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Frames and Riesz bases of twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$

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In this paper, twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$ are introduced. Characterizations of orthonormal system and Bessel sequence of twisted translates in $L^2(\mathbb{R}^{2n})$ are obtained in terms of Weyl transform. These results appear to be totally different from the classical case of characterizations of orthonormal systems and Bessel sequences of translates in $L^2(\mathbb{R}^n)$. Further, characterizations of frames, Riesz basis in twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$ are also obtained.

1. Introduction

A closed subspace $V \subset L^2(\mathbb{R})$ is called shift-invariant if $f \in V \Longrightarrow \tau_k f \in V$ for any $k \in \mathbb{Z}$. Shift-invariant spaces play an important role in modern analysis for the past two decades because of their rich underlying theory and their applications in time frequency analysis, approximation theory, numerical analysis, digital signal and image processing and so on. A shift-invariant space serves as a universal model for sampling problem as it includes a large class of functions whether bandlimited or not by appropriately choosing a generator. For a study of sampling in shift-invariant spaces, we refer to [1].

Shift-invariant spaces are also useful in obtaining a decomposition of a large class of functions. One such illustration is from the theory of wavelets, where $L^2(\mathbb{R})$ is decomposed into shift-invariant subspaces using the procedure called multiresolution analysis. A multiresolution analysis is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ such that

(1) $V_j \subseteq V_{j-1}\ \forall j \in \mathbb{Z}$.
(2) $\overline{\bigcup V_j} = L^2(\mathbb{R})$ and $\bigcap V_j = \{0\}$.

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(3) $f \in V_j \Longleftrightarrow f(2x) \in V_{j-1}$.
(4) $f \in V_j \Longleftrightarrow f(x-k) \in V_j \ \forall\, k \in \mathbb{Z}$.
(5) There exists a scaling function $\varphi \in V_0$ such that $\{\varphi(x-k) : k \in \mathbb{Z}\}$ forms an orthonormal basis for V_0.

Let W_j denote the orthogonal complement of V_j in V_{j-1}. Then one can show that $L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j$. Here the spaces V_j, W_j are shift-invariant and V_0 is usually called the principal shift-invariant space. We refer to [8,12] for further details. Characterizations of shift-invariant spaces in $L^2(\mathbb{R}^n)$ in terms of range functions was studied by Bownik in [4].

A well known result on translates of a function in $L^2(\mathbb{R})$ states that the collection $\{\tau_k\varphi : k \in \mathbb{Z}\}$ forms an orthonormal system in $L^2(\mathbb{R})$ iff $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi+k)|^2 = 1$ a.e. $\xi \in \mathbb{T}$. Similarly there are interesting characterizations of a Bessel sequence, frame, Riesz basis of shift-invariant spaces in terms of Fourier transform. We refer to [6,11] in this direction. Frames were first studied by Duffin and Schaeffer in [9] in connection with non-uniform sampling of bandlimited functions. The study of shift-invariant spaces and frames have been extended to locally compact abelian groups in [5,13] and non-abelian compact group in [14].

A simple and a natural example of a non-abelian non-compact group is the famous Heisenberg group $\mathbb{H}^n$. It is a nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$, where the group operation is defined by $(z,t)(w,s) = (z+w, t+s+\frac{1}{2}\mathrm{Im} z.\overline{w})$ and the Haar measure is the Lebesgue measure $\mathrm{d}z\mathrm{d}t$ on $\mathbb{C}^n \times \mathbb{R}$. By Stone–von Neumann theorem, every infinite dimensional irreducible unitary representation on the Heisenberg group is unitarily equivalent to the representation $\pi_\lambda, \lambda \in \mathbb{R}^\star$, where π_λ is defined by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{2\pi i\lambda t} e^{2\pi i\lambda(x.\xi + \frac{1}{2}x.y)}\varphi(\xi+y),$$

where $z = x+iy$ and $\varphi \in L^2(\mathbb{R}^n)$. In order to study shift-invariant spaces on $\mathbb{H}^n$, we need to make use of the representation theory of $\mathbb{H}^n$. The group Fourier transform on $\mathbb{H}^n$ is defined to be

$$\widehat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)dzdt$$

for $f \in L^1(\mathbb{H}^n)$. Further $L^1(\mathbb{H}^n)$ turns out to be a non-commutative Banach algebra under group convolution $*$. Define $f^\lambda(z) = \int_{\mathbb{R}} f(z,t)e^{2\pi i\lambda t}dt$ to be the inverse Fourier transform of f in the t-variable. Thus $\widehat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z,0)dz$. Hence it becomes natural to consider the operator of the form

$$W_\lambda(f) = \int_{\mathbb{C}^n} f(z)\pi_\lambda(z,0)dz.$$

Further since $\widehat{f}(\lambda) = W_\lambda(f^\lambda)$, for the t-variable the group Fourier transform is nothing but the Euclidean Fourier transform. In many problems on $\mathbb{H}^n$, an important technique is to take the partial Fourier transform in the t-variable to reduce matters to the case of $\mathbb{C}^n$. We can also easily see that

$$(f*g)^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z-w)g^\lambda(w)e^{\frac{i\lambda}{2}\mathrm{Im} z.\overline{w}}dw.$$

The right hand side is called the λ-twisted convolution of f^λ and g^λ. Now writing $\pi_1(z,0) = \pi(z)$, we can define the Weyl transform $W(f)$ as

$$W(f) = \int_{\mathbb{C}^n} f(z)\pi(z)dz$$

and for $f, g \in L^1(\mathbb{C}^n)$ the twisted convolution is defined to be

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\pi i Imz.\overline{w}}dw.$$

Then $L^1(\mathbb{C}^n)$ turns out to be non-commutative Banach algebra under twisted convolution. The Weyl transform of a function $f \in L^1(\mathbb{C}^n)$ can be explicitly written as

$$W(f)\varphi(\xi) = \int_{\mathbb{C}^n} f(z)e^{2\pi i(x.\xi+\frac{1}{2}x.y)}\varphi(\xi+y)dz, \quad \varphi \in L^2(\mathbb{R}^n), \quad z = x+iy,$$

which maps $L^1(\mathbb{C}^n)$ into the space of bounded operators on $L^2(\mathbb{R}^n)$, denoted by $\mathcal{B}(L^2(\mathbb{R}^n))$. The Weyl transform $W(f)$ is an integral operator with kernel $K_f(\xi,\eta)$ given by

$$\int_{\mathbb{R}^n} f(x,\eta-\xi)e^{i\pi x\cdot(\xi+\eta)}dx.$$

This map W can be uniquely extended to a bijection from the class of tempered distributions $S'(\mathbb{C}^n)$ onto the space of continuous linear maps from $S(\mathbb{R}^n)$ into $S'(\mathbb{R}^n)$. Weyl transform has properties similar to that of ordinary Fourier transform. For example, if $f \in L^2(\mathbb{C}^n)$, then $W(f) \in \mathcal{B}_2(L^2(\mathbb{R}^n))$, the space of Hilbert–Schmidt operators and satisfies the Plancherel formula

$$\|W(f)\|_{\mathcal{B}_2} = \|f\|_{L^2(\mathbb{C}^n)}.$$

In fact, for $f, g \in L^2(\mathbb{C}^n)$, we have

$$\langle W(f), W(g)\rangle_{\mathcal{B}_2} = \langle f, g\rangle_{L^2(\mathbb{C}^n)} = \langle K_f, K_g\rangle_{L^2(\mathbb{C}^n)}. \tag{1.1}$$

The inversion formula is given by

$$f(z) = tr(\pi(z)^* W(f)).$$

We refer to [10,15] for further information.

In [2], the authors considered the polarized Heisenberg group $\mathbb{H}^n_{pol} = \mathbb{C}^n \times \mathbb{R}$, where the group operation is defined by

$$(z,s)(w,t) = (z+w, s+t+(Rew).(Imz))$$

and studied the characterizations of orthonormal basis, Riesz basis and frames consisting of the left translates on the group $\mathbb{H}^n_{pol}$ by introducing a bracket map on $\mathbb{H}^n_{pol}$ using the group Fourier transform $\widehat{f}$ and integer periodization of $\widehat{f}(\lambda)$ with respect to the variable λ on $\mathbb{R}^\star$, which is the unitary dual of $\mathbb{H}^n_{pol}$. In [7], the authors generalized some results of [4] to shift-invariant spaces associated with a class of nilpotent Lie groups.

