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Entropy density of spacetime from the zero point length

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ABSTRACT

It is possible to obtain gravitational field equations in a large class of theories from a thermodynamic variational principle which uses the gravitational heat density $\mathcal{S}_g$ associated with null surfaces. This heat density is related to the structure of spacetime at Planck scale, $L_P^2 = (G\hbar/c^3)$, which assigns $A_\perp/L_P^2$ degrees of freedom to any area $A_\perp$. On the other hand, it is also known that the surface term $K\sqrt{h}$ in the gravitational action correctly reproduces the heat density of the null surfaces. We provide a link between these ideas by obtaining $\mathcal{S}_g$, used in emergent gravity paradigm, from the surface term in the Einstein–Hilbert action. This is done using the notion of a nonlocal qmetric – introduced recently [arXiv:1307.5618, arXiv:1405.4967] – which allows us to study the effects of *zero-point-length* of spacetime at the transition scale between quantum and classical gravity. Computing $K\sqrt{h}$ for the qmetric in the appropriate limit directly reproduces the entropy density $\mathcal{S}_g$ used in the emergent gravity paradigm.

Thermodynamic potentials like entropy density (s), the heat density (Ts), the free energy density ($\rho - Ts$), etc. provide a link between the microscopic dynamics of molecules and macroscopic dynamics described in terms of standard thermodynamic variables like pressure, temperature, etc. Recent work has shown that the field equations of gravity, describing the evolution of spacetime, are akin to the equations describing, say, gas dynamics [1,2]. These field equations, for a large class of theories of gravity, can be obtained [3,4] by extremizing the total heat density $\mathcal{S} = \mathcal{S}_g + \mathcal{S}_m$ where $\mathcal{S}_m$ and $\mathcal{S}_g[n]$ are the matter and gravitational heat densities respectively. The latter depends on a vector field n^i of constant norm and is given by [2]

$$\mathcal{S}_g[n] \propto \left[(\nabla_i n^i)^2 - \nabla_i n^j \nabla_j n^i\right] = R_{ab} n^a n^b + (\text{tot. div.}) \qquad (1)$$

in the case of Einstein's gravity. Extremizing $\mathcal{S}_g$ with respect to all vector fields n^i simultaneously, leads to a constraint on the background metric which turns out to be identical to the field equations. If the ideas of emergent gravity paradigm are correct, we should be able to obtain this expression from a more microscopic approach. In this Letter, we will show how this can be done.

Amongst various key facts that can guide us in this task are, in particular, the possibility to describe time evolution of 3-geometry in terms of surface degrees of freedom $N_{\text{sur}} = A_\perp/L_P^2$ associated with a 2-surface with area $A_\perp$ [2], and the fact that the surface term $\mathcal{A}^E_{\text{sur}}$ in Euclidean gravitational action (given by $(8\pi)^{-1}\int K\sqrt{h}$) is closely related to the entropy density. Such facts suggest that one should be able to obtain the entropy density in Eq. (1), from $K\sqrt{h}$, in a suitable limit. The operational difficulty in this program, of course, is that L_P is not an in-built feature of the classical description of geometry. What we need is a suitable prescription to incorporate quantum gravitational effects (in particular, existence of a zero point area L_P^2), at scales reasonably bigger than L_P^2 but not totally classical; that is, we need an "effective" metric q_{ab} which acknowledges the existence of a zero point length L_P. We can then compute $K\sqrt{h}$ for this effective metric. If our ideas are correct, the entropy density this gives should match, in the appropriate limit, the one in Eq. (1). Such an effective metric q_{ab} with the necessary properties has recently been derived in Refs. [5,6]. The crucial point is to start with the geodesic interval $\sigma^2(P, p)$ between any two events P and p [7], instead of the metric tensor g_{ab}, as the key variable. The advantage in doing so is that, while we have no universal rule to understand how quantum gravity modifies the metric, there is considerable amount of evidence (see e.g., [8]) which suggests that $\sigma^2(P, p)$ is modified by

$$\sigma^2 \to \sigma^2 + L_0^2; \qquad L_0^2 = \mu^2 L_P^2 \qquad (2)$$

where μ is a factor of order unity [9]. That is, one can capture the lowest-order quantum gravitational effects by introducing a zero point length in spacetime along the lines suggested by Eq. (2).

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With this as input, one can construct a second rank symmetric *bitensor* $q_{ab}(p, P)$ which will lead to $\sigma^2 + L_0^2$ as the geodesic interval; the result turns out to be [5,6]:

$$q_{ab}(p, P; L_0^2) = A g_{ab} - \left(A - \frac{1}{A}\right) n_a n_b \tag{3}$$

where $g_{ab} = g_{ab}(p)$ is the classical metric tensor, $\sigma^2 = \sigma^2(p, P)$ is the corresponding classical geodesic interval and

$$A[\sigma; L_0] = 1 + \frac{L_0^2}{\sigma^2}; \qquad n_a = \frac{\nabla_a \sigma^2}{2\sqrt{\sigma^2}} \tag{4}$$

