Entropy of BTZ black strings in the brick wall approach

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During the last few years, exact solutions that describe black holes that are bound to a two-brane in a four dimensional anti-de Sitter bulk have been constructed. In situations wherein there is a negative cosmological constant on the brane, for large masses, these solutions are exactly the rotating BTZ black holes on the brane and, in fact, describe rotating BTZ black strings in the bulk. We evaluate the canonical entropy of a free and massless scalar field (at the Hawking temperature) around the rotating BTZ black string using the brick wall model. We explicitly show that the Bekenstein-Hawking 'area' law is satisfied *both on the brane and in the bulk*.

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I. THE BRICK WALL APPROACH TO BLACK HOLE ENTROPY

It has now been three decades since it was originally suggested by Bekenstein that black holes carry an entropy $\mathcal{S}$ that is proportional to the surface 'area' $\mathcal{A}$ of their event horizon [1]. Soon after Bekenstein's suggestion, Hawking discovered that when one considers the evolution of quantum fields around black holes, they indeed radiate thermally [2]. Moreover, the temperature of the thermal radiation, referred to as the Hawking temperature, determines the exact relation between the 'area' of the black holes and their entropy, viz. that [2]

$$\mathcal{S} = \left(\frac{\mathcal{A}}{4}\right) \tag{1}$$

with the area to be 'measured' in units of square of the Planck length. This Bekenstein-Hawking 'area' law is expected to apply to *all* black hole solutions of the Einstein's equations.

Ever since Bekenstein's suggestion and Hawking's discovery, a variety of approaches have been proposed to understand the microscopic origin of black hole entropy (for a discussion on the different approaches, see, for example, Refs. [3] and references therein). One of these approaches has been the semi-classical approach originally due to 't Hooft [4], often referred to as the brick wall model. In this approach, the black hole geometry is assumed to be a fixed classical background in which quantum fields propagate and the entropy of the black hole supposedly arises due to the statistical entropy of thermal fields outside the black hole horizon, evaluated in the WKB approximation. But, due to the infinite blue shifting of the modes in the vicinity of the black hole horizon, the density of states of the matter fields diverge and, hence, for this model to be viable, it is necessary to introduce a cut-off (called the brick wall and, obviously, the reason behind the name of the approach) above the horizon by hand. This approach has been very popular in the literature, and its popularity can be gauged by the fact that the Bekenstein-Hawking 'area' law (1) has been recovered for a variety of black hole horizons (and also for accelerated as well as cosmological horizons) in different dimensions (for an incomplete list, see Refs. [5–7]). Though popular, the brick wall approach has its drawbacks. Foremost amongst them being the fact that the cut-off has to be chosen by hand to be of the order of the Planck length in order to recover the 'area' law (1). However, it should be pointed out here that there have been proposals in the literature wherein the infinities that arise can be absorbed in the renormalization of the Newton's constant (in this context, see, for instance, Refs. [8]). Another notable limitation arises due to the fact that the entropy obtained depends on the number of matter fields that one chooses to consider.

During the last few years, scenarios have been proposed wherein the observable universe is considered to be a three-brane embedded in a higher-dimensional bulk. Of particular interest have been scenarios wherein the three-brane is embedded in a five-dimensional anti-de Sitter bulk [9]. These proposals have generated considerable interest regarding the physics of black holes in these scenarios. Quite a few of the standard black hole solutions have been found to be consistent with these backgrounds and, often, it turns out that these solutions actually describe black strings that extend into the extra dimension (see Refs. [10]; in this context, also see Refs. [11]).

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With the aim of testing the brane-world scenarios in lower dimensions, exact solutions describing black holes that are confined to a two-brane in a four-dimensional anti-de Sitter bulk have also been constructed [12, 13]. For cases wherein there is a negative cosmological constant on the brane, these solutions (for large masses of the black hole) prove to be precisely the rotating Banados-Teitleboim-Zanelli (BTZ, hereafter) black holes on the brane [14] and rotating BTZ black strings in the bulk [13].

The entropy of a Schwarzschild black hole that is bound to a three-brane in a five-dimensional anti-de Sitter bulk has recently been evaluated using the brick wall approach (see Ref. [15]; also, see Ref. [16]; for another approach analysing the same situation, see, for example, Ref. [17]). The rotating BTZ black hole exhibits the essential features of rotating black holes in higher dimensions in the sense that it is characterized by mass as well as angular momentum. Therefore, apart from providing a technically simpler framework to work with, studying the properties of the rotating BTZ black string will help us gain an insight into the properties of rotating black strings in higher dimensions. Moreover, in the earlier calculation for the Schwarzschild black hole embedded in a five dimensionsal anti-de Sitter bulk, the brick wall entropy has been shown to be proportional to the area of the event horizon *only* on the brane [16]. However, one needs to establish the Bekenstein-Hawking 'area' law not only on the brane but in the bulk as well. With these motivations, in this paper, we evaluate the canonical entropy of a massless scalar field (at the Hawking temperature) around the rotating BTZ black string using the brick wall model. As we shall show, when the contributions due to the bulk modes and the metric in the bulk are properly taken into account, we are able to recover the 'area' law both on the brane and in the bulk.

This paper is organized as follows. In the next section, we shall briefly review the features of rotating BTZ black strings that are essential for our discussion that follows. In Section III, we shall evaluate the entropy of a massless scalar field around the rotating BTZ black string using the brick wall model. In Subsection III A, we shall first consider the case of a non-rotating BTZ black string and, then, in Subsection III B, we shall extend our analysis to the rotating case. Finally, we shall close with Section IV with a brief summary of our results.

Note that we shall work in units such that $G = \hbar = c = 1$. Also, the metric signature we shall adapt will be $(-,+,+,+)$.

II. THE BTZ BLACK STRING—ESSENTIALS

In this section (which closely follows Ref. [13]), we shall summarize the construction of a BTZ black string that is bound to a two-brane in a four dimensional anti-de Sitter bulk (adS$_4$, hereafter) and point out its essential features.

