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Generalized Arcangeli's discrepancy principles for a class of regularization methods for solving ill-posed problems

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Abstract — While applying regularization procedures for obtaining stable solutions of ill-posed problems, one of the crucial step is the choice of the regularization parameter. Among the well considered discrepancy principles in the literature, Morozov's method and Arcangeli's method are widely used because of their simplicity for the purpose of applications. Although Morozov's method and their variations have been considered extensively in the literature for general class of regularization methods, the Arcangeli's method is known to have applied only for Tikhonov regularization. The reason could be the belief that it can never yield a rate better than Morozov's procedure, under any smoothness assumption on the solution. However, this belief was misplaced as it has been showed by Nair (1992) that Arcangeli's method do provide the best rate $O(\delta^{2/3})$ for Tikhonov regularization under sufficient smoothness assumption on the solution, while Morozov's method gives the rate only up to $O(\delta^{1/2})$.

The purpose of this paper is to consider a generalized form of Arcangeli's method for a general class of regularization methods for the case when there is no error on the modeling, and then extend the procedure which allow error in the modeling as well.

1. INTRODUCTION

Many inverse problems in science and engineering can be modeled as an operator equation

$$Tx = y, \tag{1.1}$$

where $T : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y. If $R(T)$, the range of T, is not closed then the problem of solving (1.1) is ill-posed in the sense that the generalized solution $\hat{x} = T^{\dagger}y$ is not stable under the

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perturbation in the data y. Here $T^\dagger$ is the generalized inverse of T. A typical example of such ill-posed equation is the Fredholm integral equation of first kind

$$\int_a^b k(s,t)x(t)\,\mathrm{d}t = y(s), \qquad a \le s \le b, \tag{1.2}$$

where $X = Y = L^2[a,b]$ and $k(\,\cdot\,,\,\cdot\,)$ is a non-degenerate kernel belongs to $L^2([a,b] \times [a,b])$.

To obtain a stable approximate solution for the ill-posed equation(1.1), one has to go for regularization methods. In a regularization method, in place of equation (1.1), one considers a family of well-posed equations. A class of such regularization methods can be generated by a family $\{g_\alpha : \alpha > 0\}$ of Borel measurable functions on $[0,b]$ for certain $b > 0$, and taking

$$x_\alpha := g_\alpha(T^*T)T^*y, \quad \alpha > 0,$$

as candidates for the approximation of $T^\dagger y$.

While applying a regularization method in practical situations, what we have at our disposal are approximations T_n, $\tilde{y}$ of T, y respectively. In such situations of *modeling error*, one considers

$$\tilde{x}_{\alpha,n} := g_\alpha(T_n^*T_n)T_n^*\tilde{y}.$$

One of the crucial points in a regularization method is to choose the regularization parameter $\alpha = \alpha(\tilde{y}, n, \delta)$ such that

$$\tilde{x}_{\alpha,n} := g_\alpha(T_n^*T_n)T_n^*\tilde{y} \to T^\dagger y$$

as $\|y - \tilde{y}\| \to 0$ and $\|T - T_n\| \to 0$.

For the above general class of regularization method, many discrepancy principle have been used in the literature as a parameter choice strategy, such as the methods due to Morozov [8], Gfrerer [7] and some of their variants (cf. [2] and references therein).

The purpose of this paper is to use a variant of the generalized Arcangeli's method [1] introduced by Schock [11] for the case of Tikhonov regularization and further investigated by Groetsch and Schock [4], Nair [9], George and Nair ([5, 6]). Although it is shown by Nair [9] that Arcangeli's method can yield the highest possible rate for Tikhonov regularization (cf. also ([5, 6]) no attempt has been made so far to extend this procedure for general class of regularization methods. This paper is the first such successful attempt.

We consider the generalized Arcangeli's discrepancy principle for the case of exact as well as for the approximation T_n of T, namely,

$$\|T\tilde{x}_\alpha - \tilde{y}\| = \delta^p/\alpha^q, \qquad p,\, q > 0$$

and

$$\|T_n\tilde{x}_{\alpha,n} - \tilde{y}\| = (\delta + \epsilon_n)^p/\alpha^q, \qquad p,\, q > 0,$$

respectively, where it is assumed that

$$\|T - T_n\| \leq \epsilon_n, \qquad \|y - \tilde{y}\| \leq \delta.$$

For the convergence and error analysis, we have to impose some conditions on g_α, $\alpha > 0$. First we assume the following on $\{g_\alpha : \alpha > 0\}$ with

$$b \geq \max\{\|T\|^2, \|T_n\|^2\} \qquad \forall n = 1, 2, \ldots.$$

Assumption (1). For some $\nu_0 > 0$ and for $0 \leq \nu \leq \nu_0$, there exists $c_\nu > 0$ such that

$$\sup_{0 \leq \lambda \leq b} \lambda^\nu |1 - \lambda g_\alpha(\lambda)| \leq c_\nu \alpha^\nu \qquad \forall \alpha > 0.$$

Assumption (2). For every $\mu \in [0, 1]$, there exists $d_\mu > 0$ such that

$$\sup_{0 \leq \lambda \leq b} \lambda^{1/2} |g_\alpha(\lambda)| \leq d_\mu \alpha^{-1/2} \qquad \forall \alpha > 0.$$

These assumptions are by now very standard in the literature and general enough to include many regularizations methods such as the ones given below (cf. [10] for the verification of conditions (1) and (2)).

