Fluctuations in the current and energy densities around a magnetic flux carrying cosmic string

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Abstract

We calculate the fluctuations in the current and energy densities for the case of a quantized, minimally coupled, massless, complex scalar field around a straight and infinitesimally thin cosmic string carrying magnetic flux. At zero temperature, we evaluate the fluctuations in the current and energy densities for arbitrary flux and deficit angle. At a finite temperature, we evaluate the fluctuations in the energy density for the special case wherein the flux is absent and the deficit angle equals π. We find that, quite generically, the dimensionless ratio of the variance to the mean-squared values of the current and energy densities are of order unity which suggests that the fluctuations around cosmic strings can be considered to be large.

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I. INTRODUCTION

During the last two decades or so, there has been considerable interest in literature in delineating the domain of validity of semiclassical gravity wherein the backreaction of a quantum field on a classical gravitational background is assumed to be given by the expectation value of the stress-energy tensor of the quantum field (see Refs. [1,2]; for a detailed discussion and further references, see, for e.g., Ref. [3]). With this motivation in mind, the fluctuations in the stress-energy densities of quantum fields have been evaluated in a variety of situations in flat spacetime [4–6] and, to a relatively limited extent, in certain curved spacetimes as well [5–7]. The different possible ways of regularizing the four-point functions that one encounters when evaluating the fluctuations in the stress-energy density have been considered [4–6,8–12] and the possible implications of these results on the validity of semiclassical gravity have also been discussed in some detail (see Refs. [4–6]; for a critical outlook, see Refs. [11,12]).

Apart from the cases that have already been considered in literature, the non-trivial spacetime around a cosmic string [13–16] provides another interesting (and easily tractable) situation to study the fluctuations in the stress-energy densities of quantum fields. The mean values of the stress-energy tensor for different quantum fields around a cosmic string have been evaluated at both zero [17–26] and at a finite temperature [27–30]. In fact, situations wherein the cosmic string carries a non-zero magnetic flux have been considered as well [20,23–25,31]. It is well-known that a Aharanov-Bohm solenoid induces a non-zero current around it [32–37]. Therefore, a flux carrying cosmic string, in addition to inducing a non-zero stress-energy density, will also induce a non-zero current. Our aim in this paper is to evaluate the fluctuations in the current and energy densities of a quantum field around a flux carrying cosmic string. The quantum field we shall consider is a minimally coupled, massless, complex scalar field and we shall evaluate the fluctuations in two situations. At zero temperature, we shall evaluate the fluctuations in the current and energy densities for arbitrary flux and deficit angle. At a finite temperature, we shall evaluate the fluctuations in the energy density for the special case wherein the flux is absent and the deficit angle equals π.

This paper is organized as follows. In the following section, we shall briefly re-derive the two-point function for the quantized, massless, complex scalar field around the flux carrying cosmic string at zero temperature. We shall also obtain the finite temperature two-point function for the special case wherein the flux is absent and the deficit angle of the cosmic string equals π. Using these two-point functions, we shall obtain the mean values of the four-current and stress-energy densities. In Sec. III, using Wick's theorem, we shall express the four-point functions of the quantum field in terms of the two-point functions and evaluate the fluctuations in the current and energy densities. We shall also discuss the regularization procedure we have adopted in order to obtain finite expressions for the fluctuations. In Sec. IV, we shall evaluate the relative magnitude of the variance (i.e. the mean-squared deviations) with respect to the mean-squared values of the current and energy densities and briefly comment on the results we have obtained. We shall close with Sec. V wherein we motivate the need to evaluate the mean values and the fluctuations around cosmic strings using smeared fields or at separated points rather than evaluating them at the same spacetime point as we have done in this paper.

Before we proceed further, the following comments on our conventions and notations are in order. We shall set $\hbar = G = c = 1$ and we shall denote the single unit of electric charge as e. We shall work in $(3+1)$ dimensions with a Lorentzian metric signature of $(+,-,-,-)$. For the sake of convenience and clarity in notation, we shall denote the set of four coordinates x^μ as $\tilde{x}$ and we shall write the derivatives $(\partial/\partial x)$ simply as ∂_x. Finally, we shall denote complex conjugation and Hermitian conjugation by an asterisk and a dagger, respectively.

II. TWO-POINT FUNCTIONS AND MEAN VALUES

The spacetime around a straight and infinitesimally thin cosmic string that is oriented along the z-direction is described by the line element [13–16]

$$ds^2 = dt^2 - d\rho^2 - \alpha^2 \rho^2 d\phi^2 - dz^2, \tag{2.1}$$

where $\rho > 0$, $0 \le \phi < (2\pi)$ and $\alpha = (1 - 4\bar{\mu})$, $\bar{\mu}$ being the mass per unit length of the string. The line-element (2.1), though locally flat, is not so globally. The presence of the string leads to conical singularity and, as a result, the spacetime exhibits an azimuthal deficit angle of $[2\pi(1-\alpha)]$. Clearly, $\alpha = 1$ corresponds to Minkowski spacetime.

