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DUAL EXTREMUM PRINCIPLES FOR A CLASS OF INTERFACE PROBLEMS*

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1. Introduction. Dual extremum principles [1] have been used to obtain error bounds for a wide class of boundary value problems.

In this note, the dual extremum principles are formulated associated with the set of operator equations of the form

$$L_j\varphi_j = T_j^* k_j T_j \varphi_j = f_j(\varphi_j) \quad \text{in } D_j, \quad j = 1, 2, \tag{1}$$

with the boundary conditions

$$\varphi_j = \alpha_j \quad \text{on } \Gamma_j = \partial D_j - \Gamma, \quad j = 1, 2, \tag{2}$$

and the interface matching conditions

$$\sigma_1(\varphi_2 - \varphi_1 - g) = 0 \quad \text{on } \Gamma, \tag{3}$$

$$\sigma_1^* k_1 T_1 \varphi_1 - \sigma_2^* k_2 T_2 \varphi_2 = h(x, y) \quad \text{on } \Gamma. \tag{4}$$

Hereafter, whenever the subscript "j" appears, it should be understood that $j = 1, 2$ unless otherwise specified. D_j is a closed convex region in R^n, $n = 1, 2, 3$, with boundary ∂D_j, $\Gamma = \partial D_1 \cap \partial D_2$ being the interface. T_j^*: $H_j^u \to H_j^\varphi$ on D_j is the formal adjoint operator of T_j: $H_j^\varphi \to H_j^u$ on D_j such that

$$(u, T_j\varphi)_j = \langle T_j^* u, \varphi\rangle_j + (u, \sigma_j\varphi)_{\partial D_j}; \tag{5}$$

σ_j^*: $H_j^u \to H_j^\varphi$ on ∂D_j is the adjoint operator of σ_j: $H_j^\varphi \to H_j^u$ such that

$$(u, \sigma_j\varphi) = \langle \sigma_j^*, u, \varphi\rangle \quad \text{on } \partial D_j \tag{6}$$

and $f_j(\varphi_j)$: $H_j^\varphi \to H_j^\varphi \cdot H_j^\varphi$ and H_j^u are Hilbert spaces of functions defined on D_j with the innerproducts $\langle\ ,\ \rangle_j$ and $(\ ,\ )_j$, respectively, and α_j and g and h are known functions defined on the boundary and the interface, respectively. These kinds of problems occur in many branches of mathematical physics such as heat conduction [2], electromagnetic theory [3] and fluid dynamics [4]. When $T_j = \text{grad}$, $g = 0$ and k_j is the thermal conductivity of the medium D_j, the problem (1)–(4) reduces to finding the steady state temperature distribution φ_j in the media which are in perfect contact, $h(x, y)$ being the source function

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defined on the interface [2]. The interface problem (1)–(4) reduces to finding the magnetic vector potential φ_j in a medium such that the magnetic vector potential and the magnetic intensity at the interface are continuous when T_j = curl and k_j^{-1} is the magnetic permeability of the medium D_j. After deriving the bounding functionals to the extremizing functional, an error estimate is obtained for the approximate solution in terms of the bounding functionals. The theory is illustrated by the steady state heat conduction in a composite medium and the approximate solution is obtained by the method of Kantorovich [5].

2. Variational formulation. Consider the action functional

$$\begin{aligned} I(u_1, u_2, \varphi_1, \varphi_2) = \sum_{j=1}^{2} \Big[(u_j, T_j\varphi_j)_j - W_j(u_j, \varphi_j) \\ + \langle \sigma_j^* u_j, \alpha_j - \varphi_j \rangle_{\Gamma_j} \Big] + \langle \sigma_1^* u_1, \varphi_2 - \varphi_1 - g \rangle_\Gamma - \langle h, \varphi_2 \rangle_\Gamma \quad (7) \\ = \sum_{j=1}^{2} \Big[\langle T_j^* u_j, \varphi_j \rangle_j - W_j(u_j, \varphi_j) + \langle \sigma_j^* u_j, \alpha_j \rangle_{\Gamma_j} \Big] \\ + \langle \varphi_2, \sigma_1^* u_1 - \sigma_2^* u_2 - h \rangle_\Gamma - \langle \sigma_1^* u_1, g \rangle_\Gamma, \quad (8) \end{aligned}$$

where

$$W_j(u_j, \varphi_j) = \frac{1}{2}\left(\frac{u_j}{k_j}, u_j \right) + \langle 1, F_j(\varphi_j) \rangle \tag{9}$$

and

$$F_j(\varphi_j) = \int^{\varphi_j} f_j(t)\, dt.$$

A necessary and sufficient condition for stationary behaviour of I at (u_j, φ_j) is that the Frechet derivatives

$$\frac{\delta I}{\delta u_j} = \frac{\delta I}{\delta \varphi_j} = 0. \tag{10}$$