The concept of the bracket map has been generalized in [3] to include non-abelian discrete group Γ using its unitary representations and L^1 space over the non-commutative measurable space vNa(Γ), which is the compact dual of Γ whose underlying space is a group von Neumann algebra. Using this bracket map characterizations of orthonormal basis, Riesz basis, frames were obtained for shift-invariant spaces in a Hilbert space $\mathcal{H}$ given by the action of a non-abelian countable discrete group Γ.

In this paper we wish to obtain characterizations of orthonormal basis, frames, Riesz basis etc. of twisted translates in terms of Weyl transform. Further, here we consider the integer periodization in the variable ξ of the kernel $K(\xi, \eta)$ of the Weyl transform. From the Heisenberg point of view, this approach is based on periodization associated with projective representation of $\mathbb{C}^n$ instead of looking into integer periodization in the unitary dual of the group which corresponds to the central variable as in [2].

As mentioned earlier, the aim of this paper is to introduce twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$ and obtain characterizations of frames, Riesz basis etc. in terms of Weyl transform. Unlike the classical case, it is surprising to notice that the results become totally different when we move to this setting. A characterization like $\{\tau_k\varphi : k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$ iff $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + k)|^2 = 1$ a.e. $\xi \in \mathbb{T}$ becomes false in our setting. In fact we show that if the twisted translates of φ form an orthonormal system in $L^2(\mathbb{R}^{2n})$, then $\sum_{m\in\mathbb{Z}^n} \int_{\eta\in\mathbb{R}^n} |K_\varphi(\xi + m, \eta)|^2 d\eta = 1$ a.e. $\xi \in \mathbb{T}^n$. But the converse need not be true. This is illustrated with a counter example. Then naturally we go in for an additional condition, which we call "condition C" to obtain the required result in the converse direction.

The paper is organized as follows. In Section 2, we provide the required definitions and prove the basic results. In Section 3, we obtain characterizations of orthonormal system and Bessel sequence of twisted translates in terms of Weyl transform in $L^2(\mathbb{R}^{2n})$. In Section 4, we study Parseval frames in twisted shift-invariant spaces. We also obtain a decomposition theorem for a twisted shift-invariant space in $L^2(\mathbb{R}^{2n})$ using the Parseval frames. In Section 5, we study frames, canonical dual frames and Riesz basis in twisted shift-invariant spaces.

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space.

Definition 2.1. A sequence $\{f_k : k \in \mathbb{Z}\}$ in $\mathcal{H}$ is called a Bessel sequence for $\mathcal{H}$ if there exists a constant $B > 0$ such that

$$\sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Definition 2.2. A sequence $\{f_k : k \in \mathbb{Z}\}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

In particular if $A = B = 1$, then $\{f_k : k \in \mathbb{Z}\}$ is called a Parseval frame. Let $S : \mathcal{H} \longrightarrow \mathcal{H}$ be defined by $Sf := \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle f_k$. Then S is bounded, invertible, self adjoint and positive. Further, $\{S^{-1}f_k : k \in \mathbb{Z}\}$ is also a frame for $\mathcal{H}$ and is called the canonical dual frame of $\{f_k : k \in \mathbb{Z}\}$.

Definition 2.3. A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ue_k : k \in \mathbb{Z}\}$ where $\{e_k : k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded invertible operator. Equivalently, a Riesz basis is a sequence

$\{f_k : k \in \mathbb{Z}\}$ which is complete in $\mathcal{H}$ and there exist constants $A, B > 0$ such that for every finite scalar sequence $\{c_k\}$, one has

$$A \sum_k |c_k|^2 \leq \left\| \sum_k c_k f_k \right\|^2 \leq B \sum_k |c_k|^2 . \tag{2.1}$$

It is to be noted that if (2.1) holds for all finite scalar sequences, then it holds for all $\{c_k : k \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$.

Definition 2.4. Let $\varphi \in L^2(\mathbb{R}^{2n})$. For $(k,l) \in \mathbb{Z}^{2n}$, we define twisted translation of φ, denoted by $T^t_{(k,l)}\varphi$, as

$$T^t_{(k,l)}\varphi(x,y) = e^{\pi i(x.l - y.k)}\varphi(x-k, y-l), \quad (x,y) \in \mathbb{R}^{2n}.$$

Now, we shall introduce twisted shift-invariant spaces.

Definition 2.5. For $\varphi \in L^2(\mathbb{R}^{2n})$, we define the twisted shift-invariant space of φ, denoted by $V^t(\varphi)$, as $\overline{span}\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ in $L^2(\mathbb{R}^{2n})$.

2.1. Properties of twisted translation

(1) The adjoint $(T^t_{(k,l)})^*$ of $T^t_{(k,l)}$ is $T^t_{(-k,-l)}$.
(2) $T^t_{(k_1,l_1)}T^t_{(k_2,l_2)} = e^{-\pi i(k_1.l_2 - k_2.l_1)}T^t_{(k_1+k_2,l_1+l_2)}$.
(3) $T^t_{(k,l)}$ is a unitary operator on $L^2(\mathbb{R}^{2n})$ for all $(k,l) \in \mathbb{Z}^{2n}$.
(4) The Weyl transform of $T^t_{(k,l)}\varphi$ is given by $W(T^t_{(k,l)}\varphi) = \pi(k,l)W(\varphi)$.

Proof of (4). Let $f \in L^2(\mathbb{R}^{2n})$. Then

$$\begin{aligned}
&\pi(k,l)W(\varphi)f(\xi) \\
&= e^{2\pi i(k.\xi + \frac{1}{2}k.l)}W(\varphi)f(\xi + l) \\
&= e^{2\pi i(k.\xi + \frac{1}{2}k.l)} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \varphi(x,y)e^{2\pi i[x.(\xi+l) + \frac{1}{2}x.y]}f(\xi + l + y)dxdy \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\pi i[(x+k).l - (y+l).k]}\varphi(x,y)e^{2\pi i[(x+k).\xi + \frac{1}{2}(x+k).(y+l)]}f(\xi + l + y)dxdy \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\pi i(x.l - y.k)}\varphi(x-k, y-l)e^{2\pi i(x.\xi + \frac{1}{2}x.y)}f(\xi + y)dxdy \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} T^t_{(k,l)}\varphi(x,y)\pi(z)f(\xi)dxdy \\
&= W(T^t_{(k,l)}\varphi)f(\xi). \qquad \square
\end{aligned}$$

We refer to §1.3 in [15] in this connection.

(5) For $\varphi \in L^2(\mathbb{R}^{2n})$, let $F(x,y) = e^{2\pi i x.l}T^t_{(0,k-l)}\varphi(x,y)$, $(x,y) \in \mathbb{R}^{2n}$. Then $K_\varphi(\xi + k, \eta + l) = K_F(\xi, \eta)$, where K_φ and K_F denote the kernel of $W(\varphi)$ and $W(F)$ respectively. In particular, putting $l = 0$, we have

$$K_\varphi(\xi + k, \eta) = K_{T^t_{(0,k)}\varphi}(\xi, \eta). \tag{2.2}$$

Proof.

$$\begin{aligned}K_F(\xi,\eta) &= \int\limits_{\mathbb{R}^n} F(x,\eta-\xi)e^{\pi ix.(\xi+\eta)}dx \\ &= \int\limits_{\mathbb{R}^n} e^{2\pi ix.l}T^t_{(0,k-l)}\varphi(x,\eta-\xi)e^{\pi ix.(\xi+\eta)}dx \\ &= \int\limits_{\mathbb{R}^n} e^{2\pi ix.l}e^{\pi ix.(k-l)}\varphi(x,\eta-\xi+l-k)e^{\pi ix.(\xi+\eta)}dx \\ &= \int\limits_{\mathbb{R}^n} \varphi(x,\eta+l-\xi-k)e^{\pi ix.(\eta+l+\xi+k)}dx \\ &= K_\varphi(\xi+k,\eta+l). \qquad \square\end{aligned}$$

The property (5) especially (2.2) will be used frequently in the sequel. The following lemmas will be useful in due course.

