp\left[2in\pi(z - z')/a\right]\, e^{-(4in^2\pi^2 s/a^2)}, \tag{34}$$

which in the limit of $z \to z'$ reduces to

$$K(z, z; s) = \left(\frac{1}{a}\right) \sum_{n=-\infty}^{\infty} e^{-(4in^2\pi^2 s/a^2)}. \tag{35}$$

Therefore, the complete kernel in D dimensions is given by

$$K(x, x; s) = \left(\frac{i}{(4\pi i s)^{(D-1)/2}}\right) \left(\frac{1}{a}\right) \sum_{n=-\infty}^{\infty} e^{-4in^2\pi^2 s/a^2}. \tag{36}$$

Using the Poisson sum formula (see, for e.g., Ref. [9], Part 1, p. 483) the above sum can be rewritten as

$$K(x, x; s) = \left(\frac{i}{(4\pi i s)^{D/2}}\right) \sum_{n=-\infty}^{\infty} e^{in^2a^2/4s} = \left(\frac{i}{(4\pi i s)^{D/2}}\right) \left\{1 + 2\sum_{n=1}^{\infty} e^{in^2a^2/4s}\right\}. \tag{37}$$

Substituting this expression in Eq. (7) and setting $m = 0$, we obtain that

$$\mathcal{L}_{\rm corr} = \left(\frac{1}{2(4\pi i)^{D/2}}\right) \int_0^\infty \frac{ds}{s^{D/2+1}} \left\{1 + 2\sum_{n=1}^{\infty} e^{in^2a^2/4s}\right\}. \tag{38}$$

Now, subtracting the quantity $\mathcal{L}^0_{\rm corr}$ (given by Eq. (19) with $m = 0$) from the above expression, we get

$$\bar{\mathcal{L}}_{\rm corr} = \left(\mathcal{L}_{\rm corr} - \mathcal{L}^0_{\rm corr}\right) = \left(\frac{1}{(4\pi i)^{D/2}}\right) \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s^{D/2+1}}\, e^{in^2a^2/4s}. \tag{39}$$

The integral over s can be expressed in terms of the Gamma function (see, for instance, Ref. [11], p. 934), so that

$$\bar{\mathcal{L}}_{\rm corr} = \left(\frac{\Gamma(D/2)}{(\pi a^2)^{D/2}}\right)\sum_{n=1}^{\infty} n^{-D} = \left(\frac{\Gamma(D/2)}{(\pi a^2)^{D/2}}\right)\ \zeta(D), \tag{40}$$

where $\zeta(D)$ is the Riemann zeta-function (see, for e.g., Ref. [12], p. 334).

Now, consider the case $D = 4$. For such a case, we find that

$$\bar{\mathcal{L}}_{\rm corr} = \left(\frac{\Gamma(2)}{(\pi a^2)^2}\right)\ \zeta(4) = \left(\frac{\pi^2}{90\, a^4}\right). \tag{41}$$

Also, for the case $D = 2$, $\bar{\mathcal{L}}_{\rm corr}$ is given by

$$\bar{\mathcal{L}}_{\rm corr} = \left(\frac{\Gamma(1)}{(\pi a^2)}\right)\ \zeta(2) = \left(\frac{\pi}{6a^2}\right). \tag{42}$$

2. Results with the modified weightage factor

Let us now evaluate the effective Lagrangian with the modified weightage factor. Substituting the kernel (37) in (9) and setting $m = 0$, we obtain that

$$\mathcal{L}^{\rm P}_{\rm corr} = \left(\frac{1}{2(4\pi i)^{D/2}}\right)\int_0^{\infty}\frac{ds}{s^{D/2+1}}\ e^{iL_P^2/s}\ \left\{1 + 2\sum_{n=1}^{\infty} e^{in^2a^2/4s}\right\}. \tag{43}$$

On subtracting the quantity $\mathcal{L}^{\rm P0}_{\rm corr}$ (given by Eq. (26) with m set to zero) from this expression, we get

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} = \left(\mathcal{L}^{\rm P}_{\rm corr} - \mathcal{L}^{\rm PO}_{\rm corr}\right) = \left(\frac{1}{(4\pi i)^{D/2}}\right)\sum_{n=1}^{\infty}\int_0^{\infty}\frac{ds}{s^{D/2+1}}\ e^{i(4L_P^2+n^2a^2)/4s}. \tag{44}$$

As before, the integral over s can be expressed in terms of the Gamma function with the result

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} = \left(\frac{\Gamma(D/2)}{(\pi a^2)^{D/2}}\right)\sum_{n=1}^{\infty}\left(n^2 + 4L_P^2/a^2\right)^{-D/2}. \tag{45}$$

Now, consider the case $D = 4$. For such a case

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} = \left(\frac{1}{\pi^2 a^4}\right)\sum_{n=1}^{\infty}\left(n^2 + 4L_P^2/a^2\right)^{-2}. \tag{46}$$

Using the following relation (cf. Ref. [13], Vol. I, p. 687):

$$\begin{aligned}\sum_{n=0}^{\infty}\left(n^2+\alpha^2\right)^{-2} &= \left(\frac{1}{\alpha^4}\right) + \sum_{n=1}^{\infty}(n^2+\alpha^2)^{-2}\\ &= \left(\frac{1}{2\alpha^4}\right) + \left(\frac{\pi}{4\alpha^3}\right)\ \coth(\pi\alpha) + \left(\frac{\pi^2}{4\alpha^2}\right)\ {\rm cosech}^2(\pi\alpha),\end{aligned} \tag{47}$$

we can express $\bar{\mathcal{L}}^{P}_{\rm corr}$ in a closed form as follows:

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} = \left\{-\left(\frac{1}{32\pi^2 L_P^4}\right) + \left(\frac{1}{32\pi a L_P^3}\right)\ \coth(2\pi L_P/a) + \left(\frac{1}{16a^2L_P^2}\right)\ {\rm cosech}^2(2\pi L_P/a)\right\}. \tag{48}$$

For the case $D = 2$, $\mathcal{L}^{\rm P}_{\rm corr}$ is given by

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} = \left(\frac{1}{\pi a^2}\right)\sum_{n=1}^{\infty}\left(n^2 + 4L_P^2/a^2\right)^{-1}. \tag{49}$$

Using the following relation (cf. Ref. [13]. Vol. I, p. 685)

$$\sum_{n=0}^{\infty}\left(n^2+\alpha^2\right)^{-1} = \left(\frac{1}{\alpha^2}\right) + \sum_{n=1}^{\infty}(n^2+\alpha^2)^{-1} = \left(\frac{1}{2\alpha^2}\right) + \left(\frac{\pi}{2\alpha}\right)\ \coth(\pi\alpha), \tag{50}$$

we can express $\bar{\mathcal{L}}^{\rm P}_{\rm corr}$ in a closed form as follows:

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} = \left\{-\left(\frac{1}{8\pi L_P^2}\right) + \left(\frac{1}{4aL_P}\right)\ \coth(2\pi L_P/a)\right\}. \tag{51}$$

Making use of the following series expansions (cf. Ref. [11], p. 36)

$$\coth(\pi x) = \left(\frac{1}{\pi x}\right) + \left(\frac{2x}{\pi}\right)\sum_{n=1}^{\infty}(x^2+n^2)^{-1} \tag{52}$$

and

$$\mathrm{cosech}^2(\pi x) = \left(\frac{1}{\pi^2 x^2}\right) + \left(\frac{2}{\pi^2}\right)\sum_{n-1}^{\infty}\left\{\frac{x^2-n^2}{(x^2+n^2)^2}\right\}, \tag{53}$$

we find that, in the limit of $L_P \to 0$, $\bar{\mathcal{L}}^{\rm P}_{\rm corr}$ reduces to

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} \to \left\{\left(\frac{\pi^2}{90a^4}\right) - L_P^2\left(\frac{8\pi^4}{945a^6}\right)\right\} \tag{54}$$

for the case $D = 4$ while it reduces to

$$\bar{\mathcal{L}}^{\rm P}_{\rm corr} \to \left\{\left(\frac{\pi}{6a^2}\right) - L_P^2\left(\frac{2\pi^3}{45a^4}\right)\right\} \tag{55}$$

for the case $D = 2$. The lowest order quantum gravitational corrections to the conventional results appear at $O\left(L_P^2/a^2\right)$.

III. THE EFFECTIVE POTENTIAL OF A SELF-INTERACTING SCALAR FIELD THEORY

The system considered in this section is a massive, self-interacting scalar field in 4-dimensions (i.e. D=4) described by the action

$$\begin{aligned}\mathcal{S}[\Phi] = \int d^4x\ \mathcal{L}(\Phi) &= \int d^4x\left\{\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi - \mathcal{V}(\Phi)\right\}\\ &= \int d^4x\left\{\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi - \frac{1}{2}m^2\Phi^2 - \mathcal{V}_{int}(\Phi)\right\}\end{aligned} \tag{56}$$

where m is the mass and $\mathcal{V}_{int}(\Phi)$ represents the self-interaction of the scalar field. We are interested in studying the effects of small quantum fluctuations present in the system around some classical solution Φ_c. The classical solution will be assumed to be a constant or adiabatically varying so that its derivatives can be ignored. The effect of these fluctuations will be studied by expanding the action $\mathcal{S}[\Phi]$ about the classical solution Φ_c and integrating over these fluctuations to obtain an effective potential. This effective potential will contain corrections to the original potential $\mathcal{V}(\Phi_c)$.