Let us now assume that the infinitesimally thin cosmic string carries an internal magnetic flux $\bar{\gamma}$. Such a magnetic flux can be described by the vector potential [32–37]

$$A_\mu = B\ (\partial_\mu \phi)\,, \tag{2.2}$$

where B is a constant. This vector potential is singular along the string and the constant B is related to the flux $\bar{\gamma}$ as follows:

$$\bar{\gamma} = \oint_{\mathcal{P}} A_\mu\, dx^\mu = \int_0^{2\pi} B\ d\phi = (2\pi B), \tag{2.3}$$

where $\mathcal{P}$ represents a closed path that encircles the string once.

We shall choose to work here in the gauge wherein the magnetic flux is represented by the vector potential (2.2). The advantage of working in this gauge is that the two-point function of a quantum field around the flux carrying string can be evaluated directly without any recourse to imposing additional boundary conditions on the field. In such a gauge, the symmetric two-point function in the vacuum state for the case of a massless, complex scalar field $\hat{\Phi}$ around the cosmic string is given by (for details, see App. A)

$$\begin{aligned} G_{\rm v}^{(1)}(\tilde{x}_1, \tilde{x}_2) = & \left(4\pi^2 \alpha \rho_1 \rho_2\right)^{-1} \left(\sinh(\gamma\eta/\alpha)\ e^{-[i(\theta_1 - \theta_2)/\alpha]} + \sinh\left[(1-\gamma)\eta/\alpha\right]\right) \\ & \times \Bigg(\sinh\eta\ \left[\cosh(\eta/\alpha) - \cos\left[(\theta_1 - \theta_2)/\alpha\right]\right]\Bigg)^{-1}, \end{aligned} \tag{2.4}$$

where $\theta = (\alpha\phi)$, $\gamma = (\bar{\gamma}/\gamma_0)$, γ_0 being the flux quantum $(2\pi/e)$ and η is given by Eq. (A10). It should be pointed out here that it is only the fractional part of γ that lies in the interval $0 < \gamma < 1$ that leads to non-trivial effects [32–37] and the special case of $\gamma = (1/2)$ corresponds to that of a twisted scalar field [24,29].

Now, consider the case wherein $\bar{\gamma} = 0$ and $\alpha = (1/2)$. In such a case, the two-point function $G_{\rm v}^{(1)}(\tilde{x}_1, \tilde{x}_2)$ above reduces to [27]

$$G_{\rm v}^{(1)}(\tilde{x}_1, \tilde{x}_2) = \left(\pi^2 \rho_1 \rho_2\right)^{-1} (\cosh\eta) \left(\cosh(2\eta) - \cos\left[2\left(\theta_1 - \theta_2\right)\right]\right)^{-1}. \tag{2.5}$$

This can be rewritten as [cf. Eq. (A10)]

$$\begin{aligned} G_{\rm v}^{(1)}(\tilde{x}_1, \tilde{x}_2) &\\ = \left(2\pi^2\right)^{-1} &\left(\left[\left(\rho_1^2 + \rho_2^2\right) - (t_1 - t_2)^2 + (z_1 - z_2)^2 - 2\rho_1\rho_2\, \cos\left(\theta_1 - \theta_2\right)\right]^{-1}\right.\\ &\left. + \left[\left(\rho_1^2 + \rho_2^2\right) - (t_1 - t_2)^2 + (z_1 - z_2)^2 + 2\rho_1\rho_2\, \cos\left(\theta_1 - \theta_2\right)\right]^{-1}\right). \end{aligned} \tag{2.6}$$

Notice that the first term in this expression corresponds to the two-point function in the Minkowski vacuum. The symmetric two-point function at a finite temperature β^{-1} can now be expressed as the infinite image sum of the above vacuum two-point function (see, for e.g., Ref. [38], Sec. 2.7). It is given by [27]

$$\begin{aligned} G_{\beta}^{(1)}(\tilde{x}_1, \tilde{x}_2) &\\ = \left(2\pi^2\right)^{-1} \sum_{n=-\infty}^{\infty} &\left(\left[\left(\rho_1^2 + \rho_2^2\right) - (t_1 - t_2 + in\beta)^2 + (z_1 - z_2)^2 - 2\rho_1\rho_2\, \cos\left(\theta_1 - \theta_2\right)\right]^{-1}\right.\\ &\left. + \left[\left(\rho_1^2 + \rho_2^2\right) - (t_1 - t_2 + in\beta)^2 + (z_1 - z_2)^2 + 2\rho_1\rho_2\, \cos\left(\theta_1 - \theta_2\right)\right]^{-1}\right), \end{aligned} \tag{2.7}$$

where we have used the subscript β to denote the fact that the two-point function has been evaluated at a finite temperature. Evidently, the first term corresponds to the thermal two-point function in Minkowski spacetime.