Lemma 2.1. *Let $\varphi \in L^2(\mathbb{R}^{2n})$. Then the kernel of the Weyl transform of $T^t_{(k,l)}\varphi$ satisfies the following relation. $K_{T^t_{(k,l)}\varphi}(\xi,\eta) = e^{\pi i(2\xi+l).k}K_\varphi(\xi+l,\eta)$. In other words,*

$$e^{2\pi ik.\xi}K_\varphi(\xi+l,\eta) = e^{-\pi il.k}K_{T^t_{(k,l)}\varphi}(\xi,\eta). \tag{2.3}$$

Proof. Consider

$$\begin{aligned}K_{T^t_{(k,l)}\varphi}(\xi,\eta) &= \int\limits_{\mathbb{R}^n} T^t_{(k,l)}\varphi(x,\eta-\xi)e^{\pi ix.(\xi+\eta)}dx \\ &= \int\limits_{\mathbb{R}^n} e^{\pi i[x.l-(\eta-\xi).k]}\varphi(x-k,\eta-\xi-l)e^{\pi ix.(\xi+\eta)}dx \\ &= \int\limits_{\mathbb{R}^n} e^{\pi i[(2\xi+l).k+x.(\xi+\eta+l)]}\varphi(x,\eta-\xi-l)dx \\ &= e^{\pi i(2\xi+l).k}K_\varphi(\xi+l,\eta).\end{aligned}$$

This completes the proof. $\square$

For $\varphi \in L^2(\mathbb{R}^{2n})$, we define

$$w_\varphi(\xi) = \sum_{m\in\mathbb{Z}^n}\int\limits_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2 d\eta, \quad \xi \in \mathbb{R}^n. \tag{2.4}$$

Lemma 2.2. *The function w_φ, defined in (2.4), is 1-periodic and it belongs to $L^1(\mathbb{T}^n)$. In particular w_φ is finite a.e. $\xi \in \mathbb{T}^n$. Its Fourier coefficients are $\widehat{w_\varphi}(k) = \langle \varphi, T^t_{(k,0)}\varphi\rangle$, $k \in \mathbb{Z}^n$.*

Proof. Clearly w_φ is 1-periodic. Now

$$\begin{aligned}\int\limits_{\mathbb{T}^n} w_\varphi(\xi)d\xi &= \int\limits_{\mathbb{T}^n}\sum_{m\in\mathbb{Z}^n}\int\limits_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2 d\eta d\xi \\ &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} |K_\varphi(\xi,\eta)|^2 d\eta d\xi \\ &= \|\varphi\|_2^2 < \infty.\end{aligned}$$

Now, using (2.3), the Fourier coefficients are given by

$$\begin{aligned}
\widehat{w_\varphi}(k) &= \int\limits_{\mathbb{T}^n} w_\varphi(\xi) e^{-2\pi i k.\xi} d\xi \\
&= \int\limits_{\mathbb{T}^n} \left(\sum_{m\in\mathbb{Z}^n} \int\limits_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2 d\eta \right) e^{-2\pi i k.(\xi+m)} d\xi \\
&= \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} |K_\varphi(\xi,\eta)|^2 e^{-2\pi i k.\xi} d\xi d\eta \\
&= \langle K_\varphi, K_{T^t_{(k,0)}\varphi} \rangle \\
&= \langle \varphi, T^t_{(k,0)}\varphi \rangle.
\end{aligned}$$

This completes the proof. □

3. Orthonormal system and Bessel sequence of twisted translates in $L^2(\mathbb{R}^{2n})$

Theorem 3.1. *If $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ forms an orthonormal system in $L^2(\mathbb{R}^{2n})$, then $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$.*

Proof. Let $\varphi \in L^2(\mathbb{R}^{2n})$. Using Plancherel formula (1.1) and Lemma 2.1, we get

$$\begin{aligned}
&\langle T^t_{(k_1,l_1)}\varphi, T^t_{(k_2,l_2)}\varphi \rangle \\
&= \langle \varphi, (T^t_{(k_1,l_1)})^* T^t_{(k_2,l_2)}\varphi \rangle \\
&= e^{-\pi i (k_1.l_2 - k_2.l_1)} \langle \varphi, T^t_{(k_2-k_1,l_2-l_1)}\varphi \rangle \\
&= e^{-\pi i (k_1.l_2 - k_2.l_1)} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} K_\varphi(\xi,\eta) e^{-\pi i (2\xi + l_2 - l_1).(k_2-k_1)} \overline{K_\varphi(\xi + l_2 - l_1, \eta)} d\xi d\eta \\
&= e^{-\pi i [(l_2-l_1).(k_2-k_1) + (k_1.l_2 - k_2.l_1)]} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} K_\varphi(\xi,\eta) \overline{K_\varphi(\xi + l_2 - l_1, \eta)} e^{-2\pi i (k_2-k_1).\xi} d\xi d\eta \\
&= e^{-\pi i [(l_2-l_1).(k_2-k_1) + (k_1.l_2 - k_2.l_1)]} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n} K_\varphi(\xi+m,\eta) \overline{K_\varphi(\xi + m + l_2 - l_1, \eta)} \\
&\qquad \times e^{-2\pi i (k_2-k_1).(\xi+m)} d\xi d\eta \\
&= e^{-\pi i [(l_2-l_1).(k_2-k_1) + (k_1.l_2 - k_2.l_1)]} \int\limits_{\mathbb{T}^n} \left(\sum_{m\in\mathbb{Z}^n} \int\limits_{\mathbb{R}^n} K_\varphi(\xi+m,\eta) \overline{K_\varphi(\xi + m + l_2 - l_1, \eta)} d\eta \right) \\
&\qquad \times e^{-2\pi i (k_2-k_1).\xi} d\xi. \qquad (3.1)
\end{aligned}$$

In particular, if we take $l_1 = l_2 = l$ in (3.1), we have

$$\langle T^t_{(k_1,l)}\varphi, T^t_{(k_2,l)}\varphi \rangle = e^{-\pi i (k_1-k_2).l} \int\limits_{\mathbb{T}^n} w_\varphi(\xi) e^{-2\pi i (k_2-k_1).\xi} d\xi = e^{-\pi i (k_1-k_2).l} \widehat{w_\varphi}(k_2 - k_1).$$

Since $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ forms an orthonormal system in $L^2(\mathbb{R}^{2n})$, we have $\widehat{w_\varphi}(k) = \delta_{k,0}$. Let $g(\xi) = w_\varphi(\xi) - 1$. Then $\widehat{g}(k) = \widehat{w_\varphi}(k) - \int\limits_{\mathbb{T}^n} e^{-2\pi i k.\xi} d\xi = 0$, $\forall\ k \in \mathbb{Z}^n$. Hence $g(\xi) = 0$ a.e. $\xi \in \mathbb{T}^n$, from which it follows that $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$. □

Remark 3.1. The converse need not be true. We shall illustrate it with a counter example.

Let $\varphi = \frac{1}{\sqrt{2}}\chi_{[0,1]\times[0,2]}$. Here supp $\varphi = [0,1] \times [0,2]$ and supp $T^t_{(0,1)}\varphi = [0,1] \times [1,3]$. So supp $\varphi \bigcap$ supp $T^t_{(0,1)}\varphi = [0,1] \times [1,2]$. Now

$$\langle \varphi, T^t_{(0,1)}\varphi\rangle = \int\limits_1^2 \int\limits_0^1 \varphi(\xi,\eta) e^{-\pi i \xi} \varphi(\xi, \eta - 1) d\xi d\eta = \frac{1}{2} \int\limits_1^2 \int\limits_0^1 e^{-\pi i \xi} d\xi d\eta = \frac{1}{i\pi}.$$

Hence $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^2\}$ does not form an orthonormal system in $L^2(\mathbb{R}^2)$. Now we shall show that $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$. Towards this end, it is enough to prove that $\widehat{w_\varphi}(k) = \delta_{k,0}$. From Lemma 2.2, $\widehat{w_\varphi}(0) = \|\varphi\|_2^2 = 1$ and for $k \neq 0$, $\widehat{w_\varphi}(k) = \langle \varphi, T^t_{(k,0)}\varphi\rangle = 0$, since supp $\varphi \bigcap$ supp $T^t_{(k,0)}\varphi = ([0,1] \times [0,2]) \bigcap ([k, k+1] \times [0,2]) = \emptyset$.

So, it is natural to look for a proper condition under which one can obtain the converse part. Towards this end, we have the following

Definition 3.1. A function $\varphi \in L^2(\mathbb{R}^{2n})$ is said to satisfy "condition C" if

$$\sum_{m\in\mathbb{Z}^n} \int\limits_{\mathbb{R}^n} K_\varphi(\xi + m, \eta)\overline{K_\varphi(\xi + m + l, \eta)} d\eta = 0 \ \text{ a.e. } \xi \in \mathbb{T}^n, \quad \text{for all } \ l \in \mathbb{Z}^n \setminus \{0\}.$$

Theorem 3.2. *Let $\varphi \in L^2(\mathbb{R}^{2n})$. Assume $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$. If, in addition φ satisfies condition C, then $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$.*

Proof. If $l_1 \neq l_2$, from (3.1) it follows that $\langle T^t_{(k_1,l_1)}\varphi, T^t_{(k_2,l_2)}\varphi\rangle = 0$, as φ satisfies condition C. If $l_1 = l_2 = l$, using (3.1) and the assumption $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$, we have

$$\langle T^t_{(k_1,l)}\varphi, T^t_{(k_2,l)}\varphi\rangle = e^{-\pi i (k_1 - k_2).l} \delta_{k_2 - k_1, 0}.$$

Hence $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$. □

Remark 3.2. If $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$, then φ satisfies condition C.