Working with the qmetric we can capture some of the effects of quantum gravity — especially those arising from the existence of the zero point area — *without leaving the comforts of the standard differential geometry.*

There are several non-trivial effects arising from the nonlocal description of geometry in terms of the qmetric, discussed at length in [6], with the key point being the following: Suppose $\phi(P|g)$ is some scalar computed from the metric g_{ab} and its derivatives (for example, R, $R_{ab}R^{ab}$, etc.). When we carry out the corresponding algebra using $q_{ab}(p, P)$ (with all differentiations carried out at the event p) we will end up getting a nonlocal (biscalar) $\phi(p, P; L_0^2|q)$ which depends on two events (p, P) *and* L_0^2. To obtain a local result, we now take the limit of $\sigma \to 0$ (that is, $p \to P$) keeping L_0^2 finite. The resulting $\phi(P, P; L_0^2|q)$ will show quantum gravitational residual effects due to nonzero L_0^2, essentially arising from the non-commutativity of the limits:

$$\lim_{L_0^2 \to 0} \lim_{\sigma^2 \to 0} \phi(p, P; L_0^2|q) \neq \lim_{\sigma^2 \to 0} \lim_{L_0^2 \to 0} \phi(p, P; L_0^2|q) \tag{5}$$

Note that, when we take the limit of $\sigma \to 0$ keeping L_0^2 finite, the qmetric actually diverges. So we have no assurance that we will even get anything sensible when we take the limit; surprisingly, we do. This is what yields non-trivial effects. (We refer the reader to [6] for more details.)

After this preamble, we return to our main focus, the surface term $K\sqrt{h}$ in the gravitational action. Given a fixed spacetime event P, the most natural *surface* Σ on which to evaluate this term is the one formed by events p at a constant geodesic interval $\sqrt{|\sigma^2(p, P)|} = \lambda$ from P. The intrinsic as well as extrinsic geometry of such a surface is completely determined by the geodesic structure of the background manifold, and hence is completely characterized by invariants built out of spacetime curvature. The mathematical expressions we shall need here can be found in [6], and additional geometrical aspects of *equi-geodesic* surfaces are discussed in [10].

We will use the qmetric and compute $K\sqrt{h}$ and demonstrate that it does lead to the entropy density $\mathcal{S}_g$ in Eq. (1). This is a relatively straightforward (though somewhat lengthy) computation and we shall describe the key steps. For clarity, we will work in a $D = 4$ Euclidean space (the final result is same for Lorentzian signature), and use units with $L_P = 1$ so that $L_0 = \mu$. *In the local Rindler frame around P, the origin of $t_E - x$ plane will be the horizon and hence the limit of $p \to P$ corresponds to computing a quantity on the horizon.* We want to compute $K\sqrt{h}(p, P, \mu^2)$ for the qmetric and take the limit $p \to P$ (i.e., $\lambda \to 0$) to obtain the quantum corrected entropy density.

The $[K\sqrt{h}]_q$ for the qmetric can be easily related to the corresponding quantity evaluated for the metric g_{ab} by the relation (where $\nabla_{\boldsymbol{n}} \equiv n^i \nabla_i$)

$$\left[K\sqrt{h}\right]_q = A^2 \left\{ \left[K\sqrt{h}\right]_g + \frac{3}{2}\sqrt{h}\nabla_{\boldsymbol{n}} \ln A \right\} \tag{6}$$

Series expansion of the extrinsic curvature tensor in λ yields (see [6,10])

$$K = \frac{3}{\lambda} - \frac{1}{3}\lambda\, \mathcal{S}(P) + \mathcal{O}(\lambda^2) \tag{7}$$

$$\sqrt{h} = \lambda^3 \left[1 - \frac{1}{6}\mathcal{E}(P)\,\lambda^2 + \mathcal{O}(\lambda^3)\right] \tag{8}$$

where $\mathcal{S}(P) = R_{ab}n^a n^b|_P$, and the second series is obtained from the definition $K = \partial(\ln\sqrt{h})/\partial\lambda$. In the units we are using $\sqrt{h}$ incorporates the length dimensions and $K\sqrt{h}$ has dimensions [length]^2. We also have $\nabla_{\boldsymbol{n}} A = -2\mu^2/\lambda^3$. Substituting Eq. (7) and Eq. (8) in Eq. (6), we get the result

$$\left[K\sqrt{h}\right]_q = 3A\lambda^2 - \frac{5}{6}(A\lambda^4)\, R_{ab}n^a n^b \left[1 + \frac{2}{5}\frac{\mu^2}{\lambda^2}\right] + \mathcal{O}(\lambda) \tag{9}$$

Using $A = 1 + (\mu^2/\lambda^2)$ and taking the coincidence limit $\lambda \to 0$, we get the final result

$$\begin{aligned}\lim_{\lambda \to 0} \left[K\sqrt{h}\right]_q &= 3\mu^2 - \frac{\mu^4}{3}\, R_{ab}n^a n^b \\ &= \mathcal{S}_0 - \frac{\mu^4}{3}\mathcal{S}_g \end{aligned} \tag{10}$$

with all quantities on RHS now evaluated at P. The term $\mathcal{S}_0 = 3\mu^2$ can be thought of as the zero point entropy density of the spacetime which is a new feature. Its numerical value depends on the ratio $\mu = L_0/L_P$ which we expect to be of order unity and we will comment on it towards the end. The second term is exactly the heat density used in emergent gravity paradigm.