The mean values of the four-current density and the stress-energy tensor in the vacuum state and at a finite temperature can be expressed as

$$\left\langle \hat{J}_\mu \right\rangle = \lim_{2\to 1}\ \mathcal{J}_\mu^{(1,2)}\ G^{(1)}(\tilde{x}_1, \tilde{x}_2), \tag{2.8}$$

$$\left\langle \hat{T}_{\mu\nu} \right\rangle = \lim_{2\to 1}\ \mathcal{T}_{\mu\nu}^{(1,2)}\ G^{(1)}(\tilde{x}_1, \tilde{x}_2), \tag{2.9}$$

where $G^{(1)}(\tilde{x}_1, \tilde{x}_2)$ refers to the corresponding vacuum or finite temperature two-point function. The differential operator $\mathcal{J}_\mu^{(1,2)}$ appearing in the expression above is defined as

$$\mathcal{J}_\mu^{(1,2)} \equiv (ie/2)\ \left(D_\mu^1 - D_\mu^{2*}\right), \tag{2.10}$$

where the derivative D_μ^1 [cf. Eq. (A2)] acts on the point $\tilde{x}_1$ and D_μ^{2*} on $\tilde{x}_2$. For the case of a minimally coupled, massless, complex scalar field we are considering here, the operator $\mathcal{T}_{\mu\nu}^{(1,2)}$ is given by

$$\mathcal{T}_{\mu\nu}^{(1,2)} \equiv \left(\frac{1}{2}\right) \left(\left(D_\mu^1 D_\nu^{2*} + D_\nu^1 D_\mu^{2*}\right) - \left(\frac{g_{\mu\nu}}{2}\right)\ \left[g^{\kappa\lambda}\ \left(D_\kappa^1 D_\lambda^{2*} + D_\lambda^1 D_\kappa^{2*}\right)\right]\right). \tag{2.11}$$

In the vacuum state, the mean values around the flux carrying string can be easily obtained using the two-point function (2.4). We find that the mean value of the four-current density is given by [note that $\tilde{x} \equiv (t, \rho, \phi, z)$]

$$\left\langle \hat{J}_\mu \right\rangle_{\rm v} = -\left(e\,\mathcal{C}/12\pi^2\rho^2\right)\,[0,0,1,0]\,, \tag{2.12}$$

where

$$\mathcal{C} = \left[\gamma\,(1-\gamma)\,(1-2\gamma)\,/\alpha^2\right]. \tag{2.13}$$

It should be pointed out that, in addition to the trivial cases of $\gamma = 0$ and 1, the current vanishes for $\gamma = (1/2)$ as well. It is also interesting to note that the conical singularity of the cosmic string amplifies the current by a factor of α^{-2} (compare with Refs. [32–34,36,37] wherein the case of $\alpha = 1$ is considered). The mean value of the stress-energy tensor around the string is given by [20,25]

$$\left\langle \hat{T}^\mu_\nu \right\rangle_{\rm v} = \left(720\pi^2\rho^4\right)^{-1}\left(\mathcal{A}\ {\rm diag}\ [1,1,-3,1] - \mathcal{B}\ {\rm diag}\ [1,-(1/2),(3/2),1]\right), \tag{2.14}$$

where

$$\mathcal{A} = \left(\left[1-\left(1/\alpha^4\right)\right] + 30\,\left[\gamma^2\,(1-\gamma)^2\,/\alpha^4\right]\right), \tag{2.15}$$

$$\mathcal{B} = 20\left(\left[1-\left(1/\alpha^2\right)\right] + 6\,\left[\gamma\,(1-\gamma)\,/\alpha^2\right]\right). \tag{2.16}$$

Let us now consider the mean values at a finite temperature around the string for the special case wherein $\bar{\gamma} = 0$ and $\alpha = (1/2)$. When the flux is absent, then, obviously, no current will be induced. The mean value of the stress-energy tensor can be obtained using the finite temperature two-point function (2.7). We find that the sums involved can be expressed in terms of elementary functions (cf. Ref. [39], Vol. 1, pp. 687–688) and the stress-energy density is given by [27]

$$\left\langle \hat{T}^\mu_\nu \right\rangle_\beta = \left(\mathcal{A}_\beta\ {\rm diag}\,[1,-(1/3),-(1/3),-(1/3)] + \mathcal{B}_\beta\ {\rm diag}\,[1,-1,2,0] + \mathcal{C}_\beta\ {\rm diag}\,[0,0,1,1]\right), \tag{2.17}$$

where

$$\mathcal{A}_\beta = \left(\pi^2/15\beta^4\right), \tag{2.18}$$

$$\mathcal{B}_\beta = \left(16\pi\rho^3\beta\right)^{-1}\coth\left(2\pi\rho/\beta\right) + \left(8\rho^2\beta^2\right)^{-1}{\rm cosech}^2\left(2\pi\rho/\beta\right), \tag{2.19}$$

$$\mathcal{C}_\beta = \left(\pi/2\rho\beta^3\right)\coth\left(2\pi\rho/\beta\right)\ {\rm cosech}^2\left(2\pi\rho/\beta\right). \tag{2.20}$$

Note that the first term involving $\mathcal{A}_\beta$ in the expression (2.17) above corresponds to the thermal stress-energy density in flat spacetime. It should be mentioned here that we have subtracted the contribution due to the Minkowski vacuum in order to obtain these finite expressions for the mean values.