Proof. For $l_1 \neq l_2$, let $F(\xi) = \sum_{m\in\mathbb{Z}^n} \int\limits_{\mathbb{R}^n} K_\varphi(\xi + m, \eta)\overline{K_\varphi(\xi + m + l_2 - l_1, \eta)} d\eta$. Then using (2.3), we get

$$\begin{aligned}
\widehat{F}(k) &= \int\limits_{\mathbb{T}^n} F(\xi) e^{-2\pi i k.\xi} d\xi \\
&= \int\limits_{\mathbb{T}^n} \left(\sum_{m\in\mathbb{Z}^n} \int\limits_{\mathbb{R}^n} K_\varphi(\xi + m, \eta)\overline{K_\varphi(\xi + m + l_2 - l_1, \eta)} d\eta \right) e^{-2\pi i k.(\xi + m)} d\xi \\
&= \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} K_\varphi(\xi, \eta)\overline{K_\varphi(\xi + l_2 - l_1, \eta) e^{2\pi i k.\xi}} d\xi d\eta
\end{aligned}$$

$$= \langle K_\varphi, e^{-\pi i(l_2-l_1).k} K_{T^t_{(k,l_2-l_1)}\varphi}\rangle$$
$$= e^{\pi i(l_2-l_1).k}\langle \varphi, T^t_{(k,l_2-l_1)}\varphi\rangle.$$

Since $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$ and $l_1 \neq l_2$, it follows that $\widehat{F}(k) = 0$ for all $k \in \mathbb{Z}^n$. Then $F(\xi) = 0$ a.e. $\xi \in \mathbb{T}^n$. Thus condition C is satisfied. □

Combining Theorem 3.1, Theorem 3.2 and Remark 3.2, we obtain the following result.

Theorem 3.3. *$\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$ if and only if $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$ and φ satisfies condition C.*

Remark 3.3. Condition C is weaker than the orthonormality of $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$. We shall illustrate this with an example.

Let $\varphi = \chi_{[0,2]\times[0,1]}$. Then it is easy to show that $\langle \varphi, T^t_{(1,0)}\varphi\rangle = -\frac{2}{\pi i} \neq 0$, from which it follows that $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^2\}$ is not an orthonormal system in $L^2(\mathbb{R}^2)$. For $l_1 \neq l_2$, let $F(\xi) = \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} K_\varphi(\xi+m,\eta)\overline{K_\varphi(\xi+m+l_2-l_1,\eta)}d\eta$. Following the same proof of Remark 3.2, we have $\widehat{F}(k) = e^{\pi i(l_2-l_1).k}\langle\varphi, T^t_{(k,l_2-l_1)}\varphi\rangle$. Now supp $\varphi = [0,2]\times[0,1]$ and supp $T^t_{(k,1_2-l_1)}\varphi = [k,k+2]\times[l_2-l_1, l_2-l_1+1]$. Since φ and $T^t_{(k,l_2-l_1)}\varphi$ have disjoint support, we get $\widehat{F}(k) = 0$ for all $k \in \mathbb{Z}^n$. Then $F(\xi) = 0$ a.e. $\xi \in \mathbb{T}^n$, from which it follows that φ satisfies condition C.

Theorem 3.4. *If $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Bessel sequence in $L^2(\mathbb{R}^{2n})$ with bound B, then $w_\varphi(\xi) \leq B$ a.e. $\xi \in \mathbb{T}^n$.*

Proof. We have to prove that $\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2 d\eta \leq B$ a.e. $\xi \in \mathbb{T}^n$. In order to show the result, we claim that for any $f \in L^2(\mathbb{R}^{2n})$,

$$\int_{\mathbb{T}^n}\left|\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m,\eta)}d\eta\right|^2 d\xi$$
$$\leq B\int_{\mathbb{T}^n}\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_f(\xi+m,\eta)|^2\, d\eta d\xi. \tag{3.2}$$

Since $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Bessel sequence in $L^2(\mathbb{R}^{2n})$ with bound B,

$$\sum_{k,l\in\mathbb{Z}^n}\left|\langle f, T^t_{(k,l)}\varphi\rangle\right|^2 \leq B\|f\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}^{2n}). \tag{3.3}$$

Now using Plancherel formula (1.1) and Lemma 2.1,

$$\sum_{k,l\in\mathbb{Z}^n}\left|\langle f, T^t_{(k,l)}\varphi\rangle\right|^2$$
$$= \sum_{k,l\in\mathbb{Z}^n}\left|\langle K_f, K_{T^t_{(k,l)}\varphi}\rangle\right|^2$$
$$= \sum_{k,l\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_f(\xi,\eta)e^{-\pi i(2\xi+l).k}\overline{K_\varphi(\xi+l,\eta)}d\xi d\eta\right|^2$$

$$= \sum_{k,l\in\mathbb{Z}^n} \left| \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n} K_f(\xi+m,\eta) e^{-\pi i[2(\xi+m)+l].k} \overline{K_\varphi(\xi+m+l,\eta)} d\xi d\eta \right|^2$$

$$= \sum_{l\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n} \left| \int_{\mathbb{T}^n} \left(\sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m+l,\eta)} d\eta \right) e^{-2\pi i k.\xi} d\xi \right|^2 .$$

Now for fixed l,

$$\int_{\mathbb{T}^n} \left(\sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m+l,\eta)} d\eta \right) e^{-2\pi i k.\xi} d\xi$$

is the kth Fourier coefficient for the 1-periodic function

$$\xi \mapsto \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m+l,\eta)} d\eta.$$

Thus using Parseval's formula,

$$\sum_{k,l\in\mathbb{Z}^n} |\langle f, T^t_{(k,l)}\varphi\rangle|^2 = \sum_{l\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \left| \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m+l,\eta)} d\eta \right|^2 d\xi.$$

But the right hand side of the above equation is

$$\geq \int_{\mathbb{T}^n} \left| \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m,\eta)} d\eta \right|^2 d\xi.$$

On the other hand,

$$\|f\|_2^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K_f(\xi,\eta)|^2 d\xi d\eta = \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n} |K_f(\xi+m,\eta)|^2 d\xi d\eta$$
$$= \int_{\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_f(\xi+m,\eta)|^2 d\eta d\xi.$$

Thus it follows from (3.3) that

$$\int_{\mathbb{T}^n} \left| \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m,\eta)} d\eta \right|^2 d\xi \leq \sum_{k,l\in\mathbb{Z}^n} |\langle f, T^t_{(k,l)}\varphi\rangle|^2$$
$$\leq B\|f\|^2$$
$$= B\int_{\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_f(\xi+m,\eta)|^2 d\eta d\xi,$$

thus proving our claim (3.2). In order to prove our main result, we assume by contradiction the set $M = \{\xi \in \mathbb{T}^n : \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2 d\eta > B\}$ has positive measure. Let $g(\xi) = \chi_M(\xi)$ and extend g to a 1-periodic function on $\mathbb{R}^n$. Define $f \in L^2(\mathbb{R}^{2n})$ such that $K_f(\xi,\eta) = \chi_M(\xi)K_\varphi(\xi,\eta)$. Now

$$\left|\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m,\eta)}d\eta\right|^2 = \left|\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} \chi_M(\xi)|K_\varphi(\xi+m,\eta)|^2 d\eta\right|^2$$
$$= \chi_M(\xi)\left(\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2 d\eta\right)^2,$$

and

$$\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_f(\xi+m,\eta)|^2 d\eta = \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\chi_M(\xi)K_\varphi(\xi+m,\eta)|^2 d\eta$$
$$= \chi_M(\xi)\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2 d\eta.$$

Thus

$$\int_{\mathbb{T}^n}\left|\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_\varphi(\xi+m,\eta)}d\eta\right|^2 d\xi$$
$$= \int_M \left(\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2 d\eta\right)^2 d\xi$$
$$> B\int_M \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2 d\eta d\xi$$
$$= B\int_{\mathbb{T}^n}\chi_M(\xi)\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2 d\eta d\xi$$
$$= B\int_{\mathbb{T}^n}\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_f(\xi+m,\eta)|^2 d\eta d\xi,$$

which is a contradiction to our claim (3.2). Hence M has zero measure, from which the required assertion follows. □