A solution to the Einstein's equation with a negative cosmological constant that describes an accelerating black hole in adS$_4$ is given by the line-element [18]

$$ds^2 = \left(\frac{1}{A^2(x-y)^2}\right)\left[-H(y)\,dt^2 + \left(\frac{dx^2}{G(x)}\right) + \left(\frac{dy^2}{H(y)}\right) + G(x)\,d\phi^2\right], \tag{2}$$

where the functions $G(x)$ and $H(y)$ are given by

$$G(x) = \left(1 + \kappa x^2 - 2mAx^3\right), \quad H(y) = \left(\lambda - \kappa y^2 + 2mAy^3\right), \tag{3}$$

with $\kappa = 0, \pm 1$, and m and A are parameters that are related to the mass and the acceleration of the black hole. For $\lambda > -1$, the line-element (2) is called the adS C-metric and it satisfies the relation

$$R_{AB} = -\left[(3/\ell_4^2)\,g_{AB}\right], \quad \text{where} \quad \ell_4 = \left(A\sqrt{\lambda+1}\right)^{-1}, \tag{4}$$

and the bulk cosmological constant is given by $\Lambda_4 = -(3/\ell_4^2)$. When $m = 0$, the line-element (2) can be written in terms of the variables

$$r = \left(\frac{\sqrt{y^2 + \lambda\,x^2}}{A(x-y)}\right) \quad \text{and} \quad \rho = \left(\frac{1 + \kappa\,x^2}{y^2 + \lambda\,x^2}\right)^{1/2} \tag{5}$$

as follows [19]:

$$ds^2 = r^2\left[-\left(\lambda\rho^2 - \kappa\right)\,dt^2 + \left(\frac{d\rho^2}{\lambda\rho^2 - \kappa}\right) + \rho^2\,d\phi^2\right] + \left(\frac{dr^2}{(r/\ell_4)^2 - \lambda}\right). \tag{6}$$

In order to introduce a brane into the adS$_4$ bulk, we need to identify a surface whose extrinsic curvature is proportional to the intrinsic metric. We can then cut the spacetime off at the surface and glue two copies of one of the sides to form the brane. As surfaces of constant r in the above line-element satisfy this property, any of these surfaces can be glued to a copy of itself to construct the two-brane. Also, since the constant r surface has a constant curvature, there will be a cosmological constant induced on the brane that is given by $\Lambda_3 = -(\lambda/r^2)$. Therefore, for $\lambda > 0$, the geometry of the brane will be that of three-dimensional anti-de Sitter spacetime (i.e. adS_3). If we now assume that $\kappa = 1$, then the line-element (6) describes a massive BTZ black hole on the brane, and a BTZ black string in the bulk, *provided* we make a periodic identification of ϕ with a period, say, (2π).

To introduce a negative cosmological constant in the bulk, one introduces a second brane so that the fourth dimension is compactified. A convenient choice here is to fix one brane at $x = 0$ and the second brane at a constant y position such that $H'(y) = 0$, again a simple choice being $y = 0$. The region $x \geq 0$ and $y \geq 0$ is glued to a copy of itself along the boundaries $x = 0$ and $y = 0$. These boundaries correspond to $r = (1/A)$ and $r = (\sqrt{\lambda}/A)$ and both surfaces have positive intrinsic curvature and positive tensions. The two branes are therefore chosen to be located at $L_1 = (1/A)$ and $L_2 = (\sqrt{\lambda}/A)$.

The effects of rotation can be included by starting with the following line-element that describes an accelerating and rotating black hole in adS_4 [18]

$$\begin{aligned} ds^2 \;=\; \left(\frac{1}{A^2(x-y)^2}\right)\Biggl[-\left(\frac{H(y)}{(1+a^2x^2y^2)}\right)\left(dt + ax^2 d\phi\right)^2 + \left(1+a^2x^2y^2\right)\left[\left(\frac{dy^2}{H(y)}\right) + \left(\frac{dx^2}{G(x)}\right)\right] \\ + \left(\frac{G(x)}{(1+a^2x^2y^2)}\right)\left(d\phi - ay^2 dt\right)^2\Biggr], \qquad (7) \end{aligned}$$

where the functions $G(x)$ and $H(y)$ are now given by

$$G(x) = \left(1 + \kappa x^2 - 2mAx^3 + \lambda a^2 x^4\right), \quad H(y) = \left(\lambda - \kappa y^2 + 2mAy^3 + a^2 y^4\right) \qquad (8)$$

and, as before, $\kappa = 0, \pm 1$. The above line-element also satisfies the relation (4) and the adS C-metric (2) can be obtained by setting $a = 0$. When $m = 0$, the line-element (7) describes adS_4 and in terms of the variable r [as defined in Eq. (5)] and the variable ρ now defined as

$$\rho = \left(\frac{1 + \kappa x^2 - a^2x^2y^2}{y^2 + \lambda x^2}\right)^{1/2}, \qquad (9)$$

the line-element reduces to

$$ds^2 = r^2\left[-\left[\lambda\rho^2 - \kappa + (a^2/\rho^2)\right]\, dt^2 + \left(\frac{d\rho^2}{\lambda\rho^2 - \kappa + (a^2/\rho^2)}\right) + \rho^2\, \left[d\phi - (a\, dt/\rho^2)\right]^2\right] + \left(\frac{dr^2}{(r/\ell_4)^2 - \lambda}\right). \qquad (10)$$

For $\lambda > 0$, $\kappa = 1$, and a brane located at a constant r, the line-element above describes the geometry of a rotating BTZ black hole on the brane and, the entire line-element, therefore, describes the geometry of a rotating BTZ black string.

It is useful to note here that the issue of stability of higher dimensional black strings can be addressed in this scenario as analytical solutions to both the bulk and the brane are available. In fact, it has been shown that the configuration of two branes in the four dimensional adS bulk is stable as long as the minimum transverse size of the black string remains larger than the four dimensional adS scale. When the transverse size reaches this value, the black string breaks up and forms localized black holes (for a detailed discussion, see Ref. [13]). This geometry is therefore useful to study effects of extra dimensions on black hole entropy as, it closely follows, in spirit, the two brane Randall-Sundrum scenario. The main difference in this model from the Randall-Sundrum scenario is that both the branes here have a positive tension.

III. ENTROPY OF THE BTZ BLACK STRING

In this section, we shall evaluate the canonical entropy of a free and massless scalar field (at the Hawking temperature) around the BTZ black string using the brick wall approach. In Subsection III A, we shall consider the case of the non-rotating BTZ black string and, in Subsection III B, we shall extend our analysis to the rotating case.