III. FOUR-POINT FUNCTIONS AND FLUCTUATIONS

In the coincidence limit, the correlation functions of the four-current and the stress-energy densities are given by

$$2\left\langle \hat{J}_\mu \hat{J}_\nu \right\rangle \equiv \left\langle \left(\hat{J}_\mu \hat{J}_\nu + \hat{J}_\nu \hat{J}_\mu\right) \right\rangle = \lim_{4\to 3\to 2\to 1} \mathcal{J}_\mu^{(1,2)}\ \mathcal{J}_\nu^{(3,4)}\ \mathcal{G}\left(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right), \tag{3.1}$$

$$2\left\langle \hat{T}_{\mu\nu} \hat{T}_{\kappa\lambda} \right\rangle \equiv \left\langle \left(\hat{T}_{\mu\nu} \hat{T}_{\kappa\lambda} + \hat{T}_{\kappa\lambda} \hat{T}_{\mu\nu}\right) \right\rangle = \lim_{4\to 3\to 2\to 1} \mathcal{T}_{\mu\nu}^{(1,2)}\ \mathcal{T}_{\kappa\lambda}^{(3,4)}\ \mathcal{G}\left(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right), \tag{3.2}$$

where $\mathcal{G}\left(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right)$ is the four-point function defined as

$$\begin{aligned}
\mathcal{G}\left(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right)
&= \Big\langle \left(\hat{\Phi}(\tilde{x}_1)\, \hat{\Phi}^\dagger(\tilde{x}_2) + \hat{\Phi}^\dagger(\tilde{x}_2)\, \hat{\Phi}(\tilde{x}_1)\right) \left(\hat{\Phi}(\tilde{x}_3) \hat{\Phi}^\dagger(\tilde{x}_4) + \hat{\Phi}^\dagger(\tilde{x}_4) \hat{\Phi}(\tilde{x}_3)\right) \\
&\quad + \left(\hat{\Phi}(\tilde{x}_3)\, \hat{\Phi}^\dagger(\tilde{x}_4) + \hat{\Phi}^\dagger(\tilde{x}_4)\, \hat{\Phi}(\tilde{x}_3)\right) \left(\hat{\Phi}(\tilde{x}_1)\, \hat{\Phi}^\dagger(\tilde{x}_2) + \hat{\Phi}^\dagger(\tilde{x}_2)\, \hat{\Phi}(\tilde{x}_1)\right) \Big\rangle.
\end{aligned} \tag{3.3}$$

At both zero and at a finite temperature β^{-1}, using Wick's theorem, we can write

$$\begin{aligned}
&\Big\langle \left(\hat{\Phi}(\tilde{x}_1)\, \hat{\Phi}^\dagger(\tilde{x}_2) + \hat{\Phi}^\dagger(\tilde{x}_2)\, \hat{\Phi}(\tilde{x}_1)\right) \left(\hat{\Phi}(\tilde{x}_3)\, \hat{\Phi}^\dagger(\tilde{x}_4) + \hat{\Phi}^\dagger(\tilde{x}_4)\, \hat{\Phi}(\tilde{x}_3)\right) \Big\rangle \\
&\quad = \left\langle \hat{\Phi}(\tilde{x}_1)\, \hat{\Phi}^\dagger(\tilde{x}_2) + \hat{\Phi}^\dagger(\tilde{x}_2)\, \hat{\Phi}(\tilde{x}_1) \right\rangle \left\langle \left(\hat{\Phi}(\tilde{x}_3) \hat{\Phi}^\dagger(\tilde{x}_4) + \hat{\Phi}^\dagger(\tilde{x}_4) \hat{\Phi}(\tilde{x}_3)\right) \right\rangle \\
&\quad\quad + 4 \left\langle \hat{\Phi}(\tilde{x}_1)\, \hat{\Phi}^\dagger(\tilde{x}_4) \right\rangle \left\langle \hat{\Phi}^\dagger(\tilde{x}_2)\, \hat{\Phi}(\tilde{x}_3) \right\rangle.
\end{aligned} \tag{3.4}$$

(For a discussion on Wick's theorem in the vacuum state, see Ref. [40] and, for a discussion at a finite temperature, see Ref. [41].) It ought to be emphasized here that the relation (3.4) above is valid *only* when the expectation values are evaluated in the vacuum state or at a finite temperature. This relation will not be valid, for instance, if the quantum field is assumed to be in a n-particle state or, for that matter, in a generalized squeezed state. Therefore, in the vacuum state or at a finite temperature, the expressions (3.1) and (3.2) reduce to