Let $\Phi = (\Phi_c + \phi)$, where Φ_c is the classical solution. The above action, when Taylor expanded about the classical solution, yields,

$$\begin{aligned}\mathcal{S}[\Phi] &\simeq S[\Phi]|_{\Phi=\Phi_c} + \left(\frac{\delta S[\Phi]}{\delta\Phi}\right)_{\Phi=\Phi_c}\phi + \frac{1}{2}\ \left(\frac{\delta^2 S[\Phi]}{\delta\Phi^2}\right)_{\Phi=\Phi_c}\phi^2\\ &= \int d^4x\Bigg\{\left(\frac{1}{2}\partial^\mu\Phi_c\partial_\mu\Phi_c - \frac{1}{2}m^2\Phi_c^2 - \mathcal{V}_{int}(\Phi_c)\right)\\ &\qquad + \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\mathcal{V}''_{int}(\Phi_c)\,\phi^2\right)\Bigg\},\end{aligned} \tag{57}$$

where

$$\mathcal{V}''_{int}(\Phi_c) \equiv \left(\frac{\partial^2 \mathcal{V}_{int}(\Phi)}{\partial \Phi^2}\right)_{\Phi=\Phi_c} \tag{58}$$

and we have set $(\delta S[\Phi]/\delta\Phi)_{\Phi=\Phi_c} = 0$, since Φ_c is the classical solution. The correction to the classical action $\mathcal{S}[\Phi_c]$ is obtained by integrating the degrees of freedom corresponding to the variable ϕ.

Varying the action (57) with respect to ϕ leads to the following equation of motion:

$$\left(\partial_\mu \partial^\mu + m^2 + \mathcal{V}''_{int}(\Phi_c)\right)\phi = 0. \tag{59}$$

From this equation the operator $\hat{H}$ can be easily identified to be

$$\hat{H} \equiv \left(\partial_\mu \partial^\mu + \mathcal{V}''_{int}(\Phi_c)\right). \tag{60}$$

Since $\mathcal{V}''_{int}(\Phi_c)$ is a constant, the kernel corresponding to the operator $\hat{H}$ above can be immediately written down as

$$K(x, x; s) = \left(\frac{1}{16\pi^2 i s^2}\right) \exp - \left[i\mathcal{V}''_{int}(\Phi_c)s\right]. \tag{61}$$

Using the above Kernel, the effective potential will be evaluated by the conventional standard approach first and then by using the path integral duality principle.

A. Conventional result

In the conventional approach, we substitute the kernel given in Eqn. (61) into Eqn. (7) to obtain,

$$\mathcal{L}_{\rm corr} = -\left(\frac{1}{32\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, e^{-i\alpha s} \qquad \text{where} \qquad \alpha = \left(m^2 + \mathcal{V}''_{int}(\Phi_c)\right). \tag{62}$$

The above integral is quadratically divergent near the origin. In the conventional approach, this integral is evaluated (after performing a Euclidean rotation first) with the lower limit set to a small cutoff value Λ with the limit $\Lambda \to 0$ considered subsequently. Only the leading terms that dominate in this limit need to be retained. Therefore, by performing repeated partial integrations, we get,

$$\mathcal{L}_{\rm corr} = \left(\frac{1}{64\pi^2}\right)\left\{\left[\frac{1}{\Lambda^2} - \frac{\alpha}{\Lambda} - \alpha^2 \ln(\Lambda\mu)\right] - \alpha^2 \ln\left(\frac{\alpha}{\mu}\right) - \gamma\alpha^2\right\} \tag{63}$$

where μ is an arbitrary finite parameter introduced to keep the argument of the logarithms dimensionless and γ is the Euler-Mascheroni constant. The last two terms within the curly brackets, namely, $\alpha^2 \ln(\alpha/\mu)$ and $\gamma\alpha^2$, do not depend on Λ and hence are finite in the limit $\Lambda \to 0$. The term inside the square brackets however diverges. There are linear, quadratic and logarithmically divergent terms. The quadratically divergent term is independent of α and being just an infinite constant can be dropped while the other two divergent terms depend on α and thus cannot be ignored.

For an arbitrary $\mathcal{V}_{int}$, no sense can be made out of the above expression for $\mathcal{L}_{\rm corr}$. One can only confine ourselves to those $\mathcal{V}_{int}$ for which the divergent quantities $\Lambda^{-1}\alpha$ and $\alpha^2\ln(\Lambda\mu)$ have the same form as the original $\mathcal{V}(\Phi)$. If that is the case, the divergent terms can be absorbed into constants that determine the form of $\mathcal{V}(\Phi)$ and the theory can then be suitably reinterpreted. It is clear that for non-polynomial $\mathcal{V}(\Phi)$, the above criteria will not be satisfied. Therefore, assume that $\mathcal{V}(\Phi)$ is an nth-degree polynomial in Φ with constants $\lambda_1, \lambda_2, \ldots, \lambda_n$.

$$\mathcal{V}(\Phi) = \sum_{k=1}^{n} \lambda_k \Phi^k. \tag{64}$$

Then $\alpha = \mathcal{V}''$ will be a $(n-2)$ degree polynomial and α^2 will be a $2(n-2)$ polynomial. If the expression for $\mathcal{L}_{\rm corr}$ is not to have terms originally not present in $\mathcal{V}(\Phi)$, we must have

$$2(n-2) \leq n \tag{65}$$

implying $n \leq 4$. Therefore, only if $\mathcal{V}(\Phi)$ is a polynomial of quartic degree or less can the divergences be absorbed and the theory reinterpreted. As an example, consider the case,

$$\mathcal{V}_{int}(\Phi) = \frac{1}{4!}\lambda\Phi^4. \tag{66}$$

Then

$$\mathcal{V}''_{int}(\Phi_c) = \frac{1}{2}\lambda\Phi_c^2 \qquad \text{and} \qquad \alpha = \left(m^2 + \frac{1}{2}\lambda\Phi_c^2\right). \tag{67}$$

Carrying out the analysis, we get,

$$\begin{aligned}\mathcal{V}_{\rm eff} &= \mathcal{V}(\Phi_c) - \mathcal{L}_{\rm corr} \\ &= \frac{1}{2}m^2\Phi_c^2 + \frac{1}{4!}\lambda\Phi_c^4 - \mathcal{L}_{\rm corr} \\ &= \frac{1}{2}m_{\rm corr}^2\Phi_c^2 + \frac{1}{4!}\lambda_{\rm corr}\Phi_c^4 + \mathcal{V}_{\rm finite}\end{aligned} \tag{68}$$

where

$$\begin{aligned}m_{\rm corr}^2 &= m^2 + \frac{\lambda}{32\pi^2}\left(\frac{1}{2\Lambda} + m^2\ln(\Lambda\mu)\right) \\ \lambda_{\rm corr} &= \lambda + \frac{3\lambda^2}{32\pi^2}\ln(\Lambda\mu) \\ \mathcal{V}_{\rm finite} &= \frac{1}{64\pi^2}\left(m^2 + \frac{\lambda}{2}\Phi_c^2\right)^2\left\{\ln\left[\frac{1}{\mu}\left(m^2 + \frac{\lambda}{2}\Phi_c^2\right)\right] + \gamma\right\}\end{aligned} \tag{69}$$

The original potential $\mathcal{V}(\Phi)$ had two constants m and λ, which were the coefficients of Φ^2 and Φ^4. In $\mathcal{V}_{\rm eff}(\Phi_c)$ these are replaced by two other constants $m_{\rm corr}$ and $\lambda_{\rm corr}$ which are functions of m, λ and an arbitrary finite parameter μ. These constants also contain divergent terms involving Λ. Using renormalization group techniques, one can interpret $\mathcal{V}(\Phi)$ suitably. We will not discuss these techniques in this paper.

B. Results with the modified weightage factor

With the modified weightage factor, substituting the kernel given in Eq. (61) in Eq. (9), we obtain that

$$\mathcal{L}_{\rm corr}^{\rm P} = -\left(\frac{1}{32\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, e^{iL_P^2/s}\, e^{-i\alpha s} \qquad \text{where} \qquad \alpha = \left(m^2 + \mathcal{V}''_{int}(\Phi_c)\right). \tag{70}$$

The above integral can be easily performed and the resulting $\mathcal{L}_{\rm corr}^{\rm P}$ can be expressed in a closed form as follows (see, for instance, Ref. [11], p. 340):

$$\mathcal{L}_{\rm corr}^{\rm P} = \left(\frac{\alpha}{16\pi^2 {L_P}^2}\right) K_2\left(2L_P\sqrt{\alpha}\right), \tag{71}$$

where $K_2(2L_P\sqrt{\alpha})$ is the modified Bessel function. Since

$$\mathcal{L}_{\rm corr}^{\rm P0} = -\left(\frac{1}{32\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, e^{-im^2 s}\, e^{iL_P^2/s} = \left(\frac{m^2}{16\pi^2 L_P^2}\right) K_2\left(2L_P m\right), \tag{72}$$

on subtracting this quantity from $\mathcal{L}_{\rm corr}^{\rm P}$, we obtain that

$$\begin{aligned}\bar{\mathcal{L}}_{\rm corr}^{\rm P} &= \left(\mathcal{L}_{\rm corr}^{\rm P} - \mathcal{L}_{\rm corr}^{\rm P0}\right) \\ &= \left(\frac{1}{16\pi^2 L_P^2}\right)\left\{\alpha\, K_2\left(2L_P\sqrt{\alpha}\right) - m^2\, K_2\left(2L_P m\right)\right\}.\end{aligned} \tag{73}$$

Since

$$\mathcal{L}_{\rm eff}^{\rm P} = \left\{ \frac{1}{2}\partial^\mu \Phi_c \partial_\mu \Phi_c - \mathcal{V}_{\rm eff}^{\rm P} \right\}$$
$$= \left\{ \left(\frac{1}{2}\partial^\mu \Phi_c \partial_\mu \Phi_c - \frac{1}{2} m^2 \Phi_c^2 - \mathcal{V}_{int}(\Phi_c) \right) + \bar{\mathcal{L}}_{\rm corr}^{\rm P} \right\}, \tag{74}$$

we have

$$\mathcal{V}_{\rm eff}^{\rm P} = \left\{ \left(\frac{1}{2} m^2 \Phi_c^2 + \mathcal{V}_{int}(\Phi_c) \right) - \left(\frac{1}{16\pi^2 L_P^2} \right) \left(\alpha \; K_2\left(2L_P\sqrt{\alpha}\right) - m^2 \; K_2(2L_P m) \right) \right\}. \tag{75}$$

The expression for the effective potential given above is applicable for arbitrary $\mathcal{V}_{int}$. This is in contrast to the conventional approach where the divergences appearing in $\mathcal{L}_{\rm corr}$ forced the potential to be a polynomial of quartic degree or less. Further, there is no need to introduce an arbitrary parameter μ as was required in the conventional approach. The need for such a parameter arose because of the cutoff Λ that was introduced by hand in order to isolate the divergent terms appearing in $\mathcal{L}_{\rm corr}$. The introduction of a fundamental length L_P in the theory dispenses with such a need.