A. The non-rotating case

As we had mentioned in the introductory section, in the brick wall approach, the entropy of black holes is attributed to the statistical entropy of matter fields that are in thermal equilibrium with the black hole. A simple choice for the matter field would be a free and massless scalar field, say, Φ that satisfies the wave equation

$$\Box\Phi = 0. \tag{11}$$

In the spacetime of a BTZ black string described by the line-element (6), the modes of the scalar field Φ can be decomposed as follows:

$$\Phi\,(t,\rho,\phi,r) = \left[e^{-iEt}\, e^{im\phi}\, R_{Em\mu}(\rho)\, F_\mu(r)\right], \tag{12}$$

where the functions $R_{Em\mu}$ and F_μ satisfy the differential equations

$$\left(\frac{1}{\rho\, N^2(\rho)}\right)\frac{d}{d\rho}\left[\rho\, N^2(\rho)\left(\frac{dR_{Em\mu}}{d\rho}\right)\right] + k_\rho^2\, R_{Em\mu} = 0 \tag{13}$$

and

$$\left(\frac{1}{r^3 P(r)}\right)\frac{d}{dr}\left[r^3\, P(r)\left(\frac{dF_\mu}{dr}\right)\right] + k_r^2\, F_\mu = 0, \tag{14}$$

respectively. The quantities N, P, k_ρ and k_r in the above two differential equations are given by the expressions

$$N^2(\rho) = \left(\lambda\,\rho^2 - \kappa\right), \qquad P^2(r) = \left[(r/\ell_4)^2 - \lambda\right], \tag{15}$$

$$k_\rho^2(E,m,\mu) = \left(\frac{1}{N^4(\rho)}\right)\left(E^2 - \left(\frac{mN(\rho)}{\rho^2}\right)^2 - \left[\mu\, N(\rho)\right]^2\right), \qquad k_r^2(\mu) = \left(\frac{\mu}{rP(r)}\right)^2. \tag{16}$$

The free energy $\mathcal{F}$ of a scalar field at the inverse temperature β can be written as (see, for example, Ref. [4])

$$\mathcal{F} = \left(\frac{1}{\beta}\right)\int dE\,\left(\frac{dg(E)}{dE}\right)\,\ln\left[1 - e^{-\beta E}\right], \tag{17}$$

where $g(E)$ denotes the number of the modes of the field upto the energy E and the integral is to be carried out over all allowed values of E. In general, it turns out to be a difficult task to determine the number of states $g(E)$ of a quantum field exactly around a black hole. (It can be evaluated exactly only in some special cases of two-dimensional black holes; see, for instance, Ref. [20].) However, under certain conditions, it can be evaluated in the WKB approximation—a procedure that is often referred to in the literature as the brick-wall model [4].

Let us now introduce an infrared and an ultraviolet cutoff so that the scalar field Φ vanishes at $\rho = (\rho_H + \epsilon)$ and at $\rho = \mathcal{R}$, where $\epsilon \ll \rho_{\rm H}$ and $\mathcal{R} \gg \rho_{\rm H}$, $\rho_{\rm H} = \sqrt{\kappa/\lambda}$ being the horizon of the black string, i.e. where $N = 0$. (As we shall see, the entropy associated with the quantum field diverges as $\epsilon \to 0$ and, the ultraviolet cutoff, viz. $\epsilon \to 0^+$, is referred to as the brick-wall above the horizon.) In the WKB limit, for a given m and μ, the number of radial modes upto a given energy E is given by (cf. Refs. [4, 15, 16])

$$n_\rho(E,m,\mu) = \left(\frac{1}{\pi}\right)\int\limits_{(\rho_H+\epsilon)}^{\mathcal{R}} d\rho\;\bar{k}_\rho(E,m,\mu), \tag{18}$$

where $\bar{k}_\rho = k_\rho$ when $k_\rho^2 > 0$, and $\bar{k}_\rho = 0$ when $k_\rho^2 \le 0$. Similarly, the number of modes in the direction of the bulk can be written as

$$n_r(\mu) = \left(\frac{1}{\pi}\right)\int\limits_{L_1}^{L_2} dr\;\bar{k}_r(\mu), \tag{19}$$

where, as in the radial direction, $\bar{k}_r = k_r$ when k_r^2 is positive, and zero otherwise. Then, in the WKB limit, the total number of states $g(E)$ of the field not exceeding the energy E is given by

$$g(E) = \int d\mu\left(\frac{dn_r(\mu)}{d\mu}\right)\int dm\; n_\rho(E,m,\mu). \tag{20}$$

It should be pointed out here that the spectrum of energy E will actually be discrete due to the boundary conditions imposed on the field at $\rho = (\rho_H + \epsilon)$ and at $\rho = \mathcal{R}$. However, the gap between these energy levels will be small if $\mathcal{R}$ is assumed to be large. Therefore, in what follows, we shall integrate (rather than sum) over the allowed states.

The free energy of the quantized, free and massless scalar field can now be written as

$$\mathcal{F} \approx \left(\frac{1}{\beta}\right) \int dE \int d\mu \left(\frac{dn_r}{d\mu}\right) \int dm \left(\frac{dn_\rho}{dE}\right) \ln\left[1 - e^{-\beta E}\right] \tag{21}$$

which, on integrating over E by parts, reduces to [16]

$$\mathcal{F} = -\int dE \int dm \int d\mu \left(\frac{dn_r}{d\mu}\right) \left(\frac{n_\rho}{e^{\beta E} - 1}\right). \tag{22}$$

From the expressions (16) and (19), we find that

$$\left(\frac{dn_r}{d\mu}\right) = \left(\frac{1}{\pi}\right) \int_{L_1}^{L_2} dr \left(\frac{d\bar{k}_r}{d\mu}\right) = \left(\frac{1}{\pi}\right) \int_{L_1}^{L_2} \frac{dr}{r\,P}. \tag{23}$$