$$\begin{aligned}
&\left(\left\langle \hat{J}_\mu \hat{J}_\nu \right\rangle - \left\langle \hat{J}_\mu \right\rangle \left\langle \hat{J}_\nu \right\rangle\right) \\
&\quad = \lim_{4\to 3\to 2\to 1} 2\ \mathcal{J}_\mu^{(1,2)}\ \mathcal{J}_\nu^{(3,4)}\ \left[G^+\left(\tilde{x}_1, \tilde{x}_4\right)\ G^-\left(\tilde{x}_3, \tilde{x}_2\right) + G^-\left(\tilde{x}_1, \tilde{x}_4\right)\ G^+\left(\tilde{x}_3, \tilde{x}_2\right)\right],
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\left(\left\langle \hat{T}_{\mu\nu} \hat{T}_{\kappa\lambda} \right\rangle - \left\langle \hat{T}_{\mu\nu} \right\rangle \left\langle \hat{T}_{\kappa\lambda} \right\rangle\right) \\
&\quad = \lim_{4\to 3\to 2\to 1} 2\ \mathcal{T}_{\mu\nu}^{(1,2)}\ \mathcal{T}_{\kappa\lambda}^{(3,4)}\ \left[G^+\left(\tilde{x}_1, \tilde{x}_4\right)\ G^-\left(\tilde{x}_3, \tilde{x}_2\right) + G^-\left(\tilde{x}_1, \tilde{x}_4\right)\ G^+\left(\tilde{x}_3, \tilde{x}_2\right)\right],
\end{aligned} \tag{3.6}$$

where $G^\pm\left(\tilde{x}_a, \tilde{x}_b\right)$ refer to the corresponding vacuum or finite temperature Wightman functions [cf. Eq. (A5)].

Let us now evaluate the variance (i.e. the mean-squared deviations) in the current ($\hat{J}_\phi$) and the energy ($\hat{T}_{tt}$) densities. For convenience, we shall hereafter denote $\hat{J}_\phi$ as $\hat{j}$ and $\hat{T}_{tt}$ as $\hat{\varepsilon}$. On using the two-point function (2.4) to represent the Wightman functions $G_{\rm v}^\pm\left(\tilde{x}_a, \tilde{x}_b\right)$ (which can be done with a suitable introduction of a factor of $(i\epsilon)$, where $\epsilon \to 0^+$), it can be shown that the variance in the current and energy densities in the vacuum state are given by

$$\left(\left\langle\hat{j}^2\right\rangle_{\rm v} - \left\langle\hat{j}\right\rangle_{\rm v}^2\right) = \left(e/24\pi^2\rho^2\right)^2\,\left[2\mathcal{C}^2 - \left(\alpha^2\mathcal{A}\mathcal{B}/400\right)\right] \tag{3.7}$$

and

$$\left(\left\langle\hat{\varepsilon}^2\right\rangle_{\rm v} - \left\langle\hat{\varepsilon}\right\rangle_{\rm v}^2\right) = \left(1440\pi^2\rho^4\right)^{-2}\,\left[12\mathcal{A}^2 + \left(9\mathcal{B}^2/2\right) + 6\mathcal{A}\mathcal{B} + \left(7200\,\mathcal{C}^2/\alpha^2\right)\right], \tag{3.8}$$

where $\mathcal{C}$ is given by Eq. (2.13) and $\mathcal{A}$ and $\mathcal{B}$ are given by Eqs. (2.15) and (2.16). For the case $\bar{\gamma} = 0$ and $\alpha = (1/2)$, the mean-squared deviations in the energy density at a finite temperature can similarly be evaluated using the two-point function (2.7). We find that

$$\left(\left\langle\hat{\varepsilon}^2\right\rangle_\beta - \left\langle\hat{\varepsilon}\right\rangle_\beta^2\right) = \left[\left(\mathcal{A}_\beta^2/3\right) + \left(3\mathcal{B}_\beta^2/2\right) + \left(\mathcal{C}_\beta^2/2\right) + \left(\mathcal{A}_\beta\mathcal{B}_\beta/3\right) + \mathcal{B}_\beta\mathcal{C}_\beta - \left(\mathcal{A}_\beta\mathcal{C}_\beta/3\right)\right], \tag{3.9}$$

where $\mathcal{A}_\beta$, $\mathcal{B}_\beta$ and $\mathcal{C}_\beta$ are given by Eqs. (2.18), (2.19) and (2.20), respectively.

At this stage of our discussion, it is important that we comment on the procedure we have adopted here to regularize the four-point functions and their derivatives in the limit when all the four points coincide. Towards the end of the last section, we had mentioned that, in order to obtain divergence free expressions for the mean values, we had regularized the quantities involving the two-point functions and their derivatives (in the coincidence limit) by subtracting the corresponding contribution due to the Minkowski vacuum. Since, using Wick's theorem, we can express the four-point functions (both in the vacuum state and at a finite temperature) in terms of the two-point functions, the divergences in the four-point functions and their derivatives (when all the four points coincide) essentially involve the divergences of the two-point functions and their derivatives. Therefore, evidently, if we use the regularized two-point functions to evaluate the corresponding four-point functions, then the resulting four-point functions will be free of the divergences in the coincidence limit [4–7,11,12]. Indeed, this is the regularization procedure we have adopted to obtain divergence free expressions for the mean-squared deviations.