Theorem 3.5. *Let $w_\varphi(\xi) \le B$ a.e. $\xi \in \mathbb{T}^n$. In addition, if φ satisfies condition C, then $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Bessel sequence in $L^2(\mathbb{R}^{2n})$.*

Proof. Let $\{c_{k,l}\}_{(k,l)\in\mathbb{Z}^{2n}}$ be a finite sequence. It is enough to show that there exists a constant $0 < B < \infty$ such that $\left\|\sum_{(k,l)\in\mathcal{F}} c_{k,l}T^t_{(k,l)}\varphi\right\|_2^2 \le B\sum_{(k,l)\in\mathcal{F}}|c_{k,l}|^2$, where $\mathcal{F}$ denotes a finite set. Consider

$$\left\|\sum_{(k,l)\in\mathcal{F}} c_{k,l}T^t_{(k,l)}\varphi\right\|_2^2 = \left\|K_{(\sum_{k,l} c_{k,l}T^t_{(k,l)}\varphi)}\right\|_2^2$$
$$= \left\|\sum_{k,l} c_{k,l}K_{T^t_{(k,l)}\varphi}\right\|_2^2$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| \sum_{k,l} c_{k,l} K_{T^t_{(k,l)}\varphi}(\xi,\eta) \right|^2 d\xi d\eta$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| \sum_{k,l} c_{k,l} e^{\pi i(2\xi+l).k} K_{\varphi}(\xi+l,\eta) \right|^2 d\xi d\eta$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| \sum_{l}\sum_{k} c_{k,l} e^{\pi i(2\xi+l).k} K_{\varphi}(\xi+l,\eta) \right|^2 d\xi d\eta.$$

Now we break the summation appearing in the last line and get

$$\left\| \sum_{(k,l)\in\mathcal{F}} c_{k,l} T^t_{(k,l)}\varphi \right\|_2^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{l} \left| \sum_{k} c_{k,l} e^{\pi i(2\xi+l).k} K_{\varphi}(\xi+l,\eta) \right|^2 d\xi d\eta$$
$$+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{l_1<l_2} 2\mathrm{Re} \sum_{k} c_{k,l_1} e^{\pi i(2\xi+l_1).k} K_{\varphi}(\xi+l_1,\eta) \sum_{k} \overline{c_{k,l_2}} e^{-\pi i(2\xi+l_2).k} \overline{K_{\varphi}(\xi+l_2,\eta)} d\xi d\eta \tag{3.4}$$

The first term on the right hand side of (3.4) is equal to

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{l} \left| \sum_{k} c_{k,l} e^{\pi i(2\xi+l).k} \right|^2 |K_{\varphi}(\xi+l,\eta)|^2 d\xi d\eta$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n}\sum_{l} \left| \sum_{k} c_{k,l} e^{\pi i[2(\xi+m)+l].k} \right|^2 |K_{\varphi}(\xi+m+l,\eta)|^2 d\xi d\eta$$
$$= \sum_{l} \int_{\mathbb{T}^n} \left| \sum_{k} c_{k,l} e^{\pi i l.k} e^{2\pi i k.\xi} \right|^2 \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_{\varphi}(\xi+m+l,\eta)|^2 d\eta d\xi$$
$$= \sum_{l} \int_{\mathbb{T}^n} \left| \sum_{k} c_{k,l} e^{\pi i l.k} e^{2\pi i k.\xi} \right|^2 \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_{\varphi}(\xi+m,\eta)|^2 d\eta d\xi \tag{3.5}$$
$$\leq B \sum_{l} \int_{\mathbb{T}^n} \left| \sum_{k} c_{k,l} e^{\pi i l.k} e^{2\pi i k.\xi} \right|^2 d\xi$$
$$= B \sum_{l}\sum_{k} |c_{k,l} e^{\pi i l.k}|^2$$
$$= B \sum_{k,l} |c_{k,l}|^2. \tag{3.6}$$

The second term on the right hand side of (3.4) is equal to

$$\sum_{l_1<l_2} 2\mathrm{Re} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} c_{k_1,l_1} \overline{c_{k_2,l_2}} e^{\pi i(2\xi+l_1).k_1} e^{-\pi i(2\xi+l_2).k_2} K_{\varphi}(\xi+l_1,\eta) \overline{K_{\varphi}(\xi+l_2,\eta)} d\xi d\eta$$
$$= 2\mathrm{Re} \sum_{l_1<l_2,k_1,k_2} c_{k_1,l_1} \overline{c_{k_2,l_2}} e^{\pi i(l_1.k_1-l_2.k_2)}$$
$$\times \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i(k_1-k_2).\xi} K_{\varphi}(\xi+l_1,\eta) \overline{K_{\varphi}(\xi+l_2,\eta)} d\xi d\eta$$

$$= 2\mathrm{Re} \sum_{l_1<l_2,k_1,k_2} c_{k_1,l_1}\overline{c_{k_2,l_2}} e^{\pi i(l_1.k_1-l_2.k_2)}$$
$$\times \int_{\mathbb{T}^n} \left(\sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_\varphi(\xi+m+l_1,\eta)\overline{K_\varphi(\xi+m+l_2,\eta)}d\eta \right) e^{2\pi i(k_1-k_2).\xi} d\xi = 0, \tag{3.7}$$

as φ satisfies condition C. Then the required assertion follows from (3.4) and (3.6). □

Remark 3.4. From the above theorem it is clear that if $w_\varphi(\xi) \le B$ a.e. $\xi \in \mathbb{T}^n$, then condition C is sufficient to prove that $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Bessel sequence. But condition C is not necessary for $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ to be a Bessel sequence. We shall illustrate this by an example.

Before providing the example, we observe the following. From (3.7), using (2.3) the second term of (3.4) is equal to

$$2\mathrm{Re} \sum_{l_1<l_2,k_1,k_2} c_{k_1,l_1}\overline{c_{k_2,l_2}} e^{\pi i(l_1.k_1-l_2.k_2)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i(k_1-k_2).\xi} K_\varphi(\xi+l_1,\eta)\overline{K_\varphi(\xi+l_2,\eta)}d\xi d\eta$$
$$= 2\mathrm{Re} \sum_{l_1<l_2,k_1,k_2} c_{k_1,l_1}\overline{c_{k_2,l_2}} e^{\pi i(l_1.k_1-l_2.k_2)} \left\langle K_\varphi \ , \ e^{-\pi i(l_2-l_1).(k_2-k_1)} K_{T^t_{(k_2-k_1,l_2-l_1)}\varphi} \right\rangle$$
$$= 2\mathrm{Re} \sum_{l_1<l_2,k_1,k_2} c_{k_1,l_1}\overline{c_{k_2,l_2}} e^{\pi i[(l_2-l_1).(k_2-k_1)+l_1.k_1-l_2.k_2]} \left\langle \varphi \ , \ T^t_{(k_2-k_1,l_2-l_1)}\varphi \right\rangle.$$

Now take $\varphi = \frac{1}{\sqrt{2}}\chi_{[0,1]\times[0,2]}$. Then supp $\varphi = [0,1]\times[0,2]$ and supp $T^t_{(k_2-k_1,l_2-l_1)}\varphi = [k_2-k_1, k_2-k_1+1]\times[l_2-l_1, l_2-l_1+2]$. For $k_1 \ne k_2$, supp $\varphi \bigcap$ supp $T^t_{(k_2-k_1,l_2-l_1)}\varphi = \emptyset$. For $l_2 - l_1 \ge 2$, supp $\varphi \bigcap$supp $T^t_{(k_2-k_1,l_2-l_1)}\varphi = \emptyset$. We have to take the sum over $k_1 = k_2$ and $l_1 < l_2$, $l_2 - l_1 = 1$. Then the second term becomes

$$2\mathrm{Re}\sum_{k,l} c_{k,l}\overline{c_{k,l+1}} e^{\pi ik}\langle\varphi, T^t_{(0,1)}\varphi\rangle = 2\mathrm{Re}\left(\sum_{k,l} c_{k,l}\overline{c_{k,l+1}} e^{\pi ik}\frac{1}{\pi i}\right)$$
$$\le \sum_{k,l}\left(\left|\frac{e^{\pi ik}}{\pi i}c_{k,l}\right|^2 + \left|c_{k,l+1}\right|^2\right)$$
$$\le \left(\frac{1}{\pi^2}+1\right)\sum_{k,l}|c_{k,l}|^2.$$

Then $\left\|\sum_{k,l} c_{k,l}T^t_{(k,l)}\varphi\right\|_2^2 \le \left(B + \frac{1}{\pi^2} + 1\right)\sum_{k,l}|c_{k,l}|^2$, showing that $\{T^t_{(k,l)}\varphi\}_{(k,l)\in\mathbb{Z}^{2n}}$ is a Bessel sequence. We have seen in Remark 3.1 that $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^2\}$ is not an orthonormal system in $L^2(\mathbb{R}^2)$ and $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$. So it follows from Theorem 3.3 that φ does not satisfy condition C.