On using this result and the definition (18) in the expression (22) for the free energy $\mathcal{F}$, we obtain that

$$\mathcal{F} = -\left(\frac{1}{\pi}\right) \int_{L_1}^{L_2} \frac{dr}{r\,P} \int_{(\rho_H+\epsilon)}^{\mathcal{R}} \frac{d\rho}{N^2} \int dE \int d\mu \int dm \left(\frac{1}{e^{\beta E} - 1}\right) \left[E^2 - (mN/\rho)^2 - (\mu\,N)^2\right]^{1/2}. \tag{24}$$

The limits of integration on m and μ are determined by the range of values for which the argument of the square root remains positive. This condition leads to the following limits for the remaining integrals:

$$0 \le m \le (\rho/N) \left[E^2 - \mu^2 N^2\right]^{1/2}, \quad 0 \le \mu \le (E/N) \quad \text{and} \quad 0 \le E \le \infty. \tag{25}$$

On carrying out the integrals over m and μ, in that order, we find that the expression for $\mathcal{F}$ simplifies to a product of two separate integrals which in turn can be easily integrated to yield the result

$$\mathcal{F} = -\left(\frac{1}{6\,\pi}\right) \int_{L_1}^{L_2} \frac{dr}{r\,P} \int_{(\rho_H+\epsilon)}^{\mathcal{R}} \frac{d\rho\,\rho}{N^4} \int_0^\infty \frac{dE\,E^3}{(e^{\beta E} - 1)} = \left(\frac{\zeta(4)}{2\,\lambda^2\,\beta^4}\right) \int_{L_1}^{L_2} \frac{dr}{[r\,P(r)]} \left[\left(\frac{1}{\mathcal{R}^2 - \rho_H^2}\right) - \left(\frac{1}{(\rho_H + \epsilon)^2 - \rho_H^2}\right)\right], \tag{26}$$

where $\zeta(4)$ denotes the Riemann ζ-function. As $\mathcal{R} \to \infty$, this expression reduces to

$$\mathcal{F} = -\left(\frac{\zeta(4)}{2\,\lambda^2\,\beta^4}\right) \int_{L_1}^{L_2} \frac{dr}{r\,P} \left(\frac{1}{(\rho_H + \epsilon)^2 - \rho_H^2}\right) \tag{27}$$

and the need for a brick-wall above the horizon is evident from this expression for the free energy $\mathcal{F}$—it would have diverged had ϵ been zero. As we had mentioned in the introductory section, this divergence arises due to the infinite blue-shifting of the modes near the horizon [4].

The entropy $\mathcal{S}$ corresponding to the free energy $\mathcal{F}$ is given by [4, 16]

$$\mathcal{S} = \beta^2 \left(\frac{\partial \mathcal{F}}{\partial \beta}\right) = \left(\frac{2\,\zeta(4)}{\lambda^2\,\beta^3}\right) \int_{L_1}^{L_2} \frac{dr}{r\,P} \left(\frac{1}{(\rho_H + \epsilon)^2 - \rho_H^2}\right). \tag{28}$$

Since the scalar field is in thermal equilibrium with the black string, the entropy $\mathcal{S}_{\rm BS}$ of the BTZ black string can be obtained by evaluating the above entropy at the inverse Hawking temperature $\beta_{\rm H}$ of the string. We find that, the entropy $\mathcal{S}_{\rm BS}$ of the black string is given by

$$\mathcal{S}_{BS} = \left(\frac{2\,\zeta(4)}{\lambda^2\,\beta_H^3}\right) \int_{L_1}^{L_2} \frac{dr}{r\,P} \left(\frac{1}{(\rho_H + \epsilon)^2 - \rho_H^2}\right) \tag{29}$$

an expression which, clearly, diverges as ϵ approaches zero.

In order to do away with this divergence, we shall now follow the 't Hooft's procedure [4] and express the entropy of the black string in terms of an invariant ultraviolet cutoff, say, $\tilde{\epsilon}$ rather than the brick wall location ϵ. The invariant cutoff $\tilde{\epsilon}$ is defined as the invariant distance between the horizon and the brick wall, viz.

$$\tilde{\epsilon} = \int_{\rho_{\rm H}}^{\rho_{\rm H}+\epsilon} d\rho\, \sqrt{g_{\rho\rho}} = r \int_{\rho_{\rm H}}^{\rho_{\rm H}+\epsilon} \frac{d\rho}{N} = \left(r/\sqrt{\lambda}\right)\, \cosh^{-1}\left(\frac{\rho_{\rm H}+\epsilon}{\rho_{\rm H}}\right). \tag{30}$$

On using this relation, we find that the entropy $\mathcal{S}_{\rm BS}$ of the black string can then be written in terms of $\tilde{\epsilon}$ as follows:

$$\mathcal{S}_{\rm BS} = \left(\frac{\lambda\, \zeta(4)\, \rho_H}{4\pi^3}\right) \int_{L_1}^{L_2} \frac{dr}{r\, P}\, \sinh^{-2}\left(\sqrt{\lambda}\, \tilde{\epsilon}/r\right), \tag{31}$$

where we have made use of the fact that $\beta_{\rm H} = (2\pi/\lambda\, \rho_{\rm H})$. If we now assume that $\tilde{\epsilon}$ is a very small quantity—i.e. the brick wall is located very close to the horizon—then, in the leading order, we have

$$\mathcal{S}_{\rm BS} \simeq \left(\frac{\lambda\, \zeta(4)\, \rho_{\rm H}}{4\, \pi^3}\right) \int_{L_1}^{L_2} \frac{dr}{r\, P} \left(\frac{r}{\sqrt{\lambda}\, \tilde{\epsilon}}\right)^2 = \left(\frac{\zeta(4)}{4\pi^3\, \tilde{\epsilon}^2}\right)\, (\rho_{\rm H}\, \mathcal{L}), \tag{32}$$

where

$$\mathcal{L} = \int_{L_1}^{L_2} \frac{dr\ r}{P} \tag{33}$$

is the invariant length of the black string along the extra dimension. We can now write the above expression for the entropy as

$$\mathcal{S}_{\rm BS} = \left(\frac{\zeta(4)}{8\pi^4\, \tilde{\epsilon}^2}\right)\, \mathcal{A}_{\rm BS}, \tag{34}$$

where $\mathcal{A}_{\rm BS} = [(2\pi\, \rho_{\rm H})\, \mathcal{L}]$ is the area of the black string horizon. Evidently, the Bekenstein-Hawking area law (1) will be satisfied if we choose $\tilde{\epsilon}$ to be of the order of Planck length.