Specialize now to the case when

$$\mathcal{V}_{int}(\Phi) = \frac{1}{4!}\lambda \Phi^4. \tag{76}$$

Then

$$\mathcal{V}_{int}''(\Phi_c) = \frac{1}{2}\lambda \Phi_c^2 \qquad \text{and} \qquad \alpha = \left(m^2 + \frac{1}{2}\lambda \Phi_c^2 \right). \tag{77}$$

For such a case, the corrections to the parameters of the theory can be obtained from $\mathcal{V}_{\rm eff}^{\rm P}$ as follows:

$$\left(m_{\rm corr}^{\rm P}\right)^2 = \left(\frac{\partial^2 \mathcal{V}_{\rm eff}^{\rm P}}{\partial \Phi_c^2} \right)_{\Phi_c=0} = \left\{ m^2 - \left(\frac{\lambda}{16\pi^2 L_P^2} \right) \; K_2(2L_P m) - \left(\frac{m\lambda}{16\pi^2 L_P} \right) \; K_2^{'}(2L_P m) \right\} \tag{78}$$

and

$$\lambda_{\rm corr}^{\rm P} = \left(\frac{\partial^4 \mathcal{V}_{\rm eff}^{\rm P}}{\partial \Phi_c^4} \right)_{\Phi_c=0} = \left\{ \lambda - \left(\frac{9\lambda^2}{32\pi^2 m L_P} \right) \; K_2^{'}(2L_P m) - \left(\frac{3\lambda^2}{16\pi^2} \right) \; K_2^{''}(2L_P m) \right\}, \tag{79}$$

where ${K_2}'$ and $K_2^{''}$ denote first and second derivatives of the modified Bessel function K_2 with respect to the argument, respectively. In the limit $L_P \to 0$, we obtain,

$$\begin{aligned}
\left(m_{\rm corr}^{\rm P}\right)^2 &= m^2 + \frac{\lambda(2\gamma-1)}{32\pi^2} m^2 + \frac{\lambda}{32\pi^2}\left[\frac{1}{L_P^2} + m^2 \ln(L_P^2 m^2) \right] \\
&\quad + \frac{\lambda m^4 L_P^2}{128\pi^2}\left[\ln(L_P^2 m^2) + (4\gamma+3) \right] \\
\lambda_{\rm corr}^{\rm P} &= \lambda + \frac{3\gamma}{16\pi^2}\lambda^2 + \frac{3\lambda^2}{32\pi^2}\ln(L_P^2 m^2) \\
&\quad + \frac{3\lambda^2 m^2 L_P^2}{32\pi^2}\left[\ln(L_P^2 m^2) + 2(\gamma-1) \right] \\
\mathcal{V}_{\rm finite}^{P} &= \frac{1}{2} m^2 \left[\frac{\lambda}{32\pi^2}\ln\left(1 + \frac{\lambda}{2m^2}\Phi_c^2\right) - \frac{\lambda}{64\pi^2} \right] \Phi_c^2 \\
&\quad + \frac{1}{4!}\left[\frac{3\lambda^2}{32\pi^2}\ln\left(1+\frac{\lambda}{2m^2}\Phi_c^2\right) - \frac{9\lambda^2}{64\pi^2} - \frac{11\lambda^2 m^2}{192\pi^2} L_P^2 - \frac{3\lambda^2 m^2}{32\pi^2} L_P^2 \ln(L_P^2 m^2) \right] \Phi_c^4 \\
&\quad + \frac{m^4}{64\pi^2}\ln\left(1+\frac{\lambda}{2m^2}\Phi_c^2\right) + \frac{L_P^2}{192\pi^2}\left(m^2 + \frac{1}{2}\lambda\Phi_c^2\right)^3 \ln\left(1+\frac{\lambda}{2m^2}\Phi_c^2\right) \\
&\quad - \frac{m^6}{192\pi^2} L_P^2 \ln(L_P^2 m^2)
\end{aligned} \tag{80}$$

where γ is the Euler-Mascheroni constant. By comparing Eqn. (69) and Eqn. (80), it is seen that the divergent terms in the expressions for $m^{\rm P}_{\rm corr}$ and $\lambda^{\rm P}_{\rm corr}$ are of the same form as those for $m_{\rm corr}$ and $\lambda_{\rm corr}$. There is a quadratically divergent term as well as a logarithmic divergence. The finite terms appearing in $\left(m^{\rm P}_{\rm corr}\right)^2$ and $\lambda^{\rm P}_{\rm corr}$ which are independent of L_P are different from that appearing in $\mathcal{V}_{\rm finite}$ in Eqn. (80). This is because the form of the cutoff used is different. It is also clear from the above expression that for $m \neq 0$, $\mathcal{V}^P_{\rm finite}$ is finite in the limit $L_P \to 0$.

IV. CORRECTIONS TO THE EFFECTIVE LAGRANGIAN FOR ELECTROMAGNETIC FIELD

The system we shall consider in this subsection consists of a complex scalar field Φ interacting with the electromagnetic field represented by the vector potential A^μ. It is described by the following action (see, for e.g., Ref. [15], p. 98):

$$\begin{aligned}\mathcal{S}[\Phi, A^\mu] &= \int d^4x\, \mathcal{L}(\Phi, A^\mu)\\ &= \int d^4x \left\{\left(\partial_\mu\Phi + iqA_\mu\Phi\right)\left(\partial^\mu\Phi^* - iqA^\mu\Phi^*\right) - m^2\Phi\Phi^* - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\},\end{aligned} \tag{81}$$

where q and m are the charge and the mass associated with a single quantum of the complex scalar field, the asterisk denotes complex conjugation and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{82}$$

We shall assume that the electromagnetic field behaves classically, hence A^μ is just a c-number, while we shall assume the complex scalar field to be a quantum field so that Φ is an operator valued distribution. Varying the action (81) with respect to the complex scalar field Φ, we obtain the following Klein-Gordon equation:

$$\left(\left(\partial_\mu + iqA_\mu\right)\left(\partial^\mu + iqA^\mu\right) + m^2\right)\Phi = 0. \tag{83}$$

From the above equation it can be easily seen that

$$\hat{H} \equiv \left(\partial_\mu + iqA_\mu\right)\left(\partial^\mu + iqA^\mu\right). \tag{84}$$

A. Conventional result

In what follows we shall evaluate the effective Lagrangian for a constant electromagnetic background. A constant electromagnetic background can be described by the following vector potential:

$$A^\mu = (-Ez, -By, 0, 0), \tag{85}$$

where E and B are constants. The electric and the magnetic fields that this vector potential gives rise to are given by: $\mathbf{E} = E\hat{z}$ and $\mathbf{B} = B\hat{z}$, where $\hat{z}$ is the unit vector along the positive x-axis. The operator $\hat{H}$ corresponding to the vector potential above is then given by

$$\hat{H} \equiv \left(\partial_t^2 - \nabla^2 - 2iqEz\partial_t - 2iqBy\partial_x - q^2E^2 + q^2B^2\right). \tag{86}$$

Exploiting the translational invariance of this operator along the t and the x coordinates, we can write the corresponding quantum mechanical kernel as follows:

$$\begin{aligned}&K(x, x; s)\\ &\quad= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dp_x}{2\pi}\\ &\qquad\qquad \times \langle y, z|\exp -\left[i\left(-\partial_y^2 + (p_x + qBy)^2 - \partial_z^2 - (\omega + qEz)^2\right)s\right]|y, z\rangle.\end{aligned} \tag{87}$$

This kernel then corresponds to that of an inverted oscillator along the z-direction (an oscillator with an imaginary frequency) and an ordinary oscillator along the y-direction. Therefore, using the kernel for a simple harmonic oscillator (see, for e.g., Ref. [8], p. 63) we obtain that

$$K(x,x;s) = \left\{ \left(\frac{1}{16\pi^2 i s^2}\right) \left(\frac{qEs}{\sinh(qEs)}\right) \left(\frac{qBs}{\sin(qBs)}\right) \right\}. \tag{88}$$

Substituting this kernel in the expression for $\mathcal{L}_{\rm corr}$ in Eq. (7), we get

$$\mathcal{L}_{\rm corr} = -\left(\frac{1}{16\pi^2}\right) \int_0^\infty \frac{ds}{s^3}\, e^{-i(m^2 - i\epsilon)s} \left(\frac{qas}{\sinh(qas)}\right) \left(\frac{qbs}{\sin(qbs)}\right), \tag{89}$$

where a and b are related to the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ by the following relations:

$$a^2 - b^2 = \mathbf{E}^2 - \mathbf{B}^2 \qquad \text{and} \qquad ab = \mathbf{E}\cdot\mathbf{B}. \tag{90}$$

We can now interpret the real part of $\mathcal{L}_{\rm corr}$ as the correction to the Lagrangian describing the classical electromagnetic background given by:

$$\mathcal{L}_{\rm em} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) = \frac{1}{2}(a^2 - b^2). \tag{91}$$

From Eq. (89), it is easy to see that the real part of $\mathcal{L}_{\rm corr}$ is given by the expression

$$\mathrm{Re}\ \mathcal{L}_{\rm corr} = -\left(\frac{1}{16\,\pi^2}\right) \int_0^\infty \frac{ds}{s^3}\ \cos(m^2 s) \left(\frac{qas}{\sinh(qas)}\right) \left(\frac{qbs}{\sin(qbs)}\right). \tag{92}$$

This quantity can be regularized by subtracting the flat space contribution which is obtained by setting both a and b in the above expression to zero. Such a regularization then leads us to the following result:

$$\mathrm{Re}\ \bar{\mathcal{L}}_{\rm corr} = -\left(\frac{1}{16\pi^2}\right) \int_0^\infty \frac{ds}{s^3}\ \cos(m^2 s) \left\{ \left(\frac{qas}{\sinh(qas)}\right) \left(\frac{qbs}{\sin(qbs)}\right) - 1 \right\}, \tag{93}$$