This leads to

$$T_j\varphi_j = \frac{\delta W_j}{\delta u_j} \text{ in } D_j, \tag{11}$$

$$\varphi_j = \alpha_j \text{ on } \Gamma_j, \tag{12}$$

$$\sigma_1(\varphi_2 - \varphi_1 - g) = 0 \text{ on } \Gamma, \tag{13}$$

and

$$T_j^* u_j = \frac{\delta W_j}{\delta \varphi_j} \text{ in } D_j, \tag{14}$$

$$\sigma_1^* u_1 - \sigma_2^* u_2 = h \text{ on } \Gamma. \tag{15}$$

Equations (11)–(15) are nothing but the canonical forms of the interface problem described by equations (1)–(4). Hence it is evident that (u_j, φ_j) is the solution of (1)–(4).

3. Dual extremum principles and error estimate. Let (u_j, φ_j) be the exact solution of (1)–(4) so that

$$\frac{\delta I}{\delta u_j} = \frac{\delta I}{\delta \varphi_j} = 0 \quad \text{at } (u_j, \varphi_j).$$

Define two sets of functions

$$S_1 = \left\{ (u_j, \varphi_j) / \frac{\delta I}{\delta u_j} = 0 \right\}, \tag{16}$$

$$S_2 = \left\{ (u_j, \varphi_j) / \frac{\delta I}{\delta \varphi_j} = 0 \right\}. \tag{17}$$

Using (7) and (8) form

$$J(B_1, B_2) = I(u_1^*, u_2^*, B_1, B_2), \quad (u_j^*, B_j) \in S_1, \tag{18}$$

and

$$G(A_1, A_2) = I(A_1, A_2, \varphi_1^*, \varphi_2^*), \quad (A_j, \varphi_j^*) \in S_2, \tag{19}$$

where $u_j^* = u_j^*(B_j)$ and $\varphi_j^* = \varphi_j^*(A_j)$. We have restricted the action I to the sets S_1 and S_2 to form $J(B_1, B_2)$ and $G(A_1, A_2)$. Each of these two restricted forms of the action is stationary at the critical point (u_j, φ_j), and because of the way they are constructed, the stationary results are said to be dual [1]. In deriving (19), the existence of f_j^{-1} is assumed. It can be easily shown that

$$G(A_1, A_2) \leqslant I(u_1, u_2, \varphi_1, \varphi_2) \leqslant J(B_1, B_2), \tag{20}$$

if $W_j(u_j, \varphi_j)$ is convex in u_j, concave in φ_j and at least one of these is definite.

It is, from equation (9), evident that $W_j(u_j, \varphi_j)$ is convex in u_j and is definite. The sufficient condition for $W_j(u_j, \varphi_j)$ to be concave in φ_j is that

$$-df_j/d\varphi_j \geqslant \gamma_j \geqslant 0. \tag{21}$$

Since the exact solution (u_j, φ_j) is related through the Eqs. (9) and (11) we can take

$$A_j = k_j T_j \psi_j,$$

where ψ_j is any function satisfying the constraint

$$\sigma_1^* k_1 T_1 \psi_1 - \sigma_2^* k_2 T_2 \psi_2 = h \quad \text{on } \Gamma.$$

Consequently (20) becomes

$$G(k_1 T_1 \psi_1, k_2 T_2 \psi_2) = H(\psi_1, \psi_2) \leqslant I(u_1, u_2, \varphi_1, \varphi_2) \leqslant J(B_1, B_2) \tag{22}$$

and $\psi_j = B_j$ will be the exact solution when the equality signs hold. From (22)

$$\begin{aligned} J(B_1, B_2) - H(\psi_1, \psi_2) &\geqslant J(B_1, B_2) - I(u_1, u_2, \varphi_1, \varphi_2) \\ &= \sum_{j=1}^{2} \left\{ (T_j \xi_j, k_j T_j \xi_j) + \langle \xi_j, -df_j(\varphi_{jt}) \xi_j / d\varphi_j \rangle \right\}, \end{aligned} \tag{23}$$

where $\xi_j = B_j - \varphi_j$ and $\varphi_{jt} = tB_j + (1-t)\phi_j$, $0 \leqslant t \leqslant 1$.