IV. MAGNITUDE OF FLUCTUATIONS

A useful measure of the magnitude of fluctuations in a stochastic variable is the dimensionless ratio of the variance to the mean-squared value of the variable. For a fluctuating quantum variable that is represented by the operator $\hat{\mathcal{O}}$, such a dimensionless quantity can be defined as [4–6,8,11,12]

$$\Delta_{\mathcal{O}} = \left(\frac{\left\langle\hat{\mathcal{O}}^2\right\rangle - \left\langle\hat{\mathcal{O}}\right\rangle^2}{\left\langle\hat{\mathcal{O}}^2\right\rangle}\right), \tag{4.1}$$

where the expectation values are evaluated in a given state. The fluctuations in the quantity $\mathcal{O}$ can be considered to be large if $\Delta_{\mathcal{O}} \simeq 1$ and the fluctuations can be said to be small if $\Delta_{\mathcal{O}} \ll 1$.

From the results we have obtained in the last two sections, it is easy to show that, in the vacuum state, the quantity Δ corresponding to the current and energy densities are given by

$$\Delta_j = \Big[2\,\mathcal{C}^2 - \Big(\alpha^2 \mathcal{A}\mathcal{B}/400\Big)\Big]\,\Big[6\,\mathcal{C}^2 - \Big(\alpha^2 \mathcal{A}\mathcal{B}/400\Big)\Big]^{-1} \tag{4.2}$$

and

$$\Delta_\varepsilon = \Big[24\mathcal{A}^2 + 9\mathcal{B}^2 + 12\mathcal{A}\mathcal{B} + \Big(14400\,\mathcal{C}^2/\alpha^2\Big)\Big] \times \Big[32\mathcal{A}^2 + 17\mathcal{B}^2 - 4\mathcal{A}\mathcal{B} + \Big(14400\,\mathcal{C}^2/\alpha^2\Big)\Big]^{-1}. \tag{4.3}$$

At a finite temperature β^{-1}, the relative magnitude of the variance in the energy density with respect to the mean-squared value [for the special case of $\bar{\gamma} = 0$ and $\alpha = (1/2)$] is given by

$$\Delta_\varepsilon^\beta = \Big[\Big(\mathcal{A}_\beta^2/3\Big) + \Big(3\mathcal{B}_\beta^2/2\Big) + \Big(\mathcal{C}_\beta^2/2\Big) + (\mathcal{A}_\beta\mathcal{B}_\beta/3) + \mathcal{B}_\beta\mathcal{C}_\beta - (\mathcal{A}_\beta\mathcal{C}_\beta/3)\Big] \times \Big[\Big(4\mathcal{A}_\beta^2/3\Big) + \Big(5\mathcal{B}_\beta^2/2\Big) + \Big(\mathcal{C}_\beta^2/2\Big) + (7\mathcal{A}_\beta\mathcal{B}_\beta/3) + \mathcal{B}_\beta\mathcal{C}_\beta - (\mathcal{A}_\beta\mathcal{C}_\beta/3)\Big]^{-1}. \tag{4.4}$$

In Fig. 1, we have plotted Δ_ε for the entire range of the variables α and γ. And, in

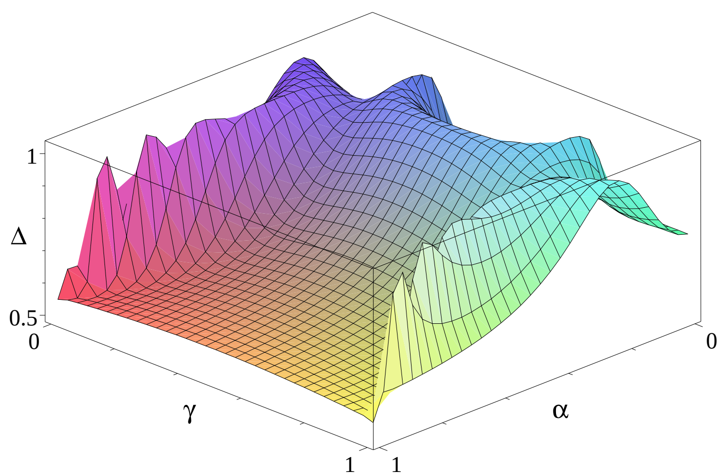

FIG. 1. Δ_ε vs. (α, γ)

Fig. 2, we have plotted Δ_ε^β for a sufficiently wide range of the variables ρ and β. It is clear from these two figures that both Δ_ε and Δ_ε^β are always of order unity which suggests that the fluctuations in the energy-density around cosmic strings can be considered to be large.

In the cases of Δ_ε and Δ_ε^β, both the numerator and the denominator remain positive definite and also prove to be of the same order of magnitude for the entire range of the variables (α, γ) and (ρ, β). As a result, Δ_ε and Δ_ε^β turn out to be of order unity. In contrast, there exist values of α and γ for which the numerator and the denominator of Δ_j vanish. In Fig. 3, we have plotted Δ_j for a small range of the variables α and γ in a region where the denominator is non-zero. Evidently, within this range, apart from those regions near the points where it vanishes identically, the magnitude of Δ_j can be said to be of order unity.