Near $s = 0$, the expression in the curly brackets above goes as $\left[-q^2 s^2 (a^2 - b^2)/6\right]$. Hence, Re $\bar{\mathcal{L}}_{\rm corr}$ is still logarithmically divergent near $s = 0$. But this divergence is proportional to the original Lagrangian $\mathcal{L}_{\rm em}$ and because of this feature we can absorb this divergence by redefining the field strengths and charge. Or, in other words, we can renormalize the field strengths and charge by absorbing the logarithmic divergence into them in the following fashion. We write

$$\mathcal{L}_{\rm eff} = \left(\mathcal{L}_{\rm em} + \mathrm{Re}\ \bar{\mathcal{L}}_{\rm corr}\right) = \left(\mathcal{L}_{\rm em} + \mathcal{L}_{\rm div}\right) + \left(\mathrm{Re}\ \bar{\mathcal{L}}_{\rm corr} - \mathcal{L}_{\rm div}\right), \tag{94}$$

where we have defined $\mathcal{L}_{\rm div}$ as follows:

$$\begin{aligned} \mathcal{L}_{\rm div} &= -\left(\frac{1}{16\pi^2}\right) \int_0^\infty \frac{ds}{s^3}\ \cos(m^2 s) \left\{ -\frac{1}{6} q^2 s^2 \left(a^2 - b^2\right) \right\} \\ &= \left(\frac{Z}{2}\right) \left(a^2 - b^2\right) = \left(\frac{Z}{2}\right) \left(\mathbf{E}^2 - \mathbf{B}^2\right) = Z\, \mathcal{L}_{\rm em} \end{aligned} \tag{95}$$

where Z is a logarithmically divergent quantity described by the integral

$$Z = \left(\frac{q^2}{48\pi^2}\right) \int_0^\infty \frac{ds}{s}\ \cos(m^2 s). \tag{96}$$

Therefore, we can write

$$\mathcal{L}_{\rm eff} = \left(\mathcal{L}_{\rm em} + \mathcal{L}_{\rm div}\right) + \left(\mathrm{Re}\ \bar{\mathcal{L}}_{\rm corr} - \mathcal{L}_{\rm div}\right) = (1 + Z)\, \mathcal{L}_{\rm em} + \mathcal{L}_{\rm finite}, \tag{97}$$

where $\mathcal{L}_{\rm finite}$ is a finite quantity described by the integral

$$\begin{aligned} \mathcal{L}_{\rm finite} &= \left(\mathrm{Re}\ \bar{\mathcal{L}}_{\rm corr} - \mathcal{L}_{\rm div}\right) \\ &= -\left(\frac{1}{16\pi^2}\right) \int_0^\infty \frac{ds}{s^3}\ \cos(m^2 s) \\ &\qquad \times \left\{ \left(\frac{qas}{\sinh(qas)}\right) \left(\frac{qbs}{\sin(qbs)}\right) - 1 + \frac{1}{6} q^2 s^2 \left(a^2 - b^2\right) \right\}. \end{aligned} \tag{98}$$

All the divergences now appear in Z. Redefining the field strengths and charges as

$$\mathbf{E}_{\rm phy} = (1 + Z)^{1/2}\, \mathbf{E} \quad ; \quad \mathbf{B}_{\rm phy} = (1 + Z)^{1/2}\, \mathbf{B} \quad ; \quad q_{\rm phy} = (1 + Z)^{-1/2}\, q, \tag{99}$$

we find that such a scaling leaves $q_{\rm phy} \mathbf{E}_{\rm phy} = q\mathbf{E}$ invariant. Thus it is possible to redefine (renormalize) the variables in the theory thereby taking care of the divergences.

B. Results with the modified weightage factor

With the modified weightage factor, we find that the quantity $\mathcal{L}_{\rm corr}^{\rm P}$ for the constant electromagnetic background is described by the following integral:

$$\mathcal{L}_{\rm corr}^{\rm P} = -\left(\frac{1}{16\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, e^{-im^2 s}\, e^{iL_P^2/s}\left(\frac{qas}{\sinh(qas)}\right)\left(\frac{qbs}{\sin(qbs)}\right). \tag{100}$$

The real part of $\mathcal{L}_{\rm corr}^{\rm P}$ is then given by

$$\text{Re}\ \mathcal{L}_{\rm corr}^{\rm P} = -\left(\frac{1}{16\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, \cos\left[m^2 s - (L_P^2/s)\right]\left(\frac{qas}{\sinh(qas)}\right)\left(\frac{qbs}{\sin(qbs)}\right). \tag{101}$$

Regularizing this quantity by subtracting the flat space contribution (viz. the quantity obtained by setting $a = b = 0$ in the above expression), we get that

$$\text{Re}\ \bar{\mathcal{L}}_{\rm corr}^{\rm P} = -\left(\frac{1}{16\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, \cos\left[m^2 s - (L_P^2/s)\right]\left\{\left(\frac{qas}{\sinh(qas)}\right)\left(\frac{qbs}{\sin(qbs)}\right) - 1\right\}. \tag{102}$$

We can now express the effective Lagrangian exactly as we had done earlier. We can define

$$\mathcal{L}_{\rm eff}^{\rm P} = \left(\mathcal{L}_{\rm em} + \text{Re}\ \bar{\mathcal{L}}_{\rm corr}^{\rm P}\right) = \left(\mathcal{L}_{\rm em} + \mathcal{L}_{\rm div}^{\rm P}\right) + \left(\text{Re}\ \bar{\mathcal{L}}_{\rm corr}^{\rm P} - \mathcal{L}_{\rm div}^{\rm P}\right), \tag{103}$$

where we can now define $\mathcal{L}_{\rm div}^{\rm P}$ as follows:

$$\begin{aligned}\mathcal{L}_{\rm div}^{\rm P} &= -\left(\frac{1}{16\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, \cos\left[m^2 s - (L_P^2/s)\right]\left\{-\frac{1}{6}q^2 s^2\left(a^2 - b^2\right)\right\} \\ &= \frac{Z^{\rm P}}{2}\left(a^2 - b^2\right) = \frac{Z^{\rm P}}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = Z^{\rm P}\,\mathcal{L}_{\rm em}.\end{aligned} \tag{104}$$

$Z^{\rm P}$ is now a finite quantity given by the integral

$$Z^{\rm P} = \left(\frac{q^2}{48\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, \cos\left(m^2 s - L_P^2/s\right) = \frac{q^2}{6\pi}\, K_0(2mL_P), \tag{105}$$

where $K_0(2mL_P)$ is the modified Bessel function of order zero. Therefore, we can write

$$\mathcal{L}_{\rm eff}^{\rm P} = \left(\mathcal{L}_{\rm em} + \mathcal{L}_{\rm div}^{\rm P}\right) + \left(\text{Re}\ \bar{\mathcal{L}}_{\rm corr}^{\rm P} - \mathcal{L}_{\rm div}^{\rm P}\right) = \left(1 + Z^{\rm P}\right)\mathcal{L}_{\rm em} + \mathcal{L}_{\rm finite}^{\rm P}, \tag{106}$$

where $\mathcal{L}_{\rm finite}^{\rm P}$ is now defined as

$$\begin{aligned}\mathcal{L}_{\rm finite}^{\rm P} &= \left(\text{Re}\ \bar{\mathcal{L}}_{\rm corr}^{\rm P} - \mathcal{L}_{\rm div}\right) \\ &= -\left(\frac{1}{16\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, \cos\left(m^2 s - L_P^2/s\right) \\ &\qquad \times\left\{\left(\frac{qas}{\sinh(qas)}\right)\left(\frac{qbs}{\sin(qbs)}\right) - 1 + \frac{1}{6}q^2 s^2\left(a^2 - b^2\right)\right\}.\end{aligned} \tag{107}$$

We can now redefine the field strengths and the charge just as we had done earlier with $Z^{\rm P}$ instead of Z. $Z^{\rm P}$ is a finite quantity for a non-zero L_P, but diverges logarithmically when L_P is set to zero. Even in the limit $L_P \to 0$, the quantity $\mathcal{L}_{\rm finite}^{\rm P}$ stays finite and the divergence appears only in the expression for $Z^{\rm P}$.

In the limit $L_P \to 0$, we can make a rough estimation of the value of $\mathcal{L}_{\rm finite}^{\rm P}$ as follows. The quantity $\exp(-m^2 s - L_P^2/s)$ is a sharply peaked function about the value $s = L_P/m \ll 1$. Therefore, we may, without appreciable error, expand the term in the curly brackets in the above expression for $\mathcal{L}_{\rm finite}^{\rm P}$ in a Taylor series about the point $s = 0$ retaining only the first non-zero term. Keeping the limits of integration from 0 to ∞, (and performing an Euclidean rotation in order to evaluate the integral) we obtain,

$$\mathcal{L}_{\rm finite}^{\rm P} \approx \left(\frac{1}{16\pi^2}\right)\frac{2L_P^2 q^4}{360 m^2}\left(7(a^2 - b^2)^2 + 4a^2 b^2\right) K_2(2mL_P), \tag{108}$$

where K_2 is the modified Bessel function of order 2. Expanding K_2 in a series and retaining the terms of least order in L_P, we obtain,

$$\mathcal{L}_{\rm finite}^{\rm P} \approx \left(\frac{1}{16\pi^2}\right)\frac{q^4}{360 m^4}\left(7(a^2 - b^2)^2 + 4a^2 b^2\right)\left(1 - L_P^2 m^2\right). \tag{109}$$

V. CORRECTIONS TO THE THERMAL EFFECTS IN THE RINDLER FRAME

In flat spacetime, the Minkowski vacuum state is invariant only under the Poincare group, which is basically a set of linear coordinate transformations. Under a non-linear coordinate transformation the particle concept, in general, proves to be coordinate dependent. For e.g., the quantization in the Minkowski and the Rindler coordinates are inequivalent [16]. In fact the expectation value of the Rindler number operator in the Minkowski vacuum state proves to be a thermal spectrum. This result is normally obtained in literature by quantizing the field in the two coordinate systems and then evaluating the expectation value of the Rindler number operator in the Minkowski vacuum state. If we are to evaluate the quantum gravitational corrections to the Rindler thermal spectrum in such a fashion then we need to know as to how the metric fluctuations modify the normal modes of the quantum field. But, as we have mentioned earlier, we only know as to how quantum gravitational corrections can be introduced in the effective Lagrangian. Therefore, in this section, we shall first evaluate the effective Lagrangian in the Rindler coordinates and then go on to evaluate the corrections to this effective Lagrangian. The derivation of the Hawking radiation in a black hole spacetime runs along similar lines as the derivation of the Rindler thermal spectrum. Hence the results we present in this section have some relevance to the effects of metric fluctuations on Hawking radiation.