From (5), (6), and (23) it follows that

$$\begin{aligned}
2[J(B_1, B_2) - H(\psi_1, \psi_2)] \geqslant & \sum_{j=1}^{2} \{\langle T_j^* k_j T_j \xi_j, \xi_j\rangle + \langle \xi_j, -df_j(\varphi_{jt})\xi_j/d\varphi_j\rangle\} \\
& + \langle \sigma_1^* k_1 T_1 \xi_1, \xi_1\rangle_{\Gamma_1} + \langle \sigma_2^* k_2 T_2 \xi_2, \xi_2\rangle_{\Gamma_2} \\
& + \langle \sigma_1^* k_1 T_1 \xi_1, \xi_1\rangle_{\Gamma} - \langle \sigma_2^* k_2 T_2 \xi_2, \xi_2\rangle_{\Gamma} \\
\geqslant & \sum_{j=1}^{2} (\Lambda + \gamma_j)\langle \xi_j, \xi_j\rangle,
\end{aligned} \tag{24}$$

where Λ is a lower bound to the least eigenvalue of

$$T_j^* k_j T_j \theta_j = \lambda \theta_j \quad \text{in } D_j \tag{25}$$

with $\theta_j = 0$ on Γ_j and $\theta_1 = \theta_2$ and $\sigma_1^* k_1 T_1 \theta_1 = \sigma_2^* k_2 T_2 \theta_2$ on Γ. Hence the required error estimate is

$$\|\xi\|_{L^2}^2 \leqslant 2[J - H]/(\Lambda + \gamma), \tag{26}$$

where $\gamma = \min_j \gamma_j$.

4. Application to steady-state heat conduction in a composite medium. Let D be a region in the (x, y) plane consisting of two parts D_1 and D_2. The plane which occupies the region D_j is a homogeneous material of thermal conductivity k_j. We want to investigate typical boundary value problems in steady-state heat conduction for the composite medium D with the source term $f_j(\varphi_j)$, the temperature being prescribed on the boundaries. One matching condition is obtained by assuming that no sources are created on the interface. The other condition is obtained by requiring that the temperature be continuous at the interface, which is reasonable on physical grounds when D_1 and D_2 are in intimate contact.

The governing differential equation is given by

$$L_j\varphi_j = -\nabla\cdot(k_j\nabla\varphi_j) = f_j(\varphi_j) \quad \text{in } D_j, \tag{27}$$

with $\varphi_j = \alpha_j$ on Γ_j and the matching conditions $\varphi_1 = \varphi_2$ on Γ and $\hat{n}\cdot k_1\nabla\varphi_1 = \hat{n}\cdot k_2\nabla\varphi_2$ on Γ, where $\hat{n}$ is the unit outward normal vector to Γ.

The bounding functionals J and G, for $f_1(\varphi_1) = -a^2\varphi_1 + f_1(x, y)$ and $f_2(\varphi_2) = -b^2\varphi_2 + g_2(x, y)$, defined by (18) and (19) are given by

$$\begin{aligned}
2J(B_1, B_2) = & \int_{D_1} \{k_1(\nabla B_1)^2 + a^2 B_1^2 - 2f_1(x, y)B_1\}\, dx\, dy \\
& + \int_{D_2} \{k_2(\nabla B_2)^2 + b^2 B_2^2 - 2g_2(x, y)B_2\}\, dx\, dy
\end{aligned} \tag{28}$$

and

$$-2H(\psi_1, \psi_2) = a^{-2}\int_{D_1} \left\{ (L_1\psi_1)^2 + f_1^2(x, y) - 2f_1(x, y)L_1\psi_1 \right\} dx\, dy$$
$$+ k_1 \int_{D_1} (\nabla\psi_1)^2\, dx\, dy + k_2 \int_{D_2} (\nabla\psi_2)^2\, dx\, dy$$
$$+ b^{-2}\int_{D_2} \left\{ (L_2\psi_2)^2 + g_2^2(x, y) - 2g_2(x, y)L_2\psi_2 \right\} dx\, dy$$
$$-2\int_{\Gamma_1} k_1 \frac{\partial\psi_1}{\partial n}\, ds - 2\int_{\Gamma_2} \alpha_2 k_2 \frac{\partial\psi_2}{\partial n}\, ds. \tag{29}$$