B. The case of the rotating black string

In this subsection, we shall evaluate the entropy of a rotating black string following the method outlined in the last section. However, in the background of a rotating black hole, the calculation of the entropy turns out to be a bit more involved due to the presence of the super-radiant modes [7]. The contribution of the super-radiant modes to the entropy of the rotating BTZ black string needs to be taken into account carefully and, when done so, as we shall see, one regains the result for the non-rotating black string in the limit of zero angular momentum.

Recall that the rotating BTZ black string is described by the line-element (10). The modes of the free and massless scalar field Φ propagating in such background can be decomposed exactly as we did earlier in Eq. (12) in the case of the non-rotating black string. The decomposition (12) of the scalar field mode leads to the same differential equation for $R_{Em\mu}$ as before [viz. Eq. (13)], but with N and k_ρ now given by

$$N^2 = \left[\lambda\, \rho^2 - \kappa + \left(a^2/\rho^2\right)\right], \tag{35}$$

$$k_\rho^2(E,m,\mu) = \left(\frac{1}{N^4(\rho)}\right) \left(E^2 - \left(\frac{m^2}{\rho^2}\right)(\lambda\rho^2-\kappa) - \left(\frac{2amE}{\rho^2}\right) - [\mu\, N(\rho)]^2\right). \tag{36}$$

As will be evident later, it proves to be convenient to write the above expression for $k_\rho^2(E,m,\mu)$ as follows:

$$k_\rho^2(E,m,\mu) = \left(\frac{1}{N^4(\rho)}\right) \left[(E - m\,\Omega_-)\ (E - m\,\Omega_+) - [\mu\, N(\rho)]^2\right]^{1/2}, \tag{37}$$

where $\Omega_\pm = \left[(a \pm \rho\, N(\rho))\,/\rho^2\right]$. Moreover, as expected, the bulk mode F_μ satisfies the same differential equation as in the non-rotating case, viz. Eq. (14). The inner and the outer horizons of the rotating black string, say, ρ_+ and ρ_-, correspond to the points where N vanishes. They are given by

$$\rho_\pm^2 = \left(\frac{1}{2\lambda}\right)\,\left[\kappa \pm \sqrt{\kappa^2 - 4\lambda a^2}\right] \tag{38}$$

and it is useful to note that the quantities appearing in the metric can be written in terms of ρ_+ and ρ_- as follows:

$$N^2 = \left(\frac{\lambda}{\rho^2}\right)\,\left[\left(\rho^2-\rho_+^2\right)\,\left(\rho^2-\rho_-^2\right)\right], \qquad \kappa = \lambda\,\left(\rho_+^2+\rho_-^2\right) \quad \text{and} \quad a = \sqrt{\lambda}\,\left(\rho_+\,\rho_-\right). \tag{39}$$

As we mentioned above, around a rotating black hole, there exist super-radiant modes whose contribution needs to be taken into account carefully. These are modes for which $(E - m\,\Omega_{\rm H}) < 0$, where $\Omega_{\rm H}$ is the angular speed of observers with zero angular momentum (i.e. ZAMOS) at the outer horizon ρ_+. Since the angular speed of ZAMOS at a given radius is $\Omega = \left(a/\rho^2\right)$, we have $\Omega_{\rm H} = \left(a/\rho_+^2\right)$. Around the rotating black string, the free energy $\mathcal{F}$ of the scalar field Φ can be written as

$$\mathcal{F} = \left(\mathcal{F}_{\rm NS} + \mathcal{F}_{SR}\right), \tag{40}$$

where the non-super-radiant and the super-radiant contributions, viz. $\mathcal{F}_{\rm NS}$ and $\mathcal{F}_{SR}$, are given by (for a detailed discussion on this point, see Ref. [7])

$$\mathcal{F}_{\rm NS} \;=\; \left(\frac{1}{\beta}\right)\int\limits_{\notin\,{\rm SR}} dE\,\left(\frac{dg(E)}{dE}\right)\,\ln\left[1 - e^{-\beta(E - m\,\Omega_{\rm H})}\right], \tag{41}$$

$$\mathcal{F}_{\rm SR} \;=\; \left(\frac{1}{\beta}\right)\int\limits_{\in\,{\rm SR}} dE\,\left(\frac{dg(E)}{dE}\right)\,\ln\left[1 - e^{\beta(E - m\,\Omega_{\rm H})}\right]. \tag{42}$$

In these expressions, $g(E)$, as earlier, is given by Eq. (20) with n_ρ and n_r defined in Eqs. (18) and (19). Also, as in the non-rotating case, $\bar{k}_\rho = k_\rho$ when k_ρ^2 is positive, and zero otherwise. However, note that, k_ρ is now given by Eq. (36), and the brick-wall is located just beyond the outer horizon at $(\rho_+ + \epsilon)$. The limits for μ prove to be the same for the super-radiant as well as the non-super-radiant modes, and the limits are given by $0 \le \mu \le [(E - m\,\Omega_+)\,(E - m\,\Omega_-)/N]$.

Let us first consider the contribution to the free energy due to the non-super-radiant modes. It is given by

$$\mathcal{F}_{\rm NS} \approx \left(\frac{1}{\beta}\right)\int dE \int d\mu\,\left(\frac{dn_r}{d\mu}\right)\int dm\,\left(\frac{dn_\rho}{dE}\right)\,\ln\left[1 - e^{-\beta(E - m\,\Omega_{\rm H})}\right] \tag{43}$$

which, on integrating over E by parts, reduces to

$$\begin{aligned}
\mathcal{F}_{\rm NS} \;=\; &-\left(\frac{1}{\pi}\right)\int\limits_{(\rho_+ + \epsilon)}^{\mathcal{R}} \frac{d\rho}{N^2}\int dE \int d\mu\left(\frac{dn_r}{d\mu}\right)\int dm\,\left(\frac{1}{e^{\beta(E - m\,\Omega_{\rm H})} - 1}\right)\,\left[(E - m\,\Omega_-)\,(E - m\,\Omega_+) - [\mu\,N]^2\right]^{1/2}\\
&+\left(\frac{1}{\pi\beta}\right)\int\limits_{(\rho_+ + \epsilon)}^{\mathcal{R}} \frac{d\rho}{N^2}\int d\mu\left(\frac{dn_r}{d\mu}\right)\int dm\,\left[(E - m\,\Omega_-)\,(E - m\,\Omega_+) - [\mu\,N]^2\right]^{1/2}\,\ln\left[1 - e^{-\beta(E - m\,\Omega_{\rm H})}\right]_{E_{\rm min}}^{E_{\rm max}}.
\end{aligned} \tag{44}$$