The system we shall consider in this section is a massless scalar field (in 4-dimensions described by the action

$$\mathcal{S}[\Phi] = \int d^4x\sqrt{-g}\;\mathcal{L}(\Phi) = \int d^4x\sqrt{-g}\left\{\frac{1}{2}\,g_{\mu\nu}\,\partial^\mu\,\Phi\partial^\nu\Phi\right\}. \tag{110}$$

Varying this action, we obtain the equation of motion for Φ to be

$$\hat{H}\Phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\Phi = 0. \tag{111}$$

A. Conventional result

The transformations that relate the Minkowski coordinates in flat spacetime to those of an observer who is accelerating uniformly along the x-direction are given by the following relations [17]:

$$t = g^{-1}(1+g\xi)\,\sinh(g\tau)\;\;;\;\; x = g^{-1}(1+g\xi)\,\cosh(g\tau)\;\;;\;\; y=y\;\;;\;\; z=z, \tag{112}$$

where g is a constant. The new coordinates (τ,ξ,y,z) are called the Rindler coordinates. In terms of the Rindler coordinates the flat spacetime line element is then given by

$$ds^2 = (1+g\xi)^2\,d\tau^2 - d\xi^2 - dy^2 - dz^2. \tag{113}$$

Therefore, in the Rindler coordinates, the operator $\hat{H}$ as defined in Eq. (111) is given by

$$\hat{H} \equiv \left\{\frac{1}{(1+g\xi)^2}\partial_\tau^2 - \frac{1}{(1+g\xi)}\partial_\xi\left[(1+g\xi)\partial_\xi\right] - \partial_y^2 - \partial_z^2\right\}, \tag{114}$$

where $\partial_x \equiv (\partial/\partial x)$. This operator is translationally invariant along the y and the z-directions. Or, in other words, the kernel corresponds to that of a free particle along these two directions. Exploiting this feature, we can write the quantum mechanical kernel as

$$K(x,x,;s|g_{\mu\nu}) = \left(\frac{1}{4\pi is}\right)\langle\tau,\xi|e^{-i\hat{H}'s}|\tau,\xi\rangle, \tag{115}$$

where

$$\hat{H}' \equiv \left(\frac{1}{(1+g\xi)^2}\partial_\tau^2 - \frac{1}{(1+g\xi)}\partial_\xi\left[(1+g\xi)\partial_\xi\right]\right). \tag{116}$$

On rotating the time coordinate τ to the negative imaginary axis, i.e. on setting $\tau = -i\tau_E$, we find that

$$\hat{H}' \equiv \left(-\frac{1}{(1+g\xi)^2}\partial_{\tau_E}^2 - \frac{1}{(1+g\xi)}\partial_\xi\left[(1+g\xi)\partial_\xi\right]\right). \tag{117}$$

Now, let $u = g^{-1}(1+g\xi)$, then

$$\hat{H}' \equiv \left(-\frac{1}{g^2 u^2}\partial^2_{\tau_E} - \frac{1}{u}\partial_u\left[u\partial_u\right]\right). \tag{118}$$

If we identify u as a radial variable and $g\tau_E$ as an angular variable then $\hat{H}'$ is similar in form to the Hamiltonian operator of a free particle in polar coordinates (in 2-dimensions). Then, for a constant ξ, the kernel corresponding to the operator $\hat{H}'$ can be written as [18,19]

$$\langle \tau', \xi | e^{-i\hat{H}'s} | \tau', \xi\rangle = \left(\frac{1}{4\pi s}\right) \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{i}{4s}(1+g\xi)^2\left(\tau - \tau' + 2\pi i n g^{-1}\right)^2\right\}. \tag{119}$$

Therefore, the complete quantum mechanical kernel corresponding to the operator $\hat{H}$ (in the coincidence limit) is given by

$$\begin{aligned} K(x,x;s|g_{\mu\nu}) &= \left(\frac{1}{16\pi^2 i s^2}\right) \sum_{n=-\infty}^{\infty} \exp\left(i\beta^2 n^2/4s\right) \\ &= \left(\frac{1}{16\pi^2 i s^2}\right) \left\{1 + 2\sum_{n=1}^{\infty} \exp\left(i\beta^2 n^2/4s\right)\right\}, \end{aligned} \tag{120}$$

where $\beta = 2\pi g^{-1}(1+g\xi)$. Substituting this kernel in the expression (7) and setting $m=0$, we find that

$$\mathcal{L}_{\rm corr} = -\left(\frac{1}{32\pi^2}\right) \int_0^\infty \frac{ds}{s^3} \left\{1 + 2\sum_{n=1}^{\infty} \exp\left(i\beta^2 n^2/4s\right)\right\}. \tag{121}$$

On regularization, i.e. on subtracting the quantity $\mathcal{L}^0_{\rm corr}$ (given by Eq. (19) with $m=0$) from the above expression, we obtain that

$$\bar{\mathcal{L}}_{\rm corr} = \left(\mathcal{L}_{\rm corr} - \mathcal{L}^0_{\rm corr}\right) = -\left(\frac{1}{16\pi^2}\right) \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s^3} \exp\left(i\beta^2 n^2/4s\right). \tag{122}$$

The integral over s can be expressed in terms of Gamma functions (see, for e.g., Ref. [11], p. 934), so that

$$\bar{\mathcal{L}}_{\rm corr} = \left(\frac{\Gamma(2)}{\pi^2\,\beta^4}\right) \sum_{n=1}^{\infty} n^{-4} = \left(\frac{\Gamma(2)}{\pi^2\,\beta^4}\right) \zeta(4) = \left(\frac{\pi^2}{90\,\beta^4}\right), \tag{123}$$

where we have made use of the fact that $\zeta(4) = (\pi^4/90)$ (cf. Ref. [12], p. 334).

Two points need to be noted regarding the above result. Firstly, $\bar{\mathcal{L}}_{\rm corr}$ corresponds to the total energy radiated by a black body at a temperature β^{-1}. Secondly, there arises no imaginary part to the effective Lagrangian which clearly implies that the thermal effects in the Rindler frame arises due to vacuum polarization and not due to particle production.

1. Spectrum from the propagator

The Feynman propagator corresponding to an operator $\hat{H}$ can be written as (cf. Eq. (1))

$$G_{\rm F}(x,x') = -i\int_0^\infty ds\, K(x,x';s|g_{\mu\nu}), \tag{124}$$

where $K(x,x';s|g_{\mu\nu})$ is given by Eq. (8). (Note that since we are considering a massless scalar field we have set $m=0$.) For the Rindler coordinates we are considering here, the propagator is obtained by substituting the kernel (120) in the expression for the propagator as given by Eq.(124). If we set $\xi = \xi' = 0$, $y = y'$ and $z = z'$, we find that the propagator is given by

$$G_{\rm F}(x,x') \equiv G(\Delta\tau) = -\left(\frac{1}{16\pi^2}\right)\int_0^\infty \frac{ds}{s^2}\sum_{n=-\infty}^{n=\infty} \exp - \left[i(\Delta\tau + i\beta n)^2/4s\right]$$
$$= \left(\frac{i}{4\pi^2}\right)\sum_{n=-\infty}^{n=\infty}\left(\Delta\tau + 2\pi i n g^{-1}\right)^{-2}, \tag{125}$$

where $\Delta\tau = (\tau - \tau')$. Also, since we have set $\xi = 0$, $\beta = (2\pi/g)$. (Compare this result with Eq. (3.66) in Ref. [20].) Fourier transforming this propagator with respect to $\Delta\tau$, we find that

$$P(\Omega) \equiv \left\vert \int_{-\infty}^{\infty} d\Delta\tau\, e^{-i\Omega\Delta\tau}\, G_{\rm F}(\Delta\tau)\right\vert$$
$$= \left\vert \left(\frac{i}{4\pi^2}\right)\int_{-\infty}^{\infty} d\Delta\tau \sum_{n=-\infty}^{n=\infty}\left(\frac{e^{-i\Omega\Delta\tau}}{(\Delta\tau + i\beta n)^{-2}}\right)\right\vert$$
$$= \left(\frac{1}{2\,\pi}\right)\left(\frac{\Omega}{e^{\beta\Omega} - 1}\right), \tag{126}$$

i.e. the resulting power spectrum is a thermal spectrum with a temperature β^{-1}.