In our further calculations we take

$D_1 = \{(x, y)/-1 < x < 0, -1 < y < 1\}$, $D_2 = \{(x, y)/0 < x < 1, -1 < y < 1\}$,

$\alpha_j = 0$, $f_1(x, y) = f_{11}(x)f_{12}(y)$, and $g_2(x, y) = g_{11}(x)g_{12}(y)$.

4.1. *Minimization of J*. Let us assume that the solution B_j is of the product form

$$B_j(x, y) = m_j(x)n(y). \tag{30}$$

By substituting (30) in (28) we see that J depends upon m_j and n. If we assume that n is an a priori known function satisfying $n(-1) = n(1) = 0$, then J, as a functional of m_j [5], is given by

$$2J(m_1, m_2) = \int_{-1}^{0}\left\{ A_{11}\left(\frac{dm_1}{dx}\right)^2 + A_{15}m_1^2(x) + A_{14}f_{11}(x)m_1(x)\right\} dx$$
$$+ \int_0^1 \left\{ B_{11}\left(\frac{dm_2}{dx}\right)^2 + B_{15}m_2^2(x) + B_{14}g_{11}(x)m_2(x)\right\} dx, \tag{31}$$

where the constants depend upon the assumed form of $n(y)$:

$$A_{11}(n) = \int_{-1}^{1} k_1 n^2(y)\, dy; \qquad B_{11}(n) = \int_{-1}^{1} k_2 n^2(y)\, dy,$$
$$A_{12}(n) = \int_{-1}^{1} k_1\left(\frac{dn}{dy}\right)^2; \qquad B_{12}(n) = \int_{-1}^{1} k_2\left(\frac{dn}{dy}\right)^2 dy,$$
$$A_{13}(n) = \int_{-1}^{1} a^2 n^2(y)\, dy; \qquad B_{13}(n) = \int_{-1}^{1} b^2 n^2(y)\, dy,$$
$$A_{14}(n) = -2\int_{-1}^{1} f_{12}(y)n(y)\, dy, \qquad B_{14}(n) = -2\int_{-1}^{1} g_{12}(y)n(y)\, dy,$$
$$A_{15}(n) = A_{12} + A_{13}, \quad \text{and} \quad B_{15} = B_{12} + B_{13}.$$

We may determine m_j so that J is minimized. This leads to the following Euler–Lagrange equations for m_j:

$$\frac{d^2m_1}{dx^2} - \eta_1^2 m_1 = \frac{A_{14}}{2A_{11}} f_{11}(x), \qquad -1 < x < 0,$$
$$\frac{d^2m_2}{dx^2} - \eta_2^2 m_2 = \frac{B_{14}}{2B_{11}} g_{11}(x), \qquad 0 < x < 1, \tag{32}$$

with the conditions

$$m_1(-1) = m_2(1) = 0, \qquad m_1(0) = m_2(0),$$

and

$$k_1 \frac{dm_1}{dx} = k_2 \frac{dm_2}{dx} \quad \text{at } x = 0, \tag{33}$$

where $\eta_1^2 = A_{15}/A_{11}$ and $\eta_2^2 = B_{15}/B_{11}$.

Maximization of H. Choose the trial function

$$\psi_j(x, y) = \eta B_j(x, y), \tag{34}$$

where η is the unknown parameter to be determined. Substituting (34) in (29) we have

$$-2H(\psi_1, \psi_2) = \eta^2[I_1 + I_2] - 2\eta I_3 + I_4,$$

where

$$I_1 = \bar{a}^2 \int_{D_1} (L_1 B_1)^2 \, dx\, dy + k_1 \int_{D_1} (\nabla B_1)^2 \, dx\, dy,$$

$$I_2 = \bar{b}^2 \int_{D_2} (L_2 B_2)^2 \, dx\, dy + k_2 \int_{D_2} (\nabla B_2)^2 \, dx\, dy,$$

$$I_3 = \bar{a}^2 \int_{D_1} f_1(x, y) L_1 B_1 \, dx\, dy + \bar{b}^2 \int_{D_2} g_2(x, y) L_2 B_2 \, dx\, dy,$$

$$I_4 = \bar{a}^2 \int_{D_1} f_1^2(x, y) \, dx\, dy + \bar{b}^2 \int_{D_2} g_2^2(x, y) \, dx\, dy.$$

$dH/d\eta = 0$ leads to $\eta = I_3/I_1 + I_2$.