It is useful to note that the second term in the above expression vanishes for the case of the non-rotating black string.) The above expression for $\mathcal{F}_{\rm NS}$ can be conveniently divided into a part with positive angular momentum states (i.e. with $m \ge 0$) and another with negative angular momentum states (i.e. with $m < 0$). For $0 \le m < \infty$, the limits of the integrals over ρ and E are given by $(\rho_+ + \epsilon) \le \rho \le \mathcal{R}$ and $(m\,\Omega_+) \le E < \infty$. Also, since $\Omega_+ > \Omega_-$, we have $E > (m\,\Omega_+)$. For the negative angular momentum states (i.e. when $\infty < m \le 0$), we find that, for $(\rho_+ + \epsilon) < \rho < \rho_{\rm erg} = \sqrt{k/\lambda}$, the limits of the integral over E is given by $0 \le E \le \infty$, whereas, for $(\rho_+ + \epsilon) \le \rho \le \rho_{\rm erg}$ the limits of the integral over E are given by $(m\,\Omega_-) \le E < \infty$. Therefore, the contribution to the free energy due to the non-super-radiant modes with positive and negative angular momentum states (which we shall denote as $\mathcal{F}_{\rm NS}^+$ and $\mathcal{F}_{\rm NS}^-$, respectively) are given by

$$\mathcal{F}_{\rm NS}^+ = -\left(\frac{1}{\pi}\right)\int d\mu\,\left(\frac{dn_r}{d\mu}\right)\int\limits_{(\rho_+ + \epsilon)}^{\mathcal{R}} d\rho \int\limits_0^\infty dm \int\limits_{(m\Omega_+)}^\infty dE\,\left(\frac{\bar{k}_\rho}{e^{\beta(E - m\Omega_{\rm H})} - 1}\right) \tag{45}$$

and

$$\mathcal{F}_{\rm NS}^{-} = -\left(\frac{1}{\pi}\right)\int d\mu\,\left(\frac{dn_r}{d\mu}\right)\int\limits_{(\rho_{+}+\epsilon)}^{\rho_{\rm erg}} d\rho\int_{-\infty}^{0} dm\int_{0}^{\infty} dE\,\left(\frac{\bar{k}_{\rho}}{e^{\beta(E-m\,\Omega_{\rm H})}-1}\right)$$
$$+\left(\frac{1}{\pi}\right)\int d\mu\,\left(\frac{dn_r}{d\mu}\right)\int\limits_{\rho_{\rm erg}}^{\mathcal{R}} d\rho\int\limits_{-\infty}^{0} dm\int\limits_{(m\,\Omega_{-})}^{\infty} dE\,\left(\frac{\bar{k}_{\rho}}{e^{\beta(E-m\,\Omega_{\rm H})}-1}\right)$$
$$-\left(\frac{1}{\pi\beta}\right)\int d\mu\,\left(\frac{dn_r}{d\mu}\right)\int\limits_{(\rho_{+}+\epsilon)}^{\rho_{erg}} d\rho\int\limits_{-\infty}^{0} dm\,\bar{k}_{\rho}(E=0)\,\ln\left(1-e^{m\,\beta\,\Omega_{\rm H}}\right). \tag{46}$$

The contribution to the free energy due to the super-radiant modes can be evaluated in a similar fashion. In this case, for $0\le m<\infty$, the limits of integration for ρ and E turn out to be: $(\rho_{+}+\epsilon)\le\rho\le\rho_{\rm erg}$ and $0\le E\le(m\Omega_{-})$. The corresponding contribution to the free energy is given by

$$\mathcal{F}_{\rm SR} = -\left(\frac{1}{\pi}\right)\int d\mu\,\left(\frac{dn_r}{d\mu}\right)\int\limits_{\rho_{+}+\epsilon}^{\rho_{erg}} d\rho\int\limits_{-\infty}^{0} dm\int\limits_{(m\Omega_{+})}^{\infty} dE\,\left(\frac{\bar{k}_{\rho}}{e^{-\beta(E-m\Omega_{\rm H})}-1}\right)$$
$$+\left(\frac{1}{\pi\beta}\right)\int d\mu\,\left(\frac{dn_r}{d\mu}\right)\int\limits_{(\rho_{+}+\epsilon)}^{\rho_{\rm erg}} d\rho\int\limits_{0}^{\infty} dm\,\bar{k}_{\rho}\,\ln\left(1-e^{\beta m\Omega_{\rm H}}\right). \tag{47}$$

For convenience, we shall club the contribution to $\mathcal{F}_{\rm NS}$ due to the modes with $E<(m\,\Omega_{-})$ with the contribution due to the super-radiant modes [7]. Moreover, for $0\le E\le(m\,\Omega_{-})$, when m is set to $-m$ in $\mathcal{F}_{\rm NS}$, we obtain a term which is the same as the contribution in the super-radiant modes, but with an opposite sign. Therefore, we shall drop these two terms hereafter. Moreover, the second expression in Eq. (44) vanishes for $\rho_{\rm erg}\le\rho\le\mathcal{R}$. Due to these reasons, we have

$$\mathcal{F}_{\rm NS}^{+} = -\left(\frac{1}{4}\right)\int\limits_{L_1}^{L_2}\frac{dr}{r\,P}\int\limits_{(\rho_{+}+\epsilon)}^{\mathcal{R}}\frac{d\rho}{N^{3}(\rho)}\int_{0}^{\infty} dm\int\limits_{(m\Omega_{+})}^{\infty} dE\,\left[\frac{(E-m\Omega_{+})(E-m\Omega_{-})}{e^{\beta(E-m\Omega_{\rm H})}-1}\right], \tag{48}$$