For applying our discrepancy principle, we would like to impose two additional conditions:

Assumption (3). There exists $\alpha_0 > 0$ and $\kappa_0 > 0$ such that

$$|1 - \lambda g_\alpha(\lambda)| \geq \kappa_0 \alpha^{\nu_0}, \qquad \forall \lambda \in [0, b], \quad \forall \alpha \leq \alpha_0.$$

Assumption (4). The function $f(\alpha) = \alpha^q[1 - \lambda g_\alpha(\lambda)]$, as a function of α, is continuous and differentiable and $f'(\alpha)$ is an increasing function.

Now let us list a few regularization methods which are special cases of the above procedure.

(a) Tikhonov Regularization:

$$(T^*T + \alpha I)x_\alpha = T^*y.$$

Here

$$g_\alpha(\lambda) = 1/(\lambda + \alpha).$$

Assumptions (1)–(4) hold with $\nu_0 = 1$, and κ_0 in (3) can be taken as greater than or equal to $1/(\alpha_0 + \|T\|^2)$.

(b) Generalized Tikhonov Regularization:

$$((T^*T)^{q+1} + \alpha^{q+1} I)x_\alpha = (T^*T)^q T^* y.$$

Here

$$g_\alpha(\lambda) = \lambda^q/(\lambda^{q+1} + \alpha^{q+1}),$$

assumptions (1)–(4) hold with $\nu_0 = q + 1$, $q \geq -1/2$, and κ_0 in (3) can be taken as greater than or equal to $1/(\alpha_0^{q+1} + \|T\|^{2(q+1)})$.

(c) Iterated Tikhonov Regularization: In this method, the k-th iterated approximation $x_\alpha^{(k)}$ is calculated from

$$(T^*T + \alpha I)x_\alpha^{(i)} = \alpha x_\alpha^{(i-1)} + T^*y, \quad i = 1, \dots, k,$$

with $x_\alpha^{(0)} = 0$. Here, with

$$g_\alpha(\lambda) = \frac{1}{\lambda}\Big[1 - \Big(\frac{\alpha}{\alpha+\lambda}\Big)^k\Big],$$

assumptions (1)–(4) hold with $\nu_0 = k$ and the constant κ_0 in (3) can be taken as any number greater than or equal to $1/(\alpha_0 + \|T\|^2)^k$.

2. ANALYSIS WITHOUT MODELING ERROR

Let $\{g_\alpha : \alpha > 0\}$ be a family of Borel measurable functions satisfying the assumptions (1), (2) of Section 1, and let

$$x_\alpha = g_\alpha(T^*T)T^*y, \tag{2.1}$$
$$\tilde{x}_\alpha = g_\alpha(T^*T)T^*\tilde{y}, \tag{2.2}$$

where $y \in D(T^\dagger)$, $\|y - \tilde{y}\| \le \delta > 0$. We also assume that $\|y\| > \delta$.

In this section we apply the generalized Arcangeli's method

$$\|T\tilde{x}_\alpha - \tilde{y}\| = \delta^p/\alpha^q, \qquad p, q > 0, \tag{2.3}$$

to choose the regularization parameter α. In order to do this, we consider a general estimate for the error $\|\hat{x} - \tilde{x}_\alpha\|$ in terms of α and δ, where

$$\hat{x} := T^\dagger y.$$

2.1. General error estimate

The following result is available in the literature (cf. Groetsch [3]). For the sake of completeness, we supply its proof as well.

Theorem 2.1. *Suppose $\hat{x} \in R((T^*T)^\nu)$, $0 \le \nu \le \nu_0$ and $\hat{x} = (T^*T)^\nu \hat{u}$ for some $\hat{u} \in X$. Then*

$$\|\hat{x} - x_\alpha\| \le c_\nu \|\hat{u}\|\alpha^\nu, \tag{2.4}$$
$$\|x_\alpha - \tilde{x}_\alpha\| \le d_{1/2}\delta/\sqrt{\alpha}. \tag{2.5}$$

Proof. By the definition of x_α and $\tilde{x}_\alpha$ and spectral theory, we have

$$\begin{aligned}\hat{x} - x_\alpha &= \hat{x} - g_\alpha(T^*T)T^*y = [I - g_\alpha(T^*T)T^*T]\hat{x}\\ &= (T^*T)^\nu[I - g_\alpha(T^*T)T^*T]\hat{u},\end{aligned}$$

$$x_\alpha - \tilde{x}_\alpha = g_\alpha(T^*T)T^*(y - \tilde{y}) = T^*g_\alpha(TT^*)(y - \tilde{y}).$$

Therefore, using the assumptions (1) and (2) on g_α, we get

$$\begin{aligned}
\|\hat{x} - x_\alpha\| &= \|(T^*T)^\nu [I - g_\alpha(T^*T)T^*T]\hat{u}\| \\
&\le \sup_{0\le\lambda\le b} \lambda^\nu |1 - \lambda g_\alpha(\lambda)|\, \|\hat{u}\| \le c_\nu \|\hat{u}\|\alpha^\nu, \\
\|x_\alpha - \tilde{x}_\alpha\| &= \|T^* g_\alpha(TT^*)(y - \tilde{y})\| = \|(TT^*)^{1/2} g_\alpha(TT^*)(y - \tilde{y})\| \\
&\le \sup_{0\le\lambda\le b} \lambda^{1/2} |g_\alpha(\lambda)|\, \|y - \tilde{y}\| \le d_{1/2}\delta/\sqrt{\alpha}.
\end{aligned}$$