B. Results with the modified weightage factor

Let us now evaluate the effective Lagrangian in the Rindler frame with the modified weightage factor. Now, substituting the kernel (120) in the expression (9) for $\mathcal{L}_{\rm corr}^{\rm P}$ and setting $m = 0$, we obtain that

$$\mathcal{L}_{\rm corr}^{\rm P} = -\left(\frac{1}{32\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, e^{iL_P^2/s}\left\{1 + 2\sum_{n=1}^{\infty}\exp\left(i\beta^2 n^2/4s\right)\right\}. \tag{127}$$

On regularization, i.e. on subtracting the quantity $\mathcal{L}_{\rm corr}^{\rm P0}$ (with $m = 0$) from $\mathcal{L}_{\rm corr}^{\rm P}$, we find that

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P} = \left(\mathcal{L}_{\rm corr}^{\rm P} - \mathcal{L}_{\rm corr}^{\rm P0}\right)$$
$$= -\left(\frac{1}{16\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, e^{iL_P^2/s}\sum_{n=1}^{\infty}\exp\left(i\beta^2 n^2/4s\right)$$
$$= -\left(\frac{1}{16\pi^2}\right)\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s^3}\,\exp\left[i(\beta^2 n^2 + 4L_P^2)/4s\right]$$
$$= \left(\frac{1}{\pi^2}\right)\sum_{n=1}^{\infty}(\beta^2 n^2 + 4L_P^2)^{-2}$$
$$= \left(\frac{1}{\pi^2\beta^4}\right)\sum_{n=1}^{\infty}\left[n^2 + (4L_P^2/\beta^2)\right]^{-2}. \tag{128}$$

Using the relation (50), we can express $\bar{\mathcal{L}}_{\rm corr}^{P}$ in a closed form as follows:

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P} = \left\{-\left(\frac{1}{32\pi^2 L_P^4}\right) + \left(\frac{1}{32\pi a L_P^3}\right)\coth(2\pi L_P/a) + \left(\frac{1}{16a^2 L_P^2}\right)\,{\rm cosech}^2(2\pi L_P/a)\right\}. \tag{129}$$

Making use of the series expansions (52) and (53), we find that, as $L_P \to 0$

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P} \to \left\{\left(\frac{\pi^2}{90\beta^4}\right) - L_P^2\left(\frac{8\pi^4}{945\beta^6}\right)\right\}. \tag{130}$$

1. Spectrum from the modified propagator

The propagator with the modified weightage factor is given by

$$G_{\rm F}^{\rm P}(x,x') = -i \int_0^\infty ds\, e^{iL_P^2/s}\, K(x,x';s|g_{\mu\nu}). \tag{131}$$

In the Rindler coordinates, we find that, if we set $\xi=\xi'=0$, $y=y'$ and $z=z'$, the modified propagator is given by

$$\begin{aligned} G_{\rm F}^{\rm P}(\Delta\tau) &= -\left(\frac{1}{16\pi^2}\right)\int_0^\infty \frac{ds}{s^2}\sum_{n=-infty}^{n=\infty} \exp -i\left[\frac{1}{4s}(\Delta\tau+i\beta n)^2 - \frac{L_P^2}{s}\right] \\ &= \left(\frac{i}{4\pi^2}\right)\sum_{n=-\infty}^{n=\infty}\left[(\Delta\tau+2\pi i n g^{-1})^{-2} - 4L_P^2\right]^{-1}, \end{aligned} \tag{132}$$

where, as before, $\Delta\tau = (\tau-\tau')$. Fourier transforming this modified propagator with respect to $\Delta\tau$, we obtain that

$$\begin{aligned} \mathcal{P}^{\rm P}(\Omega) &\equiv \left|\int_{-\infty}^{\infty} d\Delta\tau\, e^{-i\Omega\Delta\tau}\, G_{\rm F}^{\rm P}(\Delta\tau)\right| \\ &= \left|\left(\frac{i}{4\pi^2}\right)\int_{-\infty}^{\infty} d\Delta\tau \sum_{n=-\infty}^{n=\infty}\left(\frac{e^{-i\Omega\Delta\tau}}{[(\Delta\tau+i\beta n)^2 - 4L_P^2]}\right)\right| \\ &= \left(\frac{1}{2\,\pi}\right)\left(\frac{|\sin(2\Omega L_P)|}{2\Omega L_P}\right)\left(\frac{\Omega}{e^{\beta\Omega}-1}\right). \end{aligned} \tag{133}$$

This modified shows an appreciable deviation from the Planckian form only when $\beta \simeq L_P$. But when $\beta \simeq L_P$, the semiclassical approximation we are working with will anyway cease to be valid.

VI. CORRECTIONS TO THE GRAVITATIONAL LAGRANGIAN

The system we shall consider in this section consists of scalar field Φ interacting with a classical gravitational field described by the metric tensor $g_{\mu\nu}$. It is described by action

$$\begin{aligned} \mathcal{S}[g_{\mu\nu},\Phi] &= \int d^Dx\,\sqrt{-g}\,\mathcal{L}(g_{\mu,\nu},\Phi) \\ &= \int d^Dx\,\sqrt{-g}\left\{\frac{1}{16\,\pi G}\,(R-2\Lambda) + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2\Phi^2 - \frac{1}{2}\xi R\Phi^2\right\}, \end{aligned} \tag{134}$$

where R is the scalar curvature of the spacetime, Λ is the cosmological constant and G is the gravitational constant. Setting the parameter $\xi = 0$ or $\xi = (1/6)$ corresponds to a minimal or conformal coupling of the scalar field to the gravitational background respectively. We are interested in finding quantum corrections to the purely gravitational part of the total Lagrangian. This will be done as usual in the framework of the semiclassical theory by considering the one-loop effective action formalism. In the conventional derivation, divergences arise in the expression for $\mathcal{L}_{\rm corr}$. There are three divergent terms, two of which are absorbed into the cosmological constant Λ and the gravitational constant G and thus Einstein's theory is reinterpreted suitably. The third divergent term cannot be so absorbed. Extra terms will have to be introduced into the gravitational Lagrangian in order to absorb this divergence [20]. When the duality principle is used however, no divergences occur. The cosmological constant and the the gravitational constant are modified by the addition of finite terms which are seen to diverge in the limit $L_P \to 0$ thus recovering the standard result.

A. Conventional result

In what follows we shall first briefly outline the conventional approach to calculating $\mathcal{L}_{\rm corr}$ and then use the path integral duality prescription to compute the corrections to the gravitational Lagrangian.

Varying the above action with respect to Φ, we obtain that

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)+m^2+\xi R\right)\Phi=0. \tag{135}$$

Comparing this equation of motion with Eq. (6), it is easy to identify that

$$\hat{H}\equiv\left(\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)+m^2+\xi R\right). \tag{136}$$

In D-dimensions, the quantum mechanical kernel $K(x,x';s|g_{\mu\nu})$ (cf. Eq. (8)) corresponding to the operator $\hat{H}$ above, can be written as [20,21]

$$K(x,x';s|g_{\mu\nu})=\left(\frac{i}{(4\pi i s)^{D/2}}\right)\,e^{i\sigma(x,x')/2s}\,\Delta^{1/2}(x,x')\,F(x,x';s), \tag{137}$$

where

$$\sigma(x,x')=\frac{1}{2}\int_0^s ds'\left\{g_{\mu\nu}\frac{dx^\mu}{ds'}\frac{dx^\nu}{ds'}\right\}^{1/2} \tag{138}$$

is the proper arc length along the geodesic from x' to x and $\Delta^{1/2}(x,x')$ is the Van Vleck determinant given by

$$\Delta^{1/2}(x,x')=\left(-[-g(x)]^{-1/2}\,\det[\partial_\mu\partial_\nu\sigma(x,x')]\,[-g(x')]^{-1/2}\right). \tag{139}$$

The function $F(x,x';s)$ can be written down in an asymptotic expansion

$$F(x,x';s)=\sum_{n=0}^{\infty}a_n\,(is)^n=a_0+a_1(x,x')\,(is)+a_2(x,x')\,(is)^2+\ldots, \tag{140}$$

where the leading term a_0 is unity since F must reduce to unity in flat spacetime.

Substituting the quantum mechanical kernel above in the expression for $\mathcal{L}_{\rm corr}$ given by Eq. (7), we obtain that

$$\mathcal{L}_{\rm corr}=-\lim_{x\to x'}\left(\frac{\Delta^{1/2}(x,x')}{32\,\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\,e^{-im^2s}e^{i\sigma(x,x')/2s}\times\left\{1+a_1(x,x')\,(is)+a_2(x,x')\,(is)^2+\ldots\right\}. \tag{141}$$

In the coincidence limit, $\sigma(x,x')$ vanishes and one sees that the integral over the first three terms in the square brackets diverge. The integral over the remaining terms involving a_3, a_4 and so on, are finite in this limit. Therefore, the divergent part of $\mathcal{L}_{\rm corr}$ is given by

$$\mathcal{L}^{\rm div}_{\rm corr}=-\left(\frac{1}{32\,\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\,e^{-im^2s}\,\left\{1+a_1(x,x)\,(is)+a_2(x,x)\,(is)^2\right\}, \tag{142}$$

where the coefficients a_1 and a_2 are given by the relations [20]

$$a_1(x,x)=\left(\frac{1}{6}-\xi\right)R(x) \tag{143}$$

and

$$a_2(x,x)=\frac{1}{6}\left(\frac{1}{5}-\xi\right)g^{\mu\nu}R_{;\mu;\nu}(x)+\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2R^2(x)+\frac{1}{180}R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}-\frac{1}{180}R_{\mu\nu}R^{\mu\nu}. \tag{144}$$

Since a_1 and a_2 depend only on $R_{\mu\nu\lambda\rho}$ and its contractions, they are purely geometrical in nature. The divergences arise because of the ultraviolet behavior of the field modes. These short wavelengths probe only the local geometry in the neighborhood of x and are not sensitive to the global features of the spacetime and are independent of the quantum state of the field Φ. Since the divergent part of the effective Lagrangian is purely geometrical it can be regarded as the correction to the purely gravitational part of the Lagrangian. The divergence corresponding to the first term in the square brackets can be added to the cosmological constant thus regularizing it while the divergence due to the second term can be absorbed into the gravitational constant giving rise to the renormalized gravitational constant which is finite. The third term which involves a_2 contains derivatives of the metric tensor of order 4 and this term represents a correction to Einstein's theory which contains derivatives of order 2 only. Therefore one needs to introduce extra terms into the gravitational Lagrangian so that these divergences can be absorbed into suitable constants [20].