$$\mathcal{F}_{\rm NS}^{-} = -\left(\frac{1}{4}\right)\int\limits_{L_1}^{L_2}\frac{dr}{r\,P}\int\limits_{(\rho_{+}+\epsilon)}^{\rho_{\rm erg}}\frac{d\rho}{N^{3}(\rho)}\int\limits_{-\infty}^{0} dm\int\limits_{0}^{\infty} dE\left[\frac{(E-m\Omega_{+})(E-m\Omega_{-})}{e^{\beta(E-m\Omega_{\rm H})}-1}\right]$$
$$+\left(\frac{1}{4}\right)\int\limits_{L_1}^{L_2}\frac{dr}{r\,P}\int_{\rho_{\rm erg}}^{\mathcal{R}}\frac{d\rho}{N^{3}(\rho)}\int\limits_{-\infty}^{0} dm\int\limits_{(m\Omega_{-})}^{\infty} dE\,\left[\frac{(E-\Omega_{+}m)(E-\Omega_{+}m)}{e^{\beta(E-m\Omega_{\rm H})}-1}\right]$$
$$-\left(\frac{1}{4\beta}\right)\int\limits_{L_1}^{L_2}\frac{dr}{r\,P}\int\limits_{(\rho_{+}\epsilon)}^{\rho_{\rm erg}}\frac{d\rho}{N^{3}(\rho)}\int\limits_{-\infty}^{0} dm\,\left(m^{2}\,\Omega_{+}\Omega_{-}\right)\,\ln\left(1-e^{\beta m\Omega_{\rm H}}\right), \tag{49}$$

$$\mathcal{F}_{\rm SR} = -\left(\frac{1}{4}\right)\int\limits_{L_1}^{L_2}\frac{dr}{r\,P}\int\limits_{\rho_{+}}^{\rho_{\rm erg}}\frac{d\rho}{N^{3}(\rho)}\int_{-\infty}^{0} dm\int\limits_{(m\Omega_{+})}^{\infty} dE\,\left[\frac{(E-\Omega_{+}m)(E-\Omega_{+}m)}{e^{-\beta(E-m\Omega_{\rm H})}-1}\right]$$
$$+\left(\frac{1}{4\beta}\right)\int\limits_{L_1}^{L_2}\frac{dr}{r\,P}\int\limits_{(\rho_{+}+\epsilon)}^{\rho_{\rm erg}}\frac{d\rho}{N^{3}(\rho)}\int\limits_{0}^{\infty} dm\,\left(m^{2}\,\Omega_{+}\Omega_{-}\right)\left[\ln\left(1-e^{-\beta m\Omega_{\rm H}}\right)\right], \tag{50}$$

where we have made use the expression (23) for $(dn_r/d\mu)$.

Let us now first consider the contribution due to the positive angular momentum states of the the non-super-radiant modes. The E and m integrals in the expression for $\mathcal{F}_{\rm NS}^{+}$ can be separated by the coordinate transformation

$E = (m\Omega_+ x)$ and we obtain that

$$\mathcal{F}^+_{\rm NS} = -\left(\frac{1}{4}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \int\limits_{(\rho_+ + \epsilon)}^{\mathcal{R}} \frac{d\rho}{N^3(\rho)}\, \Omega_+^2 \int\limits_1^\infty dx\,(x-1)\,(\Omega_+ x - \Omega_-) \int\limits_0^\infty dm\, \left(\frac{m^3}{e^{\beta m(\Omega_+ x - \Omega_{\rm H})} - 1}\right). \tag{51}$$

On carrying out the integral over m first and then subsequently integrating over x, we obtain that

$$\mathcal{F}^+_{\rm NS} = -\left(\frac{\zeta(4)\Gamma(4)}{24\beta^4}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \int\limits_{(\rho_+ + \epsilon)}^{\mathcal{R}} \frac{d\rho}{N^3(\rho)} \left[\frac{3\Omega_+ - 2\Omega_{\rm H} - \Omega_-}{(\Omega_{\rm H} - \Omega_+)^2}\right]. \tag{52}$$

Similarly, for $\mathcal{F}^-_{\rm NS}$, we can transform variables to $E = (mx)$ and $E = (\Omega_- mx)$ for the first and second terms, respectively. We find that we can write

$$\begin{aligned}
\mathcal{F}^-_{\rm NS} \;=\; & -\left(\frac{\zeta(4)\Gamma(4)}{24\beta^4}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \int\limits_{(\rho_+ + \epsilon)}^{\rho_{\rm erg}} \frac{d\rho}{N^3(\rho)} \left(\frac{2\Omega_{\rm H}^2 + 2\Omega_+\Omega_- + (\Omega_+ + \Omega_-)\,\Omega_H}{\Omega_{\rm H}^3}\right) \\
& -\left(\frac{\zeta(4)\Gamma(4)}{24\beta^4}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \int\limits_{\rho_{\rm erg}}^{\mathcal{R}} \frac{d\rho}{N^3(\rho)} \left(\frac{\Omega_+ + 2\Omega_{\rm H} - 3\Omega_-}{(\Omega_{\rm H} - \Omega_+)^2}\right) \\
& -\left(\frac{1}{4\beta}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \int\limits_{(\rho_+ + \epsilon)}^{\rho_{\rm erg}} \frac{d\rho}{N^3(\rho)} \int\limits_{-\infty}^{0} dm\,\left(m^2\,\Omega_+\Omega_-\right)\left[\ln\left(1 - e^{\beta m \Omega_{\rm H}}\right)\right].
\end{aligned} \tag{53}$$

For the contribution due to the super-radiant modes, we can make the transformation is $E = (m\Omega_- x)$ and, after integrating over the variables m and x, we obtain that

$$\begin{aligned}
\mathcal{F}_{\rm SR} \;=\; & -\left(\frac{\zeta(4)\Gamma(4)}{24\beta^4}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \int\limits_{(\rho_+ + \epsilon)}^{\rho_{\rm erg}} \frac{d\rho}{N^3(\rho)} \left(\frac{\Omega_-^2\,(3\Omega_+\Omega_{\rm H} - \Omega_-\Omega_{\rm H} - 2\Omega_-\Omega_+)}{\Omega_{\rm H}^3\,(\Omega_{\rm H} - \Omega_+)^2}\right) \\
& +\left(\frac{1}{4\,\beta}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \int\limits_{(\rho_+ + \epsilon)}^{\rho_{\rm erg}} \frac{d\rho}{N^3(\rho)} \int\limits_0^\infty dm\,\left(m^2\,\Omega_+\Omega_-\right)\left[\ln\left(1 - e^{-\beta m \Omega_{\rm H}}\right)\right].
\end{aligned} \tag{54}$$