Thus the proof is completed. □

2.2. Discrepancy principle

We assume that the family $\{g_\alpha : \alpha > 0\}$ of Borel measurable functions satisfy all the assumptions (1), (2), (3), (4) of Section 1. Further, we assume that

$$y \in R(T), \qquad \|y\| > \delta.$$

We make use of the following notations :

$$r_\alpha(\lambda) := I - \lambda g_\alpha(\lambda), \qquad f(\alpha, \tilde{y}) := \alpha^q \|T\tilde{x}_\alpha - \tilde{y}\|.$$

For functions $\varphi(\delta, n)$ and $\psi(\delta, n)$ we use the notation $\varphi(\delta, n) = O(\psi(\delta, n))$ to state the fact that there exists a positive real number c, independent of the arguments δ, n, such that

$$\varphi(\delta, n) \le c\, \psi(\delta, n).$$

Lemma 2.1. *There exists a unique $\alpha = \alpha(\tilde{y}, \delta) > 0$ satisfying the discrepancy principle* (2.3)*, and*

$$\alpha = O(\delta^{p/(q+\nu_0)}).$$

Proof. It is seen that

$$f(\alpha, \tilde{y}) := \alpha^q \|T\tilde{x}_\alpha - \tilde{y}\| = \alpha^q \|r_\alpha(TT^*)\tilde{y}\|.$$

By assumption (1) on g_α, it follows that $\{\|r_\alpha(TT^*)\| : \alpha > 0\}$ is bounded, and therefore

$$\lim_{\alpha\to 0} f(\alpha, \tilde{y}) = 0, \qquad \lim_{\alpha\to\infty} f(\alpha, \tilde{y}) = \infty.$$

Hence by intermediate value theorem and assumption (4) on g_α, there exists a unique α satisfying the discrepancy principle (2.3). By using the assumption (3) on g_α, and spectral theorem,

$$\begin{aligned}
\|T\tilde{x}_\alpha - \tilde{y}\|^2 = \|r_\alpha(TT^*)\tilde{y}\|^2 &= \int_0^{\|T\|^2} (r_\alpha(\lambda))^2 \,\mathrm{d}\|E_\lambda \tilde{y}\|^2 \\
&\ge \int_0^{\|T\|^2} (\kappa_0 \alpha^{\nu_0})^2 \,\mathrm{d}\|E_\lambda \tilde{y}\|^2 \ge (\kappa_0 \alpha^{\nu_0} \|\tilde{y}\|)^2.
\end{aligned}$$

Since $\|\tilde{y}\| = \|y - (y - \tilde{y})\| \geq \|y\| - \delta$, it follows that

$$\delta^p/\alpha^q = \|T\tilde{x}_\alpha - \tilde{y}\| \geq \kappa_0 \alpha^{\nu_0}(\|y\| - \delta),$$

so that $\alpha = O(\delta^{p/(q+\nu_0)})$. □

Lemma 2.2. *Let $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \leq \nu_0$, and $\hat{u} \in X$ be such that $\hat{x} = (T^*T)^\nu \hat{u}$. Then*

$$\|r_\alpha(TT^*)T\hat{x}\| \leq \hat{c}_\nu \|\hat{u}\| \alpha^\omega,$$

for some $\hat{c}_\nu > 0$, where

$$\omega = \min\{\nu_0, \nu + 1/2\}.$$

Proof. By spectral theory for bounded self adjoint operators and the assumption (1) on the functions g_α, $\alpha > 0$,

$$\begin{aligned} \|r_\alpha(TT^*)T\hat{x}\| &= \|Tr_\alpha(T^*T)\hat{x}\| = \|(T^*T)^{1/2} r_\alpha(T^*T)\hat{x}\| \\ &= \|(T^*T)^{1/2} r_\alpha(T^*T)(T^*T)^\nu \hat{u}\| = \|(T^*T)^{\nu+1/2} r_\alpha(T^*T)\hat{u}\| \\ &\leq \hat{c}_\nu \alpha^\omega. \end{aligned}$$

□

Theorem 2.2. *Suppose $\hat{x} \in R((T^*T)^\nu)$ with $0 < \nu \leq \nu_0$, and α is chosen according to the discrepancy principle* (2.3). *Let*

$$\omega = \min\{\nu_0,\, \nu + 1/2\}, \qquad s = \min\{1,\, p\omega/(q + \nu_0)\}.$$

If $p < 2q + s$, then

$$\begin{aligned} &\|\hat{x} - \tilde{x}_\alpha\| \to 0 \quad \text{as} \quad \delta \to 0, \\ &\|\hat{x} - \tilde{x}_\alpha\| = O(\delta^\mu), \qquad \mu = \min\{p\nu/(q+\nu_0),\, 1 - p/2q + s/2q\}. \end{aligned}$$