B. Results with the modified weightage factor

Let us now evaluate the corrections to the gravitational Lagrangian with the modified weightage factor. Substituting the kernel (137) into Eq. (9), we obtain

$$
\begin{aligned}
\mathcal{L}_{\rm corr}^{\rm P} &= -\left(\frac{1}{32\,\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\, e^{-im^2 s}\, e^{iL_P^2/s}\ \left\{1 + a_1(x,x)\,(is) + a_2(x,x)\,(is)^2 + \dots\right\} \\
&= \left(\frac{m^4}{32\,\pi)^2}\right)\Bigg\{\left(\frac{2}{L_P^2\,m^2}\right)\, K_2(2L_P m) + \left(\frac{2}{L_P\,m^3}\right)\, K_1(2L_P m) a_1(x,x) \\
&\qquad\qquad + \left(\frac{2}{m^4}\right) K_0(2L_P m) a_2(x,x) + \dots\Bigg\},
\end{aligned}
\tag{145}
$$

where K_0, K_1 and K_2 are the modified Bessel functions of orders zero, 1 and 2, respectively.

The effective Lagrangian for the classical gravitational background is therefore given by

$$
\begin{aligned}
\mathcal{L}_{\rm eff}^{\rm P} &= \left(\mathcal{L}_{\rm grav} + \mathcal{L}_{\rm corr}^{\rm P}\right) \\
&= \frac{1}{16\pi G_{\rm corr}} R - \frac{1}{8\pi G}\Lambda_{\rm corr} + \frac{1}{16\pi^2} K_0(2L_P m) a_2(x,x) + \dots
\end{aligned}
\tag{146}
$$

where,

$$
\begin{aligned}
\Lambda_{\rm corr} &= \Lambda - \frac{m^2 G}{2\pi L_P^2} K_2(2L_P m) \\
\frac{1}{G_{\rm corr}} &= \frac{1}{G} + \frac{m}{\pi L_P}\left(\frac{1}{6} - \xi\right) K_1(2L_P m)
\end{aligned}
\tag{147}
$$

In Eq. (9), the duality principle demanded that the Kernel be modified by a factor $\exp(iL_P^2/s)$ where L_P^2 is the square of the Planck length. But, as mentioned earlier, L_P could as well be replaced by ηL_P where η is a numerical factor of order unity. Therefore, we replace L_P in the above equations by ηL_P and since $L_P^2 = G$ by definition, the formula for $G_{\rm corr}$ can equivalently be written as

$$
\frac{1}{G_{\rm corr}} = \frac{1}{G}\left(1 + \frac{m\sqrt{G}}{\eta\pi}\left(\frac{1}{6} - \xi\right) K_1(2\eta\sqrt{G}m)\right)
\tag{148}
$$

In the limit $(\eta m) \to 0$, using the power series expansion for the functions $K_2(2\eta\sqrt{G}m)$ and $K_1(2\eta\sqrt{G}m)$, we can write the corrections to G and Λ as follows:

$$
\begin{aligned}
\Lambda_{\rm corr} &= \Lambda - \frac{m^2}{2\pi\eta^2}\left(\frac{1}{2\eta^2 G m^2} - \frac{1}{2}\right) \\
&= \Lambda + \frac{m^2}{4\pi\eta^2} - \frac{1}{4\pi\eta^4 G}
\end{aligned}
\tag{149}
$$

and

$$
\frac{1}{G_{\rm corr}} = \frac{1}{G}\left[1 + \frac{1}{2\pi\eta^2}\left(\frac{1}{6} - \xi\right)\right]
\tag{150}
$$

For the case when the scalar field is assumed to be coupled minimally to the gravitational background, i.e. $\xi = 0$, the correction to G reduces to,

$$
\frac{1}{G_{\rm corr}} = \frac{1}{G}\left[1 + \frac{1}{12\pi\eta^2}\right]
\tag{151}
$$

while for conformal coupling where $\xi = 1/6$, the correction to G vanishes.

VII. CORRECTIONS TO THE TRACE ANOMALY

The problem concerning the renormalization of the expectation value of energy-momentum tensor in curved spacetime is considerably more involved than the corresponding problem in Minkowski spacetime. This concerns the role of the energy-momentum tensor T_{ik} in gravity. In flat spacetime only energy differences are meaningful and therefore infinite constants like the energy of the vacuum can be subtracted out without any problem. In curved spacetime, however, energy is a source of gravity. Therefore, one is not free to rescale the zero point of the energy scale in an arbitrary manner. In the semi-classical theory of gravity, one can carry out the renormalization of $\langle T_{ik}\rangle$ in a unique way using different methods like the ζ-function renormalization technique, dimensional regularization and other methods. Since $\langle T_{ik}\rangle$ can be obtained from the effective action by functionally differentiating with respect to the metric tensor, the renormalization procedure is therefore connected with the renormalization of effective action which was described in the previous section. Upon specializing to theories where the classical action is invariant under conformal transformations, it can be shown that the trace of the *classical* energy-momentum tensor is zero. But, when the renormalized expectation value of the trace is calculated, however, it is found to be non-zero. This is the conformal or trace anomaly. It essentially arises because of the divergent terms present in the effective action. When the principle of path integral duality is applied, no divergences appear and hence one would expect the trace anomaly to vanish. But because a fixed length scale appears in the problem, the trace anomaly is still non-zero.

A. Conventional result

In this section we derive the formula relating the trace of the energy momentum tensor and the effective action. We then apply the duality principle and derive an explicit formula for the trace anomaly. In the limit of $L_P \to 0$, it is shown that the usual divergences appear which when renormalized using dimensional regularization and zeta function regularization techniques yield the usual formula for the trace anomaly.

Consider a scalar field that is coupled to a classical gravitational background as described by the action (134). The energy-momentum of such a scalar field can be obtained by varying the action with respect to the metric tensor $g_{\mu\nu}$ as follows [22]:

$$\begin{aligned} T^{\mu\nu} &\equiv \left(\frac{2}{\sqrt{-g}}\right)\left(\frac{\delta\mathcal{S}}{\delta g_{\mu\nu}}\right) \\ &= \partial^\mu\Phi\partial^\nu\Phi - \frac{1}{2}g^{\mu\nu}\partial^\alpha\partial_\alpha\Phi - \frac{1}{2}g^{\mu\nu}m^2\Phi^2 \\ &\qquad + \xi\left\{g^{\mu\nu}\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta\right)\Phi^2 - \left(\Phi^2\right)^{;\mu;\nu} + G^{\mu\nu}\Phi^2\right\}, \end{aligned} \tag{152}$$

where $G^{\mu\nu} = (R^{\mu\nu} - (1/2)g^{\mu\nu}R)$. Using the field equations (135), the trace of $T^{\mu\nu}$ is

$$T^\mu_\mu = (6\xi-1)\partial^\mu\Phi\partial_\mu\Phi + \xi(6\xi-1)R\Phi^2 + (6\xi-2)m^2\Phi^2. \tag{153}$$

For the conformally invariant case, i.e. when $\xi = 1/6$ and $m = 0$, the trace vanishes. Thus, if the action is invariant under conformal transformations of the metric, the classical energy-momentum tensor is traceless. Since conformal transformations are essentially a rescaling of lengths at each spacetime point x, the presence of a mass and therefore the existence of a fixed length scale in the theory will break the conformal invariance.

On the other hand, the trace of the renormalized expectation value of the energy momentum tensor does not vanish in the conformal limit. In the semiclassical case, the expectation value of $T^{\mu\nu}$ is given by

$$\langle T^{\mu\nu}\rangle = \frac{2}{\sqrt{-g(x)}}\frac{\delta\mathcal{S}_{\rm corr}}{\delta g_{\mu\nu}}, \tag{154}$$

where

$$\mathcal{S}_{\rm corr} = \int d^4x\sqrt{-g}\mathcal{L}_{\rm corr}. \tag{155}$$

Consider the change in $\mathcal{S}_{\rm corr}$ under an infinitesimal conformal transformation

$$g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \quad \text{with} \quad \Omega^2(x) = 1+\epsilon(x) \quad \text{and} \quad \delta g_{\mu\nu} = \epsilon(x)g_{\mu\nu} \tag{156}$$

Regarding $\mathcal{S}_{\rm corr}$ as a functional of $\Omega^2(x)$, with $g_{\mu\nu}$ being a given function, one obtains,

$$\mathcal{S}_{\rm corr}[(1+\epsilon)g_{\mu\nu}] = \mathcal{S}_{\rm corr}[g_{\mu\nu}] + \int d^4x \left. \frac{\delta\mathcal{S}_{\rm corr}[\Omega^2(x)g_{\mu\nu}]}{\delta\Omega^2(x)}\right|_{\Omega^2(x)=1} \epsilon(x) \tag{157}$$

Thus,

$$\frac{\delta\mathcal{S}_{\rm corr}[g_{\mu\nu}]}{\delta g_{\mu\nu}} g_{\mu\nu} = \left.\frac{\delta\mathcal{S}_{\rm corr}[\Omega^2(x)g_{\mu\nu}]}{\delta\Omega^2(x)}\right|_{\Omega^2(x)=1} \tag{158}$$

Therefore,

$$\langle T^{\mu}{}_{\mu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_{\rm corr}[g_{\mu\nu}]}{\delta g_{\mu\nu}} g_{\mu\nu} = \frac{2}{\sqrt{-g}}\left.\frac{\delta\mathcal{S}_{\rm corr}[\Omega^2(x)g_{\mu\nu}]}{\delta\Omega^2(x)}\right|_{\Omega^2(x)=1} \tag{159}$$

Now, $\mathcal{S}_{\rm corr}$ is given by the formula

$$\mathcal{S}_{\rm corr}[g_{\mu\nu}] = -\frac{i}{2}\int d^4x\sqrt{-g}\int_0^{\infty}\frac{ds}{s}\langle x|e^{-is\hat{H}}|x\rangle. \tag{160}$$

It can be shown that under a conformal transformation [20],

$$H(x) \to \widetilde{H}(x) = \Omega^{-3}(x)H(x)\Omega(x). \tag{161}$$

where $H(x)$ is given by Eqn. (136) in the conformal limit with $\xi = 1/6$ and $m = 0$. The corresponding relation satisfied by the operator $\hat{H}$ under a conformal transformation is

$$\hat{\widetilde{H}} = \Omega^{-1}\hat{H}\Omega^{-1} \tag{162}$$

while the trace operator Tr defined by the relation

$$Tr(\hat{H}) = \int d^4x\sqrt{-g}\langle x|\hat{H}|x\rangle \tag{163}$$

remains invariant [21]. Both the above relations can be easily proved as follows. First note the following definitions. In the abstract Hilbert space, one has the relations,

$$\begin{aligned}\langle x|\hat{H}|x'\rangle &= H(x)\delta(x-x')[-g(x')]^{-1/2}\\ 1 &= \int d^4x\sqrt{-g}|x\rangle\langle x| \qquad \text{(completeness relation)}\\ \langle x|x'\rangle &= [-g(x')]^{-1/2}\delta(x-x') \qquad \text{(Orthonormality relation)}\end{aligned} \tag{164}$$