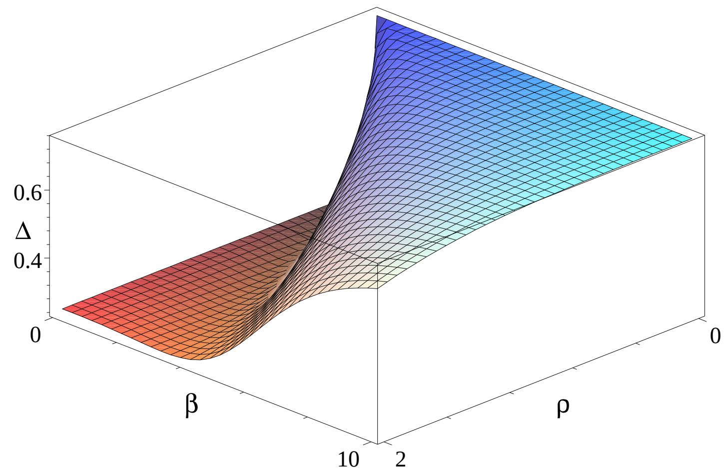

FIG. 2. Δ_ε^β vs. (ρ, β)

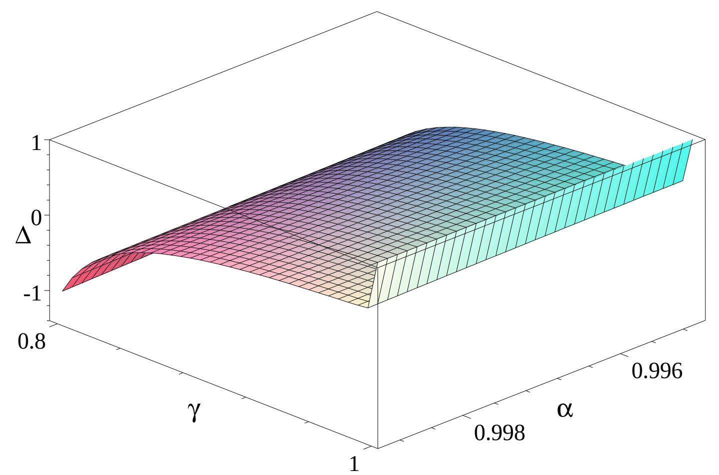

FIG. 3. Δ_j vs. (α, γ)

V. DISCUSSION

In the absence of a length scale such as the mass of the quantum field, it is only natural to expect that dimensionless quantities such as Δ_j and Δ_ε, if they do not vanish identically, they will turn out to be of order unity [11,12]. It will be worthwhile to investigate how Δ_j and Δ_ε behave for massive fields around flux carrying cosmic strings. Also, as we had discussed earlier, we had evaluated the mean values and the fluctuations at the *same* spacetime point. However, if one considers smeared or point-separated quantities rather than point-coincident ones as we have done here, then the quantities Δ_j and Δ_ε will depend on the smearing or probing scale [11,12]. It will be interesting to examine how these quantities behave as a function of the smearing scale, in particular when there is another scale present such as in the case of a massive field. We plan to address these issues in a future publication.

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APPENDIX A: EVALUATING THE TWO-POINT FUNCTION

A massless, complex scalar field Φ evolving in a classical electromagnetic and gravitational background described by the vector potential A^μ and the metric tensor $g_{\mu\nu}$ satisfies the differential equation

$$\frac{1}{\sqrt{-g}} D_\mu \left(\sqrt{-g}\, g^{\mu\nu} D_\nu \right) \Phi = 0, \tag{A1}$$

where the differential operator D_μ is defined as

$$D_\mu = \left(\partial_\mu + ieA_\mu \right). \tag{A2}$$

The normalized, positive norm modes of the field Φ evolving in a background described by the line-element (2.1) and the vector potential (2.2) are given by

$$u_{qlk_z}(\tilde{x}) = \left(q/(2\pi)^2\, (2\omega\alpha) \right)^{1/2} e^{-i\omega t}\, e^{il\phi}\, e^{ik_z z}\, J_\sigma(q\rho), \tag{A3}$$

where $\omega = (q^2 + k_z^2)^{1/2}$ and $J_\sigma(q\rho)$ is the Bessel function of order σ with $\sigma = (|(l + eB)/\alpha|)$. The completeness relation for these modes leads to the following conditions on the wave numbers: $0 \le q < \infty$, $-\infty < k_z < \infty$ and $l = 0, \pm 1, \pm 2, \ldots$.