Proof. Since $y = T\hat{x}$, it follows from the assumption (1) and Lemma 2.2 that

$$\begin{aligned} \delta^p/\alpha^q &= \|T\tilde{x}_\alpha - \tilde{y}\| = \|r_\alpha(TT^*)\tilde{y}\| \\ &\leq \|r_\alpha(TT^*)(\tilde{y} - y)\| + \|r_\alpha(TT^*)y\| \\ &\leq c_0\delta + \hat{c}_\nu\|\hat{u}\|\alpha^\omega \leq \max\{c_0, \hat{c}_\nu\|\hat{u}\|\}(\delta + \alpha^\omega), \end{aligned}$$

where ω and $\hat{c}_\nu$ are as in Lemma 2.2. Since, by Lemma 2.1, $\alpha = O(\delta^{p/q+\nu_0})$, we get

$$\delta^p/\alpha^q = O(\delta^s), \qquad s = \min\{1, p\omega/(q+\nu_0)\}.$$

Hence

$$\delta/\sqrt{\alpha} = \delta^{1-p/2q}(\delta^p/\alpha^q)^{1/2q} = O(\delta^{1-p/2q+s/2q})$$

Therefore from (2.4) and (2.5) we have

$$\|\hat{x} - \tilde{x}_\alpha\| = O(\delta^\mu), \qquad \mu = \min\{p\nu/(q+\nu_0),\, 1 - p/2q + s/2q\}.$$

From this it also follows that $\|\hat{x} - \tilde{x}_\alpha\| \to 0$ as $n \to \infty$. □

From the above theorem the following result can be easily deduced.

Corollary 2.1. *In addition to the assumptions in Theorem 2.2, suppose*

$$p\omega/(q+\nu_0) \le 1.$$

Then μ in Theorem 2.2 takes the form

$$\mu = \min\left\{\frac{p\nu}{q+\nu_0},\, 1 - \frac{p}{2(q+\nu_0)}\left[1 + \frac{\nu_0-\omega}{q}\right]\right\}. \tag{2.6}$$

Moreover,

$$\|\hat{x} - \tilde{x}_\alpha\| = O(\delta^{p\nu/(q+\nu_0)}) \qquad \textit{whenever} \quad \frac{p}{q+\nu_0} \le \frac{2}{2\nu+1+(\nu_0-\omega)/q}. \tag{2.7}$$

In particular if

$$\nu_0 - 1/2 \le \nu \le \nu_0, \qquad p/(q+\nu_0) = 2/(2\nu_0+1)$$

then

$$\|\hat{x} - \tilde{x}_\alpha\| = O(\delta^{2\nu/(2\nu_0+1)}). \tag{2.8}$$

3. ANALYSIS WITH MODELING ERROR

In this section, we carry out the analysis when there is a modeling error, that is, we have only an approximation T_n of T. We assume that (T_n) is a sequence of bounded operators such that $\|T - T_n\| \le \epsilon_n$, where (ϵ_n) is a sequence of positive real numbers such that $\epsilon_n \to 0$ as $n \to \infty$. In this case, in place of (2.1) and (2.2) we take

$$x_{\alpha,n} = g_\alpha(T_n^*T_n)T_n^*y, \tag{3.1}$$

$$\tilde{x}_{\alpha,n} = g_\alpha(T_n^*T_n)T_n^*\tilde{y}. \tag{3.2}$$

As in (2.5) it is seen that

$$\|x_{\alpha,n} - \tilde{x}_{\alpha,n}\| = \|g_\alpha(T_n^*T_n)T_n^*(y-\tilde{y})\| \le d_{1/2}\delta/\sqrt{\alpha}. \tag{3.3}$$

For choosing the regularization parameter we use the discrepancy principle

$$\|T_n\tilde{x}_{\alpha,n} - \tilde{y}\| = (\delta+\epsilon_n)^p/\alpha^q, \qquad p,\, q > 0. \tag{3.4}$$

3.1. General error estimate

We shall make use of the following result proved in Vanikko and Veretennikov [12].

Lemma 3.1. *Suppose that $A, A_n : X \to X$ are bounded positive self adjoint operators with (A_n) uniformly bounded. Then*

$$\|A^\ell - A_n^\ell\| \le a_\ell\, \|A - A_n\|^{\min\{1,\,\ell\}}, \qquad \ell > 0,$$

where a_ℓ is independent of n.

We make use of the above lemma to deduce the following.

Lemma 3.2. *Let $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le \nu_0$, and $\hat{u} \in X$ be such that $\hat{x} = (T^*T)^\nu \hat{u}$. Then*

$$\|r_\alpha(T^*T)\hat{x}\| \le c_\nu \|\hat{u}\|\alpha^\nu,$$
$$\|r_\alpha(T_n^*T_n)\hat{x}\| \le \|\hat{u}\|(c_\nu\alpha^\nu + 2\,c_0 a_\nu \sqrt{b}\,\epsilon_n^{\min\{\nu,\,1\}}),$$

where c_ν as in assumption (1) and a_ν as in Lemma 3.1.