4. Parseval frames of twisted shift-invariant spaces

Let $\varphi \in L^2(\mathbb{R}^{2n})$ be such that φ satisfies condition C. Suppose $A^t(\varphi) =$ span $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ and $V^t(\varphi) = \overline{A^t(\varphi)}$. Consider $f \in A^t(\varphi)$ i.e., $f = \sum_{(k',l')\in\mathcal{F}} c_{k',l'}T^t_{(k',l')}\varphi$, where $\mathcal{F}$ is a finite set. Define $r(\xi) = \{r_{l'}(\xi)\}_{l'\in\mathbb{Z}^n}$ for $\xi \in \mathbb{T}^n$, where

$$r_{l'}(\xi) = \sum_{k'} c_{k',l'} e^{\pi i(2\xi+l').k'}. \tag{4.1}$$

Then we have the following proposition.

Proposition 4.1. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ be such that φ satisfies condition C. Then the map $f \longmapsto r$ initially defined on $A^t(\varphi)$ can be extended to an isometric isomorphism of $V^t(\varphi)$ onto $L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$.*

Proof. Here $f \in A^t(\varphi)$ i.e., $f = \sum\limits_{(k',l')\in\mathcal{F}} c_{k',l'} T^t_{(k',l')}\varphi$, where $\mathcal{F}$ is a finite set. Using Lemma 2.1 and (4.1), we have

$$K_f(\xi,\eta) = \sum_{(k',l')\in\mathcal{F}} c_{k',l'} K_{T^t_{(k',l')}\varphi}(\xi,\eta) = \sum_{k',l'} c_{k',l'} e^{\pi i(2\xi+l').k'} K_\varphi(\xi+l',\eta)$$
$$= \sum_{l'} r_{l'}(\xi) K_\varphi(\xi+l',\eta). \tag{4.2}$$

Define $r(\xi) = \{r_{l'}(\xi)\}_{l'\in\mathbb{Z}^n}$ for $\xi \in \mathbb{T}^n$. Conversely, suppose $r(\xi) = \{r_{l'}(\xi)\}_{l'\in\mathbb{Z}^n}$ for $\xi \in \mathbb{T}^n$, where $\{r_{l'}(\xi)\}$ is given by (4.1) for finitely many l'. Notice that $r_{l'}(\xi)$ is a trigonometric polynomial for each $l' \in \mathbb{Z}^n$. Define $f = \sum\limits_{(k',l')\in\mathcal{F}} c_{k',l'} T^t_{(k',l')}\varphi$. Then $f \in A^t(\varphi)$ and $K_f(\xi,\eta) = \sum\limits_{l'} r_{l'}(\xi) K_\varphi(\xi+l',\eta)$. Thus we see a one to one correspondence between $A^t(\varphi)$ and the collection of functions of the form r. Further, as φ satisfies condition C, from (3.4) and (3.5), using (4.1) and (2.4), we get

$$\begin{aligned}\|f\|_2^2 &= \sum_{l'} \int_{\mathbb{T}^n} \left| \sum_{k'} c_{k',l'} e^{\pi i(2\xi+l').k'} \right|^2 \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2 d\eta d\xi \\ &= \int_{\mathbb{T}^n} \sum_{l'} |r_{l'}(\xi)|^2 w_\varphi(\xi) d\xi \\ &= \int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi) d\xi \\ &= \|r\|^2_{L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n),w_\varphi)}.\end{aligned}$$

Thus $f \longmapsto r$ is an isometry. Then by density argument, this isometry can be extended to the whole of $V^t(\varphi)$ onto $L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. More explicitly, this means $f \in V^t(\varphi)$ iff $K_f(\xi,\eta) = \sum\limits_{l'\in\mathbb{Z}^n} r_{l'}(\xi) K_\varphi(\xi+l',\eta)$, where $r(\xi) = \{r_{l'}(\xi)\}_{l'\in\mathbb{Z}^n}$ and $r(\xi) \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. □

Theorem 4.1. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ be such that φ satisfies condition C. Then $\{T^t_{k,l}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Parseval frame for $V^t(\varphi)$ iff $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^{n\cdot}$ on Ω_φ where $\Omega_\varphi = \{\xi \in \mathbb{T}^n : w_\varphi(\xi) \neq 0\}$.*

Proof. Let $f \in A^t(\varphi)$ *i.e.*, $f = \sum\limits_{(k',l')\in\mathcal{F}} c_{k',l'} T^t_{(k',l')}\varphi$, where $\mathcal{F}$ is a finite set. Then from (4.2), $K_f(\xi,\eta) = \sum\limits_{l'} r_{l'}(\xi) K_\varphi(\xi+l',\eta)$, where $r_{l'}(\xi)$ is given by (4.1).

Now, using (4.2), Lemma 2.1 and Plancherel formula (1.1), we get

$$\begin{aligned}\langle f, T^t_{(k,l)}\varphi\rangle &= \langle K_f, K_{T^t_{(k,l)}\varphi}\rangle \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_f(\xi,\eta) \overline{K_{T^t_{(k,l)}\varphi}(\xi,\eta)} d\xi d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{l'} r_{l'}(\xi) K_\varphi(\xi+l',\eta) e^{-\pi i(2\xi+l).k} \overline{K_\varphi(\xi+l,\eta)} d\xi d\eta\end{aligned}$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n}\sum_{l'} r_{l'}(\xi+m) K_\varphi(\xi+m+l',\eta) \times e^{-\pi i[2(\xi+m)+l].k}\overline{K_\varphi(\xi+m+l,\eta)}d\xi d\eta$$

$$= e^{-\pi i l.k}\int_{\mathbb{T}^n}\sum_{l'}\Big(\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} K_\varphi(\xi+m+l',\eta)\overline{K_\varphi(\xi+m+l,\eta)}d\eta\Big) \times r_{l'}(\xi)e^{-2\pi i k.\xi}d\xi$$

$$= e^{-\pi i l.k}\int_{\mathbb{T}^n}\sum_{l'} r_{l'}(\xi)F^{l,l'}(\xi)e^{-2\pi i k.\xi}d\xi,$$

where $F^{l,l'}(\xi) = \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} K_\varphi(\xi+m+l',\eta)\overline{K_\varphi(\xi+m+l,\eta)}d\eta$.

Let $F^l(\xi) = \sum_{l'} r_{l'}(\xi)F^{l,l'}(\xi)$. Then

$$\sum_{k,l\in\mathbb{Z}^n}\left|\langle f, T^t_{(k,l)}\varphi\rangle\right|^2$$

$$= \sum_{k,l\in\mathbb{Z}^n}\left(\int_{\mathbb{T}^n}\sum_{l'} r_{l'}(\xi)F^{l,l'}(\xi)e^{-2\pi i k.\xi}d\xi\right)\left(\overline{\int_{\mathbb{T}^n}\sum_{l'} r_{l'}(\xi')F^{l,l'}(\xi')e^{-2\pi i k.\xi'}d\xi'}\right)$$

$$= \sum_{k,l\in\mathbb{Z}^n}\left(\int_{\mathbb{T}^n} F^l(\xi)e^{-2\pi i k.\xi}d\xi\right)\left(\overline{\int_{T^n} F^l(\xi')e^{-2\pi i k.\xi'}d\xi'}\right)$$

$$= \sum_{k,l\in\mathbb{Z}^n}\widehat{F^l}(k)\overline{\widehat{F^l}(k)}$$

$$= \sum_{k,l\in\mathbb{Z}^n}\left|\widehat{F^l}(k)\right|^2$$

$$= \sum_{l\in\mathbb{Z}^n}\int_{\mathbb{T}^n}|F^l(\xi)|^2 d\xi$$

$$= \sum_{l\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\left|\sum_{l'} r_{l'}(\xi)F^{l,l'}(\xi)\right|^2 d\xi.$$

Since φ satisfies condition C, $F^{l,l'}(\xi) = 0$ a.e. $\xi\in\mathbb{T}^n$, for $l_1 \neq l_2$. Further, using (2.4) we can observe that $F^{l,l}(\xi) = w_\varphi(\xi)$. Hence we get

$$\sum_{k,l\in\mathbb{Z}^n}\left|\langle f, T^t_{(k,l)}\varphi\rangle\right|^2 = \int_{\mathbb{T}^n}\left(\sum_{l'}|r_{l'}(\xi)|^2\right)|w_\varphi(\xi)|^2 d\xi = \int_{\mathbb{T}^n}\|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)}|w_\varphi(\xi)|^2 d\xi. \tag{4.3}$$

Again since φ satisfies condition C, it follows from the proof of Proposition 4.1 that

$$\|f\|_2^2 = \int_{\mathbb{T}^n}\sum_{l'}|r_{l'}(\xi)|^2 w_\varphi(\xi)d\xi = \int_{\mathbb{T}^n}\|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)}w_\varphi(\xi)d\xi. \tag{4.4}$$

Now, suppose $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Parseval frame for $V^t(\varphi)$. Then it follows from (4.3), (4.4) and Proposition 4.1 that

$$\int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} |w_\varphi(\xi)|^2 \, d\xi = \int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi) d\xi, \quad \forall\, r \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi).$$

This means, $\int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi)[\chi_{\Omega_\varphi}(\xi) - w_\varphi(\xi)] = 0 \; \forall\, r \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$.