Since the completeness and orthonormality relations remain invariant under a conformal transformation, we get,

$$|x\rangle \to |\widetilde{x}\rangle = \Omega^{-2}(x)|x\rangle. \tag{165}$$

Now, using Eqn. (161) and the definitions given in Eqn. (164),

$$\begin{aligned}\widetilde{H}(x)\delta(x-x') &= \Omega^{-3}(x)H(x)\Omega(x)\delta(x-x')\\ &= \Omega^{-3}(x)\Omega(x')H(x)\delta(x-x')\\ &= \Omega^{-3}(x)\Omega(x')\sqrt{-g(x')}\langle x|\hat{H}|x'\rangle\\ &= \sqrt{-g(x')}\langle x|\Omega^{-3}\hat{H}\Omega|x'\rangle.\end{aligned} \tag{166}$$

We also have,

$$\begin{aligned}\widetilde{H}(x)\delta(x-x') &= \sqrt{-\widetilde{g}(x')}\langle\widetilde{x}|\hat{\widetilde{H}}|\widetilde{x}'\rangle\\ &= \sqrt{-g(x')}\langle x|\Omega^{-2}\hat{\widetilde{H}}\Omega^2|x'\rangle\end{aligned} \tag{167}$$

Comparing the above two equations, it is easy to prove the relation given in Eqn. (162). Similarly, using the invariance of the completeness relation under a conformal transformation, the relation given in Eqn. (163) can also be proved.

Using the above results it is easy to show that

$$Tr(e^{-is\tilde{\hat{H}}}) = Tr(e^{-is\Omega^{-1}\hat{H}\Omega^{-1}}) = Tr(e^{-is\Omega^{-2}\hat{H}}) \tag{168}$$

Using the above formula for $\mathcal{S}_{\text{corr}}$, and the above results, one finds that under the infinitesimal transformation given in Eqn. (156),

$$\mathcal{S}_{\text{corr}}[\Omega^2 g_{\mu\nu}] = -\frac{i}{2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds}{s}\langle x|e^{-is\Omega^{-2}\hat{H}}|x\rangle \tag{169}$$

The above expression for $\mathcal{S}_{\text{corr}}$ is clearly divergent at $s = 0$ as shown in the previous section. Making a change of variable $s \to s' = s\Omega^{-2}$, it appears that

$$\mathcal{S}_{\text{corr}}[\Omega^2 g_{\mu\nu}] = -\frac{i}{2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds'}{s'}\langle x|e^{-is'\hat{H}}|x\rangle \equiv \mathcal{S}_{\text{corr}}[g_{\mu\nu}] \tag{170}$$

But such a change of variable is not valid since the integral is divergent. To make sense of such an integral, one resorts to various techniques to determine the trace anomaly. As shown in [20,21], using the ζ function approach, the trace anomaly is shown to be equal to $(4\pi)^{-2}a_2(x,x)$.

B. Results with the modified weightage factor

But, if one modifies the propagator $\langle x|e^{-is\hat{H}}|x'\rangle$ using the duality prescription, then the expression for $\mathcal{S}_{\text{corr}}[\Omega^2 g_{\mu\nu}]$ becomes,

$$\mathcal{S}_{\text{corr}}[\Omega^2 g_{\mu\nu}] = -\frac{i}{2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds}{s}e^{iL_P^2/s}\langle x|e^{-is\Omega^{-2}\hat{H}}|x\rangle \tag{171}$$

The equation (171) now has no divergences. Making a change of variable $s \to s' = s\Omega^{-2}$,which is valid now, one obtains,

$$\mathcal{S}_{\text{corr}}[\Omega^2 g_{\mu\nu}] = -\frac{i}{2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds'}{s'}e^{(iL_P^2/\Omega^2 s')}\langle x|e^{-is'\hat{H}}|x\rangle \tag{172}$$

Using the formula for the trace of the energy momentum tensor (159), one finally obtains

$$\langle T^\mu{}_\mu\rangle = -L_P^2\int_0^\infty \frac{ds}{s^2}e^{iL_P^2/s}\langle x|e^{-is\hat{H}}|x\rangle. \tag{173}$$

Using the formula for the propagator given in Eqn. (137) in 4 dimensions and evaluating the integral over s,

$$\langle T^\mu{}_\mu\rangle = \frac{2L_P^2}{(4\pi)^2}\sum_{n=0}^\infty a_n\left(\frac{L_P}{m}\right)^{n-3}K_{(n-3)}(2L_P m) \tag{174}$$

where K_n is the usual modified Bessel function of order n. If the limit $L_P \to 0$ is considered, then the expression above reduces to

$$\lim_{L_P\to 0}\langle T^\mu{}_\mu\rangle = \lim_{L_P\to 0}\frac{2}{(4\pi)^2}\left[\frac{1}{L_P^4} + \frac{a_1 - m^2}{2L_P^2}\right] + \frac{1}{2(4\pi)^2}(2a_2 - 2m^2 a_1 + m^4) \tag{175}$$

The terms present in the square brackets represent the divergences that are present in the evaluation of the energy-momentum tensor without using the duality principle. These divergences need to be regularized by other methods like the ζ function approach mentioned earlier. The finite part that remains is the last term that, in the conformal limit, reduces to $(4\pi)^{-2}a_2(x,x)$. Therefore, in the limit of $L_P \to 0$, the standard result is recovered.

VIII. DISCUSSION

In this paper, we had evaluated the quantum gravitational corrections to some of the standard quantum field theoretic results using the 'principle of path integral duality'. We find that, the main feature of this duality principle, as we had stressed earlier in the introductory section, is that it is able provide an ultra-violet cut-off at the Planck energy scales thereby rendering the theory finite. One key feature of thisapproach is that the prescription is completely Lorentz invariant. Hence we could obtain finite but Lorentz invariant results for otherwise divergent expressions.

The obvious drawbacks of the approach are the following:

(i) The prescription of path integral duality is essentially an ad-hoc prescription. It is not backed by a theoretical framework which is capable of replacing the conventional quantum field theory at the present juncture. Hence, the prescription only tells us how to modify the kernels and Green's functions. To obtain any result with the prescription of path integral duality we have to first relate the result to the kernel or Green's function, modify the kernel and thereby obtain the final result.

In spite of this constraint we have been able to show in this paper that concrete computations can be done and specific results can be obtained. As regards the ad-hocness of the prescription, it should be viewed as the first step in the approach to quantum gravity based on a general physical principle. It's relation with zero-point length and the emergence of analogous duality principles in string theories, for example, makes one hopeful that it can be eventually put on firmer foundation.

(ii) The modified kernel (based on the principle of path integral duality) may not be obtainable from the standard framework of field theory based on unitarity, microscopic causality and locality. (We have no rigorous proof that this is the case; however, it is quite possible since standard field theories based on the above principles are usually divergent.)

It is not clear to us whether such principles will be respected in the fully quantum gravitational regime. It is very likely that the continuum field theory which we are accustomed to will be drastically modified at Planck scales. If that is the case, it is quite conceivable that the quantum gravitational corrections also leave a trace of the breakdown of continuum field theory even when expressed in such a familiar language. As an example, consider an attempt to study and interpret quantum mechanics in terms of classical trajectories. Any formulation will lead to some contradictions like, for example, the breakdown of differentiability for the path. This arises because we are attempting to interpret physical principles using an inadequate formalism.

The next logical step will be to attempt to derive path integral duality from a deeper physical principle using appropriate mathematical methods. This should throw more light on, for example, the connection between path integral duality and zero-point length which at the moment remains a mystery. We hope to address it in a future publication.

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[1] T. Padmanabhan, Class. Quantum Grav. **4**, L107 (1987).
[2] T. Padmanabhan, Ann. Phys. (N. Y.) **165**, 38 (1985).
[3] T. Padmanabhan, Phys. Rev. Lett. **78**, 1854 (1997).
[4] T. Padmanabhan, 'The hypothesis of path integral duality I: corrections to the Feynman propagator', Submitted to Phys. Rev. D.
[5] J. W. Schwinger, Phys. Rev. D **82**, 664 (1951).
[6] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York (1980).
[7] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Course of Theoretical Physics, Volume 3, Pergamon Press, New York (1977).
[8] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals*, (McGraw-Hill, Singapore, 1965).
[9] P. M. Morse and H. Feshbach, *Methods of theoretical physics*, McGraw-Hill, New York (1953).
[10] L. S. Schulman, *Techniques and Applications of Path Integration* (John Wiley, New York, 1981).
[11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
[12] G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1985).

[13] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series* (Gordon and Breach Science Publishers, New York, 1986).
[14] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Spacetime* (Cambridge University Press, Cambridge, England, 1989).
[15] L. H. Ryder, *Quantum Field Theory*, (Cambridge University Press, Cambridge, England, 1985).
[16] S. A. Fulling, Phys. Rev. D **7**, 2850 (1973).
[17] W. Rindler, Am. J. Phys. **34**, 1174 (1966).
[18] S. M. Christensen and M. J. Duff, Nucl. Phys. B**146**, 11 (1978).
[19] T. Padmanabhan, Bull. Astr. Soc. India **13**, 225 (1985).
[20] N. D. Birrell and P. c. W. Davies, *Quantum Field in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[21] L. Parker, 'Aspects of quantum field theory in curved spacetime: effective action and the energy-momentum tensor' in *Recent Developments in Gravitation*, edited by S. Deser and M. Levy (Plenum, New York, 1979).
[22] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Course of Theoretical Physics, Volume 2, Pergamon Press, New York (1975).