Proof. By the assumption (1) on the functions g_α, and spectral theory, we have

$$\begin{aligned}\|r_\alpha(T^*T)\hat{x}\| &= \|r_\alpha(T^*T)(T^*T)^\nu\hat{u}\| = \|(T^*T)^\nu r_\alpha(T^*T)\hat{u}\|\\ &\le \sup_{0\le\lambda\le b} \lambda^\nu |r_\alpha(\lambda)|\,\|\hat{u}\| \le c_\nu\|\hat{u}\|\alpha^\nu.\end{aligned}$$

For the next estimate, observe that

$$\begin{aligned}\|r_\alpha(T_n^*T_n)\hat{x}\| &= \|r_\alpha(T_n^*T_n)(T^*T)^\nu\hat{u}\|\\ &\le \|r_\alpha(T_n^*T_n)[(T^*T)^\nu - (T_n^*T_n)^\nu]\hat{u}\| + \|r_\alpha(T_n^*T_n)(T_n^*T_n)^\nu\hat{u}\|.\end{aligned}$$

By assumption (1) on the functions g_α, we have

$$\|r_\alpha(T_n^*T_n)(T_n^*T_n)^\nu\hat{u}\| = \|(T_n^*T_n)^\nu r_\alpha(T_n^*T_n)\hat{u}\| \le c_\nu\alpha^\nu\|\hat{u}\|,$$

and assumption (1) and Lemma 3.1 with $A = T^*T$, $A_n = T_n^*T_n$ and $\ell = \nu$, gives

$$\begin{aligned}&\|r_\alpha(T_n^*T_n)[(T^*T)^\nu - (T_n^*T_n)^\nu]\hat{u}\| \le c_0 a_\nu \|T^*T - T_n^*T_n\|^{\min\{\nu,\,1\}}\|\hat{u}\|\\ &\quad\le c_0 a_\nu\|\hat{u}\| 2\max\{\|T\|,\, \|T_n\|\}\|T - T_n\|^{\min\{\nu,\,1\}} \le 2\,c_0 a_\nu\|\hat{u}\|\sqrt{b}\,\epsilon_n^{\min\{\nu,\,1\}}.\end{aligned}$$

Thus

$$\|r_\alpha(T_n^*T_n)\hat{x}\| \le c_\nu\|\hat{u}\|\alpha^\nu + 2\,c_0 a_\nu\|\hat{u}\|\sqrt{b}\,\epsilon_n^{\min\{\nu,\,1\}}$$

□

Theorem 3.1. *Suppose $\hat{x} \in R((T^*T)^\nu)$, $0 \le \nu \le \nu_0$ and $y \in D(T^\dagger)$. Then*

$$\|\hat{x} - \tilde{x}_{\alpha,n}\| \le c(\alpha^\nu + (\delta + \epsilon_n)/\sqrt{\alpha} + \epsilon_n^{\min\{\nu,\,1\}}), \tag{3.5}$$

where $c > 0$ is independent of α, δ, n.

Proof. With x_α, $\tilde{x}_\alpha$, $x_{\alpha,n}$, $\tilde{x}_{\alpha,n}$ as in (2.1), (2.2), (3.1), (3.2) we have

$$\|\hat{x} - \tilde{x}_{\alpha,n}\| \leq \|\hat{x} - x_\alpha\| + \|x_\alpha - x_{\alpha,n}\| + \|x_{\alpha,n} - \tilde{x}_{\alpha,n}\| \tag{3.6}$$

Recall from (2.4) and (3.3) that

$$\|\hat{x} - x_\alpha\| \leq c_\nu \|\hat{u}\| \alpha^\nu, \qquad \|x_{\alpha,n} - \tilde{x}_{\alpha,n}\| \leq d_{1/2}\delta/\sqrt{\alpha}.$$

Also we note that

$$\begin{aligned} x_\alpha - x_{\alpha,n} &= g_\alpha(T^*T)T^*y - g_\alpha(T_n^*T_n)T_n^*y \\ &= (g_\alpha(T^*T)T^*T - I)\hat{x} + (I - g_\alpha(T_n^*T_n)T_n^*T)\hat{x} \\ &= -r_\alpha(T^*T)\hat{x} + r_\alpha(T_n^*T_n)\hat{x} + g_\alpha(T_n^*T_n)T_n^*(T_n - T)\hat{x}. \end{aligned}$$

Recall from Lemma 3.2 that

$$\begin{aligned} \|r_\alpha(T^*T)\hat{x}\| &\leq c_\nu \|\hat{u}\| \alpha^\nu, \\ \|r_\alpha(T_n^*T_n)\hat{x}\| &\leq \|\hat{u}\|(c_\nu \alpha^\nu + 2\, c_0 a_\nu \sqrt{b}\, \epsilon_n^{\min\{\nu,\,1\}}). \end{aligned}$$

Now, by the assumption (2) and spectral theory we have

$$\begin{aligned} &\|g_\alpha(T_n^*T_n)T_n^*(T_n - T)\hat{x}\| = \|T_n^* g_\alpha(T_nT_n^*)(T_n - T)\hat{x}\| \\ &\quad = \|(T_nT_n^*)^{1/2} g_\alpha(T_nT_n^*)(T_n - T)\hat{x}\| \leq \|(T_nT_n^*)^{1/2} g_\alpha(T_nT_n^*)\|\, \|(T_n - T)\hat{x}\| \\ &\qquad \leq \sup_{0<\lambda\leq b} \lambda^{1/2}|g_\alpha(\lambda)|\, \|(T_n - T)\hat{x}\| \leq d_{1/2}\|\hat{x}\|\epsilon_n/\sqrt{\alpha}. \end{aligned}$$

Thus we get

$$\|x_\alpha - x_{\alpha,n}\| \leq 2\, \|\hat{u}\| c_\nu \alpha^\nu + 2\, \|\hat{u}\| c_0 a_\nu \sqrt{b}\, \epsilon_n^{\min\{\nu,\,1\}} + d_{1/2}\|\hat{x}\|\epsilon_n/\sqrt{\alpha}.$$