We need not explicitly carry out the integral containing the logarithmic term, as a similar term arise in the contribution due to the non-super-radiant modes with an opposite sign and, hence will not contribute in the final expression for the free energy. Morover, in evaluating the integrals in the above expressions, we drop terms which vanish in the limit of $\epsilon \to 0$, as we are interested only in the divergent part. On retaining the leading order terms in ϵ, we find that the various contributions to the free energy are given by

$$\mathcal{F}^+_{\rm NS} \;=\; -\frac{1}{2}\left(\frac{\zeta(4)}{\beta^4}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \left(\frac{\rho_+}{\lambda^2(\rho_+^2 - \rho_-^2)^3}\right) \left[\left(\frac{\rho_+^3\,(\rho_+^2 - \rho_-^2)}{2\rho_+\epsilon}\right) - \left(\rho_-^3 + 3\rho_-\rho_+^2\right) + 2\rho_-\rho_+^2\sqrt{2\epsilon\rho_+}\right], \tag{55}$$

$$\mathcal{F}^-_{\rm NS} \;=\; -\frac{1}{2}\left(\frac{\zeta(4)}{\beta^4}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \left(\frac{1}{\lambda^2(\rho_+^2 - \rho_-^2)^3}\right) \left[\frac{\rho_-}{\rho_+}(\rho_-^4 + \rho_-^2\rho_+^2 + 4\rho_+^4)(\rho_+^2 - \rho_-^2) + 3\,(\rho_+\,\rho_-)^2\,\frac{\left(\rho_+^2 - \rho_-^2\right)^{3/2}}{\sqrt{2\epsilon\rho_+}}\right], \tag{56}$$

$$\mathcal{F}_{\rm SR} \;=\; -\frac{1}{2}\left(\frac{\zeta(4)}{\beta^4}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \left(\frac{1}{\lambda^2(\rho_+^2 - \rho_-^2)^3}\right) \left[\rho_+^4 + \frac{3\rho_+^6}{\rho_-^2} + \frac{\rho_+^4(\rho_+^2 - \rho_-^2)}{2\rho_+\epsilon} - \frac{3\rho_+^5}{\rho_-}\frac{\sqrt{\rho_+^2 - \rho_-^2}}{\sqrt{2\epsilon\rho_+}}\right]. \tag{57}$$

On combining the contributions due to the positive and negative angular momentum states' to the non-super-radiant modes, we obtain the corresponding entropy (evaluated at the Hawking temperature) to be

$$\mathcal{S}_{\rm NS} = \left(\frac{\lambda\zeta(4)}{8\pi^3}\right) \int\limits_{L_1}^{L_2} \frac{dr}{r\,P} \left[-\rho_+ - \frac{3\rho_+^3}{\rho_-^2} + \frac{\rho_+(\rho_+^2 - \rho_-^2)}{2\rho_+\epsilon} + \frac{3\rho_+^2}{\rho_-}\frac{\sqrt{\rho_+^2 - \rho_-^2}}{\sqrt{2\epsilon\rho_+}}\right]. \tag{58}$$

Similarly, the entropy associated with the superradiant modes can be obtained to be

$$\mathcal{S}_{\rm SR}=\left(\frac{\lambda\zeta(4)}{8\pi^3}\right)\int\limits_{L_1}^{L_2}\frac{dr}{r\,P}\,\left[\rho_+ +\frac{3\rho_+^3}{\rho_-^2}+\frac{\rho_+(\rho_+^2-\rho_-^2)}{2\rho_+\epsilon}-\frac{3\rho_+^2}{\rho_-}\frac{\sqrt{\rho_+^2-\rho_-^2}}{\sqrt{2\epsilon\rho_+}}\right] \tag{59}$$

so that the the total entropy of the black string, evaluated at the Hawking temperature, is given by

$$\mathcal{S}_{\rm BS}=\left(\frac{\lambda\,\zeta(4)}{8\pi^3}\right)\int\limits_{L_1}^{L_2}\frac{dr}{r\,P}\,\left(\frac{\rho_+^2-\rho_-^2}{\epsilon}\right) \tag{60}$$

On subsituting the expression (35) for N in the definition (30), we find that, up to the leading order in ϵ, the invariant cut-off is given by

$$\tilde{\epsilon}=\left(\frac{r\,\sqrt{2\rho_+\epsilon}}{\sqrt{\lambda\,\left(\rho_+^2-\rho_-^2\right)}}\right). \tag{61}$$

Therefore, the entropy of the rotating black string can be expressed in terms of the invariant cut off as follows:

$$\mathcal{S}_{\rm BS}=\left(\frac{\zeta(4)}{4\pi^3\,\tilde{\epsilon}^2}\right)\,(\rho_+\,\mathcal{L})=\left(\frac{\zeta(4)}{8\,\pi^4\,\tilde{\epsilon}^2}\right)\,\mathcal{A}_{\rm BS} \tag{62}$$

where $\mathcal{L}$ is given by Eq. (33) and $\mathcal{A}_{\rm BS}=[(2\pi\,\rho_+)\,\mathcal{L}]$ denotes the area of the rotating string. Note that the integral over the extra dimensional bulk is same as in the non-rotating case. Moreover, in the non-rotating limit, ρ_+ reduces to $\rho_{\rm H}$, so that we recover the results we had obtained in the previous section.

IV. DISCUSSION

In this work, we have evaluated the entropy of BTZ black strings that are confined to a two-brane in a four dimensional anti-de Sitter bulk using the brick-wall approach. We find that the Bekenstein-Hawking 'area' law is satisfied both on the brane as well as in the bulk. It will be worthwhile to construct a generic proof that shows that the contribution due to the bulk modes does not affect the entropy-area relation for an an arbitrary black string (in this context, see, for example, Refs. [21]). We hope to discuss this issue in a future publication.

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