In other words, $\int_{\Omega_\varphi} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi)[1 - w_\varphi(\xi)] d\xi = 0 \; \forall\, r \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. Hence for all $r \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$, we get

$$\int_{M_1} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi)[1 - w_\varphi(\xi)] d\xi + \int_{M_2} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi)[1 - w_\varphi(\xi)] d\xi = 0, \tag{4.5}$$

where $M_1 = \{\Omega_\varphi : w_\varphi(\xi) \geq 1\}$ and $M_2 = \{\Omega_\varphi : w_\varphi(\xi) < 1\}$. Define $r^{(i)}(\xi) = (\dots, 0, \dots, 0, 1, 0, \dots, 0, \dots)$ with 1 in the 0th position, for $\xi \in M_i$ and $r^{(i)}(\xi) = (\dots, 0, 0, 0, \dots)$, for $\xi \in M_i^c$, where $i = 1, 2$. Then $\int_{\mathbb{T}^n} \|r^{(i)}(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi) d\xi = \int_{M_i} w_\varphi(\xi) d\xi < \infty$, thus proving that $r^{(i)} \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$ for $i = 1, 2$. Put $r = r^{(1)}$. Then it follows from (4.5) that $\int_{M_1} w_\varphi(\xi)[1 - w_\varphi(\xi)] = 0$, from which it follows that $w_\varphi(\xi) = 1$ a.e. $\xi \in M_1$. Similarly, if we take $r = r^{(2)}$, then $w_\varphi(\xi) = 1$ a.e. $\xi \in M_2$. Hence $w_\varphi(\xi) = 1$ a.e. $\xi \in \Omega_\varphi$.

Converse is almost trivial. In fact, if we assume that $w_\varphi(\xi) = 1$ a.e. $\xi \in \Omega_\varphi$, then $\int_{\Omega_\varphi} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi)[1 - w_\varphi(\xi)] d\xi = 0 \; \forall\, r \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. Then retracing the steps back, one gets $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Parseval frame for $V^t(\varphi)$. □

Corollary 4.1. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ be such that φ satisfies condition C. Let $\psi \in L^2(\mathbb{R}^{2n})$ be such that*

$$K_\psi(\xi, \eta) = \begin{cases} K_\varphi(\xi, \eta) w_\varphi(\xi)^{-\frac{1}{2}}, & \xi \in \Omega_\varphi, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Then $\{T^t_{k,l}\psi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Parseval frame for $V^t(\varphi)$.

Proof. From Proposition 4.1, we have $f \in V^t(\varphi)$ iff $K_f(\xi, \eta) = \sum_{l' \in \mathbb{Z}^n} r_{l'}(\xi) K_\varphi(\xi + l', \eta)$ where $r(\xi) = \{r_{l'}(\xi)\}_{l' \in \mathbb{Z}^n}$ and $r(\xi) \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. Take $r^{(0)}(\xi) = (\dots, 0, \dots, 0, w_\varphi(\xi)^{-\frac{1}{2}}, 0, \dots 0, \dots)$ with $w_\varphi(\xi)^{-\frac{1}{2}}$ in the 0th position, for $\xi \in \Omega_\varphi$ and $r^{(0)}(\xi) = (\dots, 0, 0, 0, \dots)$, otherwise. Thus

$$\int_{\mathbb{T}^n} \|r^{(0)}(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi) d\xi = \int_{\Omega_\varphi} d\xi < \infty,$$

thus proving that $r^{(0)} \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. Further, from (4.6), $K_\psi(\xi, \eta) = K_\varphi(\xi, \eta) r_0(\xi)$, where

$$r_0(\xi) = \begin{cases} w_\varphi(\xi)^{-\frac{1}{2}}, & \xi \in \Omega_\varphi, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that $\psi \in V^t(\varphi)$. Further, $w_\psi(\xi) = \chi_{\Omega_\varphi}(\xi)$. Then the result follows from Theorem 4.1. □

This result helps us to obtain a decomposition theorem for a twisted shift-invariant space in $L^2(\mathbb{R}^{2n})$.

Definition 4.1. Let $V^t(\varphi)$ be a twisted shift-invariant space. If the system $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Parseval frame for $V^t(\varphi)$, then the function φ is called Parseval frame generator of $V^t(\varphi)$.

Theorem 4.2. *If V is a twisted shift-invariant space in $L^2(\mathbb{R}^{2n})$, then there exists a family of functions $\{\varphi_\alpha\}_{\alpha\in I}$ in $L^2(\mathbb{R}^{2n})$ (where I is an index set) such that $V = \bigoplus_{\alpha\in I} V^t(\varphi_\alpha)$. In addition, if all the φ_α satisfy condition C, then there exists a family of functions $\{\psi_\alpha\}_{\alpha\in I}$ in $L^2(\mathbb{R}^{2n})$ such that each ψ_α is a Parseval frame generator of $V^t(\varphi_\alpha)$. Moreover, in this case, if $f \in V$, then $\|f\|_2^2 = \sum_{\alpha\in I} \|r_\alpha\|^2_{L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n),w_{\varphi_\alpha})}$, where $r_\alpha \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_{\varphi_\alpha})$.*

Proof. The existence of φ_α will follow from a proof which is similar to that of Theorem 3.10 in [13]. Now as each φ_α satisfies condition C, from Corollary 4.1, there exists $\psi_\alpha \in V^t(\varphi_\alpha)$ such that ψ_α is a Parseval frame generator of $V^t(\varphi_\alpha)$. Now let $f \in V$. Let P_α be the orthogonal projection of $L^2(\mathbb{R}^{2n})$ onto the space $V^t(\varphi_\alpha)$. Then $f = \sum_{\alpha\in I} P_\alpha f$. As φ_α satisfies condition C for each α, it follows from Proposition 4.1 that

$$\|f\|_2^2 = \sum_{\alpha\in I} \|P_\alpha f\|_2^2 = \sum_{\alpha\in I} \int_{\mathbb{T}^n} \|r_\alpha(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_{\varphi_\alpha}(\xi) d\xi = \sum_{\alpha\in I} \|r_\alpha\|^2_{L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n),w_{\varphi_\alpha})}. \qquad \square$$

5. Frames and Riesz basis of twisted shift-invariant spaces

Theorem 5.1. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ be such that φ satisfies condition C. Then $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$ with frame bounds A, B iff $A \le w_\varphi(\xi) \le B$ a.e. $\xi \in \Omega_\varphi$, where $\Omega_\varphi = \{\xi \in \mathbb{T}^n : w_\varphi(\xi) \ne 0\}$.*

Proof. Let $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ be a frame for $V^t(\varphi)$ with frame bounds A, B. Then it follows from (4.3), (4.4) and Proposition 4.1 that

$$A\int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi) d\xi \le \int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} |w_\varphi(\xi)|^2 d\xi \le B\int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi) d\xi,$$

for all $r \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. Now we shall consider only the left hand side of the above inequality. This means, $\int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi)[A\chi_{\Omega_\varphi}(\xi) - w_\varphi(\xi)]d\xi \le 0$, which implies that $\int_{\Omega_\varphi} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi)[A - w_\varphi(\xi)]d\xi \le 0$, for all $r \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. Assume that $M = \{\xi \in \Omega_\varphi : w_\varphi(\xi) < A\}$ has positive measure. Define $r^{(0)}(\xi) = (\dots, 0, \dots, 0, 1, 0, \dots, 0, \dots)$ with 1 in the 0th position, for $\xi \in M$ and $r^{(0)}(\xi) = (\dots, 0, 0, 0, \dots)$ for $\xi \in M^c$. Then clearly $r^{(0)} \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n), w_\varphi)$. Then by taking $r = r^{(0)}$ and proceeding similarly as in the proof of Theorem 4.1, we get $\int_M w_\varphi(\xi)[A - w_\varphi(\xi)] \le 0$, which is a contradiction because $m(M) > 0$ and $w_\varphi(\xi) > 0$, $A - w_\varphi(\xi) > 0$ on M. Hence M has 0 measure, thus proving that $A \le w_\varphi(\xi)$ a.e. $\xi \in \Omega_\varphi$. Similarly, one can show that $w_\varphi(\xi) \le B$ a.e. $\xi \in \Omega_\varphi$. Converse is again trivial as in the case of Theorem 4.1. $\square$