Hence from (3.6),

$$\begin{aligned} &\|\hat{x} - \tilde{x}_{\alpha,n}\| \leq 3\, \|\hat{u}\| c_\nu \alpha^\nu + 2\, \|\hat{u}\| c_0 a_\nu \sqrt{b}\, \epsilon_n^{\min\{\nu,\,1\}} + d_{1/2}\|\hat{x}\|\epsilon_n/\sqrt{\alpha} + d_{1/2}\delta/\sqrt{\alpha} \\ &\leq \max\{3\, \|\hat{u}\| c_\nu,\, 2\, \|\hat{u}\| c_0 a_\nu \sqrt{b},\, d_{1/2}\|\hat{x}\| + d_{1/2}\}(\alpha^\nu + \epsilon_n^{\min\{\nu,\,1\}} + (\delta + \epsilon_n)/\sqrt{\alpha}). \end{aligned}$$

□

3.2. Discrepancy principle

By following the procedure in Section 2 and making use of assumptions (1)–(4) on g_α, $\alpha > 0$, it can be shown that there exists a unique $\alpha = \alpha(\tilde{y}, \delta, n)$ satisfying the discrepancy principle (3.4), and

$$\alpha = O((\delta + \epsilon_n)^{p/(q+\nu_0)}).$$

Recall the notation:

$$r_\alpha(\lambda) := 1 - \lambda g_\alpha(\lambda).$$

We shall make use of the following result.

Lemma 3.3. *Let* $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le \nu_0$, *and* $\hat{u} \in X$ *be such that* $\hat{x} = (T^*T)^\nu \hat{u}$. *Then*

$$\|r_\alpha(T_nT_n^*)T_n\hat{x}\| \le 2\, c_{1/2} a_\nu \sqrt{b}\, \|\hat{u}\|\, \alpha^{1/2} \epsilon_n^{\min\{\nu,\,1\}} + \hat{c}_\nu \|\hat{u}\| \alpha^\omega,$$

where $\hat{c}_\nu$ *as in Lemma 2.2 and* a_ν *as in Lemma 3.1.*

Proof. We observe that

$$\begin{aligned}
\|r_\alpha(T_nT_n^*)T_n\hat{x}\| &= \|T_n r_\alpha(T_n^*T_n)\hat{x}\| = \|(T_n^*T_n)^{1/2} r_\alpha(T_n^*T_n)\hat{x}\| \\
&= \|(T_n^*T_n)^{1/2} r_\alpha(T_n^*T_n)(T^*T)^\nu \hat{u}\| \\
&\le \|(T_n^*T_n)^{1/2} r_\alpha(T_n^*T_n)[(T^*T)^\nu - (T_n^*T_n)^\nu]\hat{u}\| \\
&\quad + \|(T_n^*T_n)^{1/2} r_\alpha(T_n^*T_n)(T_n^*T_n)^\nu \hat{u}\|.
\end{aligned}$$

Now by the assumption (1) and Lemma 3.1 with $A = T^*T$ and $B = T_n^*T_n$, we get

$$\begin{aligned}
&\|(T_n^*T_n)^{1/2} r_\alpha(T_n^*T_n)[(T^*T)^\nu - (T_n^*T_n)^\nu]\hat{u}\| \\
&\le c_{1/2}\alpha^{1/2} a_\nu \|T^*T - T_n^*T_n\|^{\min\{\nu,\,1\}}\, \|\hat{u}\| \\
&\le c_{1/2}\alpha^{1/2} a_\nu \max\{\|T\|, \|T_n\|\}\, \epsilon_n^{\min\{\nu,\,1\}}\, \|\hat{u}\| \le c_{1/2}\alpha^{1/2} a_\nu 2\sqrt{b}\, \epsilon_n^{\min\{\nu,\,1\}}\, \|\hat{u}\|
\end{aligned}$$

Also, by the assumption (1),

$$\|(T_n^*T_n)^{1/2} r_\alpha(T_n^*T_n)(T_n^*T_n)^\nu]\hat{u}\| = \|(T_n^*T_n)^{\nu+1/2} r_\alpha(T_n^*T_n)\hat{u}\| \le \hat{c}_\nu \|\hat{u}\| \alpha^\omega.$$

Therefore

$$\|r_\alpha(T_nT_n^*)T_n\hat{x}\| \le 2\, c_{1/2} a_\nu \sqrt{b}\, \alpha^{1/2} \epsilon_n^{\min\{\nu,\,1\}}\, \|\hat{u}\| + \hat{c}_\nu \|\hat{u}\| \alpha^\omega.$$

□

Theorem 3.2. *Suppose* $y \in R(T)$, $\hat{x} \in R((T^*T)^\nu)$ *with* $0 < \nu \le \nu_0$, *and* α *is chosen according to the discrepancy principle* (3.4). *Let*

$$\omega = \min\{\nu_0,\, \nu + 1/2\}, \quad \eta = \min\{1,\, p\omega/(q+\nu_0),\, p/(2(q+\nu_0)) + \min\{\nu, 1\}\}.$$

If $p < 2q + \eta$, *then*

$$\begin{aligned}
&\|\hat{x} - \tilde{x}_{\alpha,n}\| \to 0 \quad \text{as} \quad \delta \to 0, \quad n \to \infty, \\
&\|\hat{x} - \tilde{x}_{\alpha,n}\| = O((\delta + \epsilon_n)^\ell + \epsilon_n^{\min\{\nu,\,1\}}), \;\; \ell = \min\{p\nu/(q+\nu_0), 1 - p/2q + \eta/2q\}.
\end{aligned}$$