The symmetric two-point function $G^{(1)}(\tilde{x}_1, \tilde{x}_2)$ of the quantum field $\hat{\Phi}$ is defined as (see, for e.g., Ref. [38], Sec. 2.7)

$$G^{(1)}(\tilde{x}_1, \tilde{x}_2) = G^+(\tilde{x}_1, \tilde{x}_2) + G^-(\tilde{x}_1, \tilde{x}_2), \tag{A4}$$

where $G^+(\tilde{x}_1, \tilde{x}_2)$ and $G^-(\tilde{x}_1, \tilde{x}_2)$ are the Wightman functions given by

$$G^+(\tilde{x}_1, \tilde{x}_2) \equiv \left\langle \hat{\Phi}(\tilde{x}_1)\, \hat{\Phi}^\dagger(\tilde{x}_2) \right\rangle, \qquad G^-(\tilde{x}_1, \tilde{x}_2), \equiv \left\langle \hat{\Phi}^\dagger(\tilde{x}_2)\, \hat{\Phi}(\tilde{x}_1) \right\rangle \tag{A5}$$

and the expectation values are evaluated in a given state of the quantum field. In the vacuum state, the symmetric two-point function around the flux carrying cosmic string can be expressed in terms of the normalized modes (A3) as follows:

$$G_{\rm v}^{(1)}(\tilde{x}_1,\tilde{x}_2) = \int\limits_0^\infty dq \sum_{l=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} dk_z\ \left(q/(2\pi)^2\,\alpha\right)\ e^{il(\phi_1-\phi_2)}\ e^{ik_z(z_1-z_2)}\ J_\sigma(q\rho_1)\ J_\sigma(q\rho_2)$$
$$\times\left[\left(\frac{1}{2\omega}\right)\left(e^{-i\omega(t_1-t_2)}+e^{i\omega(t_1-t_2)}\right)\right], \qquad (A6)$$

where we have used the subscript v to indicate the fact that the expectation values have been evaluated in the vacuum state. The quantity in the square brackets in the above equation can now be written as (cf. Ref. [42], p. 307)

$$\left(\frac{1}{2\omega}\right)\left(e^{-i\omega(t_1-t_2)}+e^{i\omega(t_1-t_2)}\right) = \int\limits_0^\infty \frac{ds}{\sqrt{-(\pi i s)}}\ e^{-i\omega^2 s}\ e^{-\left[i(t_1-t_2)^2/4s\right]}. \qquad (A7)$$

On substituting this expression in Eq. (A6) and integrating over k_z, we find that

$$G_{\rm v}^{(1)}(\tilde{x}_1,\tilde{x}_2) = \sum_{l=-\infty}^{\infty} e^{il(\phi_1-\phi_2)} \int\limits_0^\infty \frac{ds}{(4\pi^2\alpha s)}\ e^{-\left(i\left[(t_1-t_2)^2-(z_1-z_2)^2\right]/4s\right)}$$
$$\times \int\limits_0^\infty dq\ q\ e^{-iq^2 s}\ J_\sigma(q\rho_1)\ J_\sigma(q\rho_2). \qquad (A8)$$

On carrying out the integrals over q and s (see, for e.g., Ref. [39], Vol. 2, p. 223 and p. 303), we obtain that

$$G_{\rm v}^{(1)}(\tilde{x}_1,\tilde{x}_2) = \sum_{l=-\infty}^{\infty} e^{il(\phi_1-\phi_2)} \int\limits_0^\infty \frac{ds}{(8\pi^2 i\alpha s^2)}\ e^{i\delta/s}\ I_\sigma(\rho_1\rho_2/2is)$$
$$= \left(4\pi^2\alpha\rho_1\rho_2\,\sinh\eta\right)^{-1} \sum_{l=-\infty}^{\infty} e^{il(\phi_1-\phi_2)}e^{-\sigma\eta}, \qquad (A9)$$

where $I_\sigma(\rho_1\rho_2/2is)$ is the modified Bessel function of order σ and

$$\cosh\eta = (2\delta/\rho_1\rho_2) \quad \text{with} \quad (4\delta) = \left[\left(\rho_1^2+\rho_2^2\right) - (t_1-t_2)^2 + (z_1-z_2)^2\right]. \qquad (A10)$$

The sum over l can now be evaluated to yield

$$G_{\rm v}^{(1)}(\tilde{x}_1,\tilde{x}_2) = \left(4\pi^2\alpha\rho_1\rho_2\right)^{-1}\left(\sinh(eB\eta/\alpha)\ e^{-i(\phi_1-\phi_2)} + \sinh\left[(1-eB)\eta/\alpha\right]\right)$$
$$\times\left(\sinh\eta\ \left[\cosh(\eta/\alpha) - \cos\left(\phi_1-\phi_2\right)\right]\right)^{-1}, \qquad (A11)$$

which is the result [viz. Eq. (2.4)] we have quoted in the text. It is easy to see that the two-point functions that have been obtained earlier in literature (see, for e.g., Ref. [25], Eq. (4.8); also see Ref. [34], Eq. (21) for the case of $\alpha = 1$) are related to the above two-point function by a suitable gauge-transforming phase factor (i.e. $\exp\left[-ieB\left(\phi_1-\phi_2\right)\right]$).

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