4.3. *Numerical results.* Numerical calculations are carried out with $k_1 = 0.18$, $k_2 = 0.14$, $f_1(x, y) = g_2(x, y) = x$ and $a^2 = b^2 = 1$. The solution of (32) and (33) and the value of η for

$$n(y) = \frac{\cosh \mu y}{\cosh \mu} - 1, \qquad \mu = 3.1379909,$$

are given by $m_j(x) = p_j \exp[\eta_j x] + q_j \exp[-\eta_j x] + c_j x$ in $[-1, 0]$ and $[0, 1]$, respectively, where

$$\begin{array}{lll} p_1 = 0.323063133, & q_1 = -0.047463513, & p_2 = 0.03837185, \\ q_2 = 0.23722777, & \eta_1 = 2.906658224, & \eta_2 = 3.167958905, \\ c_1 = -0.850763427, & c_2 = -0.920835684, & \eta = 0.9379559949. \end{array}$$

The corresponding extremum values of the functionals are found to be

$$J = -0.09806582, \qquad G = -0.20685117.$$

The lower bound to the least eigenvalue to the corresponding problem is taken to be [6] $\Lambda = 0.6908723$.

Consequently, the mean square error is given by

$$\|\xi\|_{L^2}^2 \leqslant 0.1286735$$

and one can improve the estimate by applying a more sophisticated method like the Finite Element Method.

5. Construction of Green's function. As an example of (1) and (2) with source or sink on the interface, we consider the one-dimensional heat conduction equation. If there is a unit sink at the origin and the boundary is kept at zero temperature, the one-dimensional heat conduction equation is given by [2]

$$d^2\varphi/dx^2 = \delta(x), \qquad -1 < x < 1, \tag{35}$$

with $\varphi(-1) = 0$, $\varphi(1) = 0$. Note that

$$\langle \delta(x), \varphi(x) \rangle = \lim_{\varepsilon \to 0} \frac{1}{2} [\langle \delta(x-\varepsilon), \varphi(x) \rangle + \langle \delta(x+\varepsilon), \varphi(x) \rangle].$$

If $\varphi = \varphi_j, j = 1, 2, 3$ in $[-1, -\varepsilon]$, $[-\varepsilon, \varepsilon]$ and $[\varepsilon, 1]$, respectively, (35) reduces to $d^2\varphi_j/dx^2 = 0$, $j = 1, 2, 3$, in $(-1, -\varepsilon)$, $(-\varepsilon, \varepsilon)$ and $(\varepsilon, 1)$, respectively, with the constraints

$$\frac{d\varphi_2}{dx}(-\varepsilon) - \frac{d\varphi_1}{dx}(-\varepsilon) = \frac{1}{2}; \qquad \frac{d\varphi_3}{dx}(\varepsilon) - \frac{d\varphi_2}{dx}(\varepsilon) = \frac{1}{2};$$

$$\varphi_1(-\varepsilon) = \varphi_2(-\varepsilon); \quad \varphi_3(\varepsilon) = \varphi_2(\varepsilon); \quad \varphi_1(-1) = \varphi_3(1) = 0.$$

Application of the dual extremum principles leads to

$$\varphi_1(x) = -\frac{1}{2}(x+1), \quad -1 \leqslant x \leqslant -\varepsilon; \qquad \varphi_2(x) = \frac{1}{2}(\varepsilon - 1), \quad -\varepsilon \leqslant x \leqslant \varepsilon,$$

and

$$\varphi_3(x) = \frac{1}{2}(x-1), \qquad \varepsilon \leqslant x \leqslant 1,$$

which is the exact solution when $\varepsilon \to 0$.

6. Concluding remarks. Dual extremum principles can be formulated for nonconvex and multiply-connected domains by subdividing the domain into convex domains and using the matching conditions on the cuts. Green's function, in two dimensions for the heat conduction equation, can be constructed by just extending the principle used in the case of one dimension.

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