Proof. Using Lemma 3.3 and assumption (1) we have,

$$\begin{aligned}
(\delta + \epsilon_n)^p/\alpha^q &= \|T_n\tilde{x}_{\alpha,n} - \tilde{y}\| = \|r_\alpha(T_nT_n^*)\tilde{y}\| \\
&\le \|r_\alpha(T_nT_n^*)(\tilde{y} - y)\| + \|r_\alpha(T_nT_n^*)y\| \le c_0\delta + \|r_\alpha(T_nT_n^*)T\hat{x}\| \\
&\le c_0\delta + \|r_\alpha(T_nT_n^*)T_n\hat{x}\| + \|r_\alpha(T_nT_n^*)(T - T_n)\hat{x}\| \\
&\le c_0\delta + \|r_\alpha(T_nT_n^*)T_n\hat{x}\| + c_0\|\hat{x}\|\epsilon_n.
\end{aligned}$$

Now, using the estimate for $\|r_\alpha(T_nT_n^*)T_n\hat{x}\|$, we have

$$(\delta+\epsilon_n)^p/\alpha^q \le c_0\delta + 2\,c_{1/2}a_\nu\sqrt{b}\|\hat{u}\|\alpha^{1/2}\epsilon_n^{\min\{\nu,\,1\}} + \hat{c}_\nu\|\hat{u}\|\alpha^\omega + c_0\|\hat{x}\|\epsilon_n$$
$$\le \max\{c_0,\, c_0\|\hat{x}\|,\, \hat{c}_\nu\|\hat{u}\|,\, 2\,c_{1/2}a_\nu\sqrt{b}\|\hat{u}\|\}(\delta+\epsilon_n+\alpha^\omega+\alpha^{1/2}\epsilon_n^{\min\{\nu,\,1\}})$$

Since $\alpha = O(\delta+\epsilon_n)^{p/(q+\nu_0)})$, from the above inequality we obtain

$$(\delta+\epsilon_n)^p/\alpha^q = O((\delta+\epsilon_n)^\eta),$$

where

$$\eta = \min\{1,\, p\omega/(q+\nu_0),\, p/(2(q+\nu_0)) + \min\{\nu,1\}\}.$$

Hence

$$(\delta+\epsilon_n)/\sqrt{\alpha} = (\delta+\epsilon_n)^{1-p/2q}[(\delta+\epsilon_n)^p/\alpha^q]^{1/2q} = O((\delta+\epsilon_n)^{1-p/2q+\eta/2q}).$$

Therefore (3.5) becomes

$$\|\hat{x}-\tilde{x}_{\alpha,n}\| \le c[(\delta+e_n)^{p\nu/(q+\nu_0)} + (\delta+\epsilon_n)^{1-p/2q+s/2q} + \epsilon_n^{\min\{\nu,\,1\}}]$$

so that if $p < 2q+\eta$, then $1-p/2q+\eta/2q > 0$ and hence

$$\|\hat{x}-\tilde{x}_{\alpha,n}\| = O((\delta+\epsilon_n)^\ell + \epsilon_n^{\min\{\nu,1\}}),$$

where $\ell = \min\{p\nu/(q+\nu_0),\, 1-p/2q+\eta/2q\}$. From this it also follows that

$$\|\hat{x}-\tilde{x}_{\alpha,n}\| \to 0 \quad \text{as} \quad \delta\to 0, \quad n\to\infty.$$

□

Recall that the quantity ℓ in the above theorem involves the quantities

$$\omega := \min\left\{\nu_0,\, \nu+\frac{1}{2}\right\} \quad \text{and} \quad \eta := \min\left\{1, \frac{p\omega}{q+\nu_0}, \frac{p}{2(q+\nu_0)} + \min\{\nu,1\}\right\}.$$

Suppose ν is such that $\nu+1/2 \le \nu_0$. Then

$$\frac{p\omega}{q+\nu_0} = \frac{p}{q+\nu_0}\Big(\nu+\frac{1}{2}\Big),$$

so that in this case

$$\frac{p\omega}{q+\nu_0} \le \frac{p}{2(q+\nu_0)} + \min\{\nu,\,1\} \quad \text{whenever} \quad \frac{p}{q+\nu_0} \le \min\left\{1, \frac{1}{\nu}\right\}.$$

Next suppose that ν is such that $\nu+1/2 \ge \nu_0$. In this case observe that

$$\frac{p\omega}{q+\nu_0} = \frac{p\nu_0}{q+\nu_0} \le \frac{p}{q+\nu_0}\Big(\nu+\frac{1}{2}\Big).$$

Hence, in this case also,

$$\frac{p\omega}{q+\nu_0} \le \frac{p}{2(q+\nu_0)} + \min\{\nu,\,1\} \quad \text{whenever} \quad \frac{p}{q+\nu_0} \le \min\left\{1, \frac{1}{\nu}\right\}.$$