Theorem 5.2. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ and $a, b > 0$. Suppose $\{T^t_{(bk,al)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $L^2(\mathbb{R}^{2n})$ with frame operator S. Then the canonical dual frame also has the same structure and is given by $T^t_{(bk,al)}S^{-1}\varphi$.*

Proof. Since $\{T^t_{(bk,al)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame, the frame operator S is invertible. Now in order to show the required result, it is enough to prove that $S^{-1}T^t_{(bk,al)} = T^t_{(bk,al)}S^{-1}$ for all $(k,l) \in \mathbb{Z}^{2n}$. Let $f \in L^2(\mathbb{R}^{2n})$. Then using property (2) in Section 2.1, we have

$$\begin{aligned} ST^t_{(bk,al)}f &= \sum_{k',l' \in \mathbb{Z}^n} \langle T^t_{(bk,al)}f, T^t_{(bk',al')}\varphi\rangle T^t_{(bk',al')}\varphi \\ &= \sum_{k',l' \in \mathbb{Z}^n} \langle f, T^t_{(-bk,-al)}T^t_{(bk',al')}\varphi\rangle T^t_{(bk',al')}\varphi \\ &= \sum_{k',l' \in \mathbb{Z}^n} \langle f, e^{-\pi iab(-k.l'+k'.l)}T^t_{(b(k'-k),a(l'-l))}\varphi\rangle T^t_{(bk',al')}\varphi \\ &= \sum_{k',l' \in \mathbb{Z}^n} e^{-\pi iab(k.l'-k'.l)}\langle f, T^t_{(b(k'-k),a(l'-l))}\varphi\rangle T^t_{(bk',al')}\varphi \\ &= \sum_{k',l' \in \mathbb{Z}^n} e^{-\pi iab[k.(l'+l)-(k'+k).l]}\langle f, T^t_{(bk',al')}\varphi\rangle T^t_{(b(k'+k),a(l'+l))}\varphi, \end{aligned}$$

using a change of variables. Thus, using again property (2) of Section 2.1, we have

$$ST^t_{(bk,al)}f = \sum_{k',l' \in \mathbb{Z}^n} \langle f, T^t_{(bk',al')}\varphi\rangle T^t_{(bk,al)}T^t_{(bk',al')}\varphi.$$

Since $T^t_{(bk,al)}$ is continuous, we get

$$ST^t_{(bk,al)}f = T^t_{(bk,al)} \sum_{k',l' \in \mathbb{Z}^n} \langle f, T^t_{(bk',al')}\varphi\rangle T^t_{(bk',al')}\varphi = T^t_{(bk,al)}Sf.$$

Since S is invertible, $S^{-1}T^t_{(bk,al)} = T^t_{(bk,al)}S^{-1}$. □

Let $\{c_{k',l'}\}_{(k',l') \in \mathbb{Z}^{2n}} \in C_{00}(\mathbb{Z}^{2n})$. Define $r(\xi) = \{r_{l'}(\xi)\}_{l' \in \mathbb{Z}^n}$ where $r_{l'}(\xi)$ is given by (4.1) for finitely many l'. Clearly there is a one to one correspondence between $C_{00}(\mathbb{Z}^{2n})$ and the collection of functions of the form r. Now

$$\begin{aligned} \int_{\mathbb{T}^n} \|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} d\xi &= \int_{\mathbb{T}^n} \sum_{l'} |r_{l'}(\xi)|^2 d\xi \\ &= \sum_{l'} \int_{\mathbb{T}^n} \left| \sum_{k'} c_{k',l'} e^{\pi i(2\xi+l').k'} \right|^2 d\xi \\ &= \sum_{l'} \sum_{k'} \left| c_{k',l'} e^{\pi i l'.k'} \right|^2 = \sum_{k',l'} |c_{k',l'}|^2. \end{aligned} \tag{5.1}$$

Thus $\{c_{k',l'}\}_{(k',l') \in \mathbb{Z}^{2n}} \mapsto r$ is an isometry. Then by density argument, this isometry can be extended to the whole of $\ell^2(\mathbb{Z}^{2n})$ onto $L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$.

Theorem 5.3. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ be such that φ satisfies condition C. Then $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Riesz basis for $V^t(\varphi)$ with frame bounds A, B iff $A \le w_\varphi(\xi) \le B$ a.e. $\xi \in \mathbb{T}^n$.*

Proof. Let $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ be a Riesz basis for $V^t(\varphi)$ with bounds A and B. Then from (2.1), we have, $A \sum_{k,l \in \mathbb{Z}^n} |c_{k,l}|^2 \le \left\| \sum_{k,l \in \mathbb{Z}^n} c_{k,l} T^t_{(k,l)}\varphi \right\|_2^2 \le B \sum_{k,l \in \mathbb{Z}^n} |c_{k,l}|^2$, for all $\{c_{k,l}\}_{(k,l) \in \mathbb{Z}^{2n}} \in \ell^2(\mathbb{Z}^{2n})$. Consider only the left hand side of the above inequality. Then using (4.4), (5.1) and the discussion mentioned before this theorem, we get $A\int_{\mathbb{T}^n}\|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)}d\xi \le \int_{\mathbb{T}^n}\|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)}w_\varphi(\xi)d\xi$, for all $r\in L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n))$, which implies that $\int_{\mathbb{T}^n}\|r(\xi)\|^2_{\ell^2(\mathbb{Z}^n)}(A-w_\varphi(\xi))d\xi \le 0$, for all $r\in L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n))$. Then proceeding as in the proof of Theorem 5.1, we conclude that $A\le w_\varphi(\xi)$ a.e. $\xi\in\mathbb{T}^n$. The right hand side inequality will follow in a similar manner.

Conversely assume that $A\le w_\varphi(\xi)\le B$ a.e. $\xi\in\mathbb{T}^n$. This means that $A\le w_\varphi(\xi)\le B$ a.e. $\xi\in\Omega_\varphi$. Since φ satisfies condition C, by Theorem 5.1, $\{T^t_{(k,l)}\varphi:(k,l)\in\mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$. Then $\{T^t_{(k,l)}\varphi:(k,l)\in\mathbb{Z}^{2n}\}$ is complete in $V^t(\varphi)$. Now let $\{c_{k,l}\}$ be a finite sequence. Now since φ satisfies condition C, it follows from (3.4) and (3.5) that

$$\Big\|\sum_{k,l\in\mathcal{F}}c_{k,l}T^t_{(k,l)}\varphi\Big\|_2^2=\sum_l\int_{\mathbb{T}^n}\Big|\sum_k c_{k,l}e^{\pi il.k}e^{2\pi ik.\xi}\Big|^2\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2d\eta d\xi$$
$$=\sum_l\int_{\mathbb{T}^n}\Big|\sum_k c_{k,l}e^{\pi il.k}e^{2\pi ik.\xi}\Big|^2 w_\varphi(\xi)d\xi.$$

Thus, using our assumption $A\le w_\varphi(\xi)\le B$ a.e. $\xi\in\mathbb{T}^n$, we get

$$A\sum_l\int_{\mathbb{T}^n}\Big|\sum_k c_{k,l}e^{\pi il.k}e^{2\pi ik.\xi}\Big|^2d\xi\le\Big\|\sum_{k,l\in\mathcal{F}}c_{k,l}T^t_{(k,l)}\varphi\Big\|_2^2$$
$$\le B\sum_l\int_{\mathbb{T}^n}\Big|\sum_k c_{k,l}e^{\pi il.k}e^{2\pi ik.\xi}\Big|^2d\xi.$$

This means, $A\sum_{k,l}|c_{k,l}|^2\le\Big\|\sum_{k,l}c_{k,l}T^t_{(k,l)}\varphi\Big\|^2\le B\sum_{k,l}|c_{k,l}|^2$. Thus it follows from Definition 2.3 that $\{T^t_{(k,l)}\varphi:(k,l)\in\mathbb{Z}^{2n}\}$ is a Riesz basis for $V^t(\varphi)$. □

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