This result is significant in several ways which we shall now describe.

The most important feature of our result is that it reproduces correctly (except for an unimportant multiplicative constant) the entropy density $\mathcal{S}_g \propto R_{ab}n^a n^b$ used in emergent gravity paradigm. *This tells us that the entire program has a remarkable level of internal consistency.* There is no way one could have guessed this result a priori and, in fact, there is no assurance that the result should even be finite in the coincidence limit of $\sigma^2 \to 0$. The qmetric itself diverges when $\sigma^2 \to 0$ and its derivatives diverge faster. It is a nice and a non-trivial feature that all the lethal divergences cancel in the final result.

Second, it is rather satisfying to obtain this result from $K\sqrt{h}$ part of the action rather than from the $R\sqrt{-g}$ part of the action. Several previous works [11] have shown that there is an intimate relationship between the surface and bulk parts of the gravitational action and hence we would have expected the correct entropy density $\mathcal{S}_g$ to emerge from either of them if it emerges from one of them. This expectation is correct and indeed we have shown earlier [6] that a similar analysis with the bulk part of the action does lead to the correct entropy density. The crucial difference is that the computation in Ref. [6] leads to an additional divergent term which, however, can be regularized to give the correct final result. The computation here, starting from the surface term, however does not lead to any divergences. This is a mathematically *non-trivial* fact which arises from a *delicate cancellation* of divergences between the two terms on the right-hand side of Eq. (6). More specifically, the numerical factor and the structure of second of these terms depend on the (disformal) form of the qmetric, and an arbitrary, ad hoc deformation of geometry will *not* lead to similar cancellation of divergences (see, however, [9]).

Further, as we argued earlier, $K\sqrt{h}$ does have the natural interpretation of (being proportional to) the heat density on the horizon. Note that when we work in the Euclideanized local Rindler frame around an event P, the Rindler horizon gets mapped to the origin of the (x, t_E) plane. The coincidence limit of $p \to P$ is precisely the same as taking the horizon limit in the local Rindler frame. In this limit, $K\sqrt{h}/8\pi$ gives the entropy density. So if we had taken the limit $L_0 \to 0$ first, we would have recovered this standard result.

Finally, the most intriguing feature of our result is the discovery of "zero point entropy density" represented by the first term $\mathcal{S}_0 = 3\mu^2$. Since this is an entropy density, it tells us that the total zero point entropy in a sphere of Planck radius is given by

$$S_0 = \frac{4\pi}{3} \times 3\mu^2 = 4\pi\mu^2 \tag{11}$$

Recently, it has been shown that the cosmological constant problem can be solved within the emergent gravity paradigm if one could attribute a value 4π to the measure of degrees of freedom in the universe at Planck epoch, if the inflation took place at GUTs scale. This measure remains as a conserved quantity during the subsequent evolution and allows one to determine the numerical value of the cosmological constant (see, for details, Ref. [12]). On the other hand, if the inflation took place at Planck scale, we need $\mu^2 \approx 1.2$ (see [13]) which is quite consistent with Eq. (11).

Unfortunately, the value of μ cannot be determined from the analysis of pure gravity sector for two reasons. First of all, there can be a numerical factor multiplying $K\sqrt{h}$ to give the entropy density. In the standard approach, this term is $(1/8\pi L_P^2)K\sqrt{h}$ but it is not clear whether we should use the same expression in a microscopic theory. Second, the overall coupling between gravity and matter is undetermined until we have introduced the matter sector which we have not yet done. If we assume that the total heat density, maximized to get the field equations is the sum of gravitational and matter heat densities (with the latter being $\mathcal{S}_m = T_{ab}n^a n^b$; see e.g. [2,3]), then one can determine the value of μ. (Incidentally, the negative sign of the second term in Eq. (10) is important for the consistency of this result; the fact that it comes out right is another consistency check for this approach.) But it is possible for a microscopic approach to modify the matter sector term to $\mathcal{S}_m = \lambda T_{ab}n^a n^b$ where λ is a numerical factor. So, altogether there is a possibility of yet another undetermined numerical factor in the theory. To see its effect, let us take the gravitational entropy term as just $K\sqrt{h}$ and write the matter sector term as $\mathcal{S}_m = \lambda T_{ab}n^a n^b$ where λ is a numerical factor. Then simple algebra shows that, to reproduce Einstein's equations $G_{ab} = 8\pi T_{ab}$ with correct coefficient, we need $(\mu^4/24\pi\lambda) = (1/8\pi)$ or $\mu^2 = \sqrt{3\lambda}$. While this is in the right range to solve the cosmological constant problem, the numerical factor cannot be fixed until we have obtained the heat density of the matter sector from a similar description. But it is clear that the result in Eq. (11), which brings in a zero-point-entropy density, could provide a more detailed and microscopic justification for this idea. This issue is under investigation.

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