Thus we can conclude that, if

$$\frac{p}{q+\nu_0} \le \min\Big\{1, \frac{1}{\nu}\Big\},$$

then

$$\frac{p\omega}{q+\nu_0} \le \frac{p}{2(q+\nu_0)} + \min\{\nu,\, 1\},$$

and consequently,

$$\eta = \min\Big\{1,\, \frac{p\omega}{q+\nu_0}\Big\}.$$

Again, we have

$$\frac{p\omega}{q+\nu_0} \le 1 \quad \text{provided} \quad \frac{p}{q+\nu_0} \le \min\Big\{\frac{1}{\nu_0},\, \frac{2}{2\nu+1}\Big\}.$$

Thus, if

$$\frac{p}{q+\nu_0} \le \min\Big\{1,\, \frac{1}{\nu_0},\, \frac{2}{2\nu+1}\Big\},$$

then

$$\eta = \frac{p\omega}{q+\nu_0}, \qquad 1 - \frac{p}{2q} + \frac{\eta}{2q} = 1 - \frac{p}{2(q+\nu_0)}\Big[1 + \frac{\nu_0 - \omega}{q}\Big],$$

consequently

$$\ell = \min\Big\{\frac{p\nu}{q+\nu_0},\, 1 - \frac{p}{2(q+\nu_0)}\Big[1 + \frac{\nu_0-\omega}{q}\Big]\Big\}.$$

Summing up the above observations, from Theorem 3.2, we obtain the following result.

Corollary 3.1. *Suppose $y \in R(T)$, $\hat{x} \in R((T^*T)^\nu)$ with $0 < \nu \le \nu_0$, and α is chosen according to the discrepancy principle* (3.4)*. If in addition*

$$\frac{p}{q+\nu_0} \le \min\Big\{1,\, \frac{1}{\nu_0},\, \frac{2}{2\nu+1}\Big\}, \tag{3.7}$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,n}\| = O((\delta+\epsilon_n)^\ell), \quad \ell = \min\Big\{\frac{p\nu}{q+\nu_0},\, 1 - \frac{p}{2(q+\nu_0)}\Big[1 + \frac{\nu_0-\omega}{q}\Big]\Big\}.$$

In particular, the above conclusion holds if the condition (3.7) *is replaced by*

$$\frac{p}{q+\nu_0} \le \min\Big\{1,\, \frac{1}{\nu_0},\, \frac{2}{2\nu_0+1}\Big\}.$$

Some special cases of the above corollary are listed in the following.

Corollary 3.2. *Suppose $y \in R(T)$, $\hat{x} \in R((T^*T)^\nu)$ with $0 < \nu \le \nu_0$, and α is chosen according to the discrepancy principle* (3.4)*. If*

$$\frac{p}{q+\nu_0} \le \min\Big\{1,\, \frac{1}{\nu_0},\, \frac{2}{2\nu+1+(\nu_0-\omega)/q}\Big\}, \tag{3.8}$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,n}\| = O((\delta + \epsilon_n)^\ell), \qquad \ell = p\nu/(q + \nu_0).$$

In particular, the above conclusion holds if the condition (3.8) *is replaced by*

$$\frac{p}{q+\nu_0} \le \min\left\{1, \frac{2}{2\nu+1}\right\}, \qquad 0 \le \nu_0 - \frac{1}{2} \le \nu \le \nu_0$$

or

$$\frac{p}{q+\nu_0} \le \frac{2}{2\nu_0+1}, \qquad \nu_0 \ge \frac{1}{2}.$$

Proof. It can be seen that

$$\frac{p\nu}{q+\nu_0} \le 1 - \frac{p}{2(q+\nu_0)}\left[1 + \frac{\nu_0 - \omega}{q}\right]$$

if and only if

$$\frac{p}{q+\nu_0} \le \min\left\{1, \frac{1}{\nu_0}, \frac{2}{2\nu+1+(\nu_0-\omega)/q}\right\}$$

so that under the conditions of the corollary, the quantity ℓ in Corollary 3.1 takes the form

$$\ell = \frac{p\nu}{q+\nu_0}.$$

The first part of the particular case follows, since

$$\nu_0 - \frac{1}{2} \le \nu \le \nu_0 \quad \text{implies} \quad \omega = \nu_0, \quad \frac{2}{2\nu+1} \le \frac{1}{\nu_0}\,.$$

For the second part of the particular case, we observe the following:

$$\nu_0 - \frac{1}{2} \le \nu \le \nu_0 \quad \text{implies} \quad \frac{2}{2\nu_0+1} \le \frac{2}{2\nu+1} \le \frac{1}{\nu_0}\,,$$

$$\nu_0 - \frac{1}{2} \ge \nu \le \nu_0 \quad \text{implies} \quad \frac{2}{2\nu_0+1} \le \frac{1}{\nu_0} \le \frac{2}{2\nu+1}\,.$$

□

Remark. Note that by Corollary 3.2, if $\nu_0 \ge 1/2$, then we obtain the rate

$$\|\hat{x} - \tilde{x}_{\alpha,n}\| = O((\delta + \epsilon_n)^{2\nu/(2\nu_0+1)})$$

whenever

$$\frac{p}{q+\nu_0} = \frac{2}{2\nu_0+1} \quad \text{and} \quad \hat{x} \in R((T^*T)^\nu), \quad 0 < \nu \le \nu_0$$

resulting in the best optimal order $O((\delta + \epsilon_n)^{2\nu_0/(2\nu_0+1)})$ for the case $\nu = \nu_0$. Note that the condition $p/(q+\nu_0) = 2/(2\nu_0+1)$ includes the Arcangeli's method, i.e., the case $p = 1,\ q = 1/2$.

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