# On the critical behavior of disordered quantum magnets: The relevance of rare regions

Rajesh Narayanan
*Department of Physics and Materials Science Institute, University of Oregon, Eugene, OR 97403*

Thomas Vojta
*Department of Physics and Materials Science Institute, University of Oregon, Eugene, OR 97403*
*and Institut für Physik, TU Chemnitz, D-09107 Chemnitz, FRG*

D. Belitz
*Department of Physics and Materials Science Institute, University of Oregon, Eugene, OR 97403*

T.R. Kirkpatrick
*Institute for Physical Science and Technology, and Department of Physics,*
*University of Maryland, College Park, MD 20742*
(November 19, 2018)

The effects of quenched disorder on the critical properties of itinerant quantum antiferromagnets and ferromagnets are considered. Particular attention is paid to locally ordered spatial regions that are formed in the presence of quenched disorder even when the bulk system is still in the paramagnetic phase. These rare regions or local moments are reflected in the existence of spatially inhomogeneous saddle points of the Landau-Ginzburg-Wilson functional. We derive an effective theory that takes into account small fluctuations around *all* of these saddle points. The resulting free energy functional contains a new term in addition to those obtained within the conventional perturbative approach, and it comprises what would be considered non-perturbative effects within the latter. A renormalization group analysis shows that in the case of antiferromagnets, the previously found critical fixed point is unstable with respect to this new term, and that no stable critical fixed point exists at one-loop order. This is contrasted with the case of itinerant ferromagnets, where we find that the previously found critical behavior is unaffected by the rare regions due to an effective long-ranged interaction between the order parameter fluctuations.

## I. INTRODUCTION

The influence of static or quenched disorder on the critical properties of a system near a continuous phase transition is a very interesting problem in statistical mechanics. While it was initially suspected that quenched disorder always destroys any critical point,[1] this was soon found to not necessarily be the case. Harris[2] found a convenient criterion for the stability of a given critical behavior with respect to quenched disorder: If the correlation length exponent $\nu$ obeys the inequality $\nu \geq 2/D$, with $D$ the spatial dimensionality of the system, then the critical behavior is unaffected by the disorder. In the opposite case, $\nu < 2/D$, the disorder modifies the critical behavior.[3] This modification may either (i) lead to a new critical point that has a correlation length exponent $\nu \geq 2/D$ and is thus stable, or (ii) lead to an unconventional critical point where the usual classification in terms of power-law critical exponents looses its meaning, or (iii) lead to the destruction of a sharp phase transition. The first possibility is realized in the conventional theory of random-$T_c$ classical ferromagnets,[1] and the second one is probably realized in classical ferromagnets in a random field.[4] The third one has occasionally been attributed to the exactly solved McCoy-Wu model.[5] This is misleading, however, as has recently been emphasized in Ref. 6; there is a sharp, albeit unorthodox, transition in that model, and it thus belongs to category (ii).

Independent of the question of if and how the critical behavior is affected, disorder leads to very interesting phenomena as a phase transition is approached. Disorder in general decreases the critical temperature $T_c$ from its clean value $T_c^0$. In the temperature region $T_c < T < T_c^0$ the system does not display global order, but in an infinite system one will find arbitrarily large regions that are devoid of impurities, and hence show local order, with a small but non-zero probability that usually decreases exponentially with the size of the region. These static disorder fluctuations are known as 'rare regions', and the order parameter fluctuations induced by them as 'local moments' or 'instantons'. Since they are weakly coupled, and flipping them requires to change the order parameter in a whole region, the local moments have very slow dynamics. Griffiths[7] was the first to show that they lead to a non-analytic free energy everywhere in the region $T_c < T < T_c^0$, which is known as the Griffiths phase, or, more appropriately, the Griffiths region. In generic classical systems this is a weak effect, since the singularity in the free energy is only an essential one. An important exception is the McCoy-Wu model,[5] which is a 2-$D$ Ising model with bonds that are random along one direction, but identical along the second direction. The resulting infinite-range correlation of the disorder in one direction leads to very strong effects. As the temperature is lowered through the Griffiths region, the local moments cause the divergence of an increasing number of higher order susceptibilities, $\chi^{(n)} = \partial^n M/\partial B^n$ ($n \geq 2$), with $M$ the order parameter and $B$ the field conjugate to it, starting with large $n$. Even the average susceptibility proper, $\chi^{(1)}$, diverges at a temperature $T_\chi > T_c$, although the average order parameter does not become non-zero until the temperature reaches $T_c$. This is caused by rare fluctuations in the susceptibility distribution, which dominate the average susceptibility and make it very different from the typical or most probable one.

Surprisingly little is known about the influence of the Griffiths region and related phenomena on the critical behavior. Recent work[8] on a random-$T_c$ classical Ising model has suggested that it can be profound, even in this simple model where the conventional theory predicts standard power-law critical behavior, albeit with critical exponents that are different from the clean case. The authors of Ref. 8 have shown that the conventional theory is unstable with respect to perturbations that break the replica symmetry. By approximately taking into account the rare regions, which are neglected in the conventional theory, they found a new term in the action that actually induces such perturbations. In some systems replica symmetry breaking is believed to be associated with activated, i.e. non-power law, critical behavior. Reference 8 thus raised the interesting possibility that, as a result of rare-region effects, the random-$T_c$ classical Ising model shows activated critical behavior, as is believed to be the case for the random-field classical Ising model,[4] although in the case of the random-$T_c$ model no final conclusion about the fate of the transition could be reached.

Griffiths regions also occur in the case of quantum phase transitions (QPTs), i.e. transitions that occur at zero temperature as a function of some non-thermal control parameter.[9,10] Their consequences for the critical behavior are even less well investigated than in the classical case, with the remarkable exception of certain 1-$D$ systems. Fisher[6] has investigated quantum Ising spin chains in a transverse random field. These systems are closely related to the classical McCoy-Wu model, with time in the quantum case playing the role of the 'ordered direction' in the latter. He has found activated critical behavior due to rare regions. This has been confirmed by numerical simulations.[11] Other recent simulations[12] suggest that this type of behavior may not be restricted to 1-$D$ systems, raising the possibility that exotic critical behavior dominated by rare regions may be generic in quenched disordered quantum systems, independent of the dimensionality and possibly also of the type of disorder.

In this paper we consider this problem analytically for two QPTs in $D > 1$. We first concentrate on a simple model for a quantum antiferromagnet. Previous work,[13] which did not take into account rare regions, had found a transition with some surprising properties. One of our goals is to check whether these results survive taking into account rare regions. We find that they do not; the previously found critical fixed point is unstable with respect to the rare regions, and one finds runaway flow in all of physically accessible parameter space. We will discuss possible interpretations of this result. We then show that the critical behavior of itinerant quantum *ferro*magnets is *not* affected by the rare regions, in sharp contrast to the antiferromagnetic case. A brief report of some of our results has been given previously in Ref. 14.

The paper is organized as follows. In Sec. II we derive an effective action for an itinerant antiferromagnet in the presence of rare regions. In Sec. III we perform a one-loop renormalization group analysis of this action, and show that there is no stable critical fixed point to that order. In Sec. IV we perform an analogous analysis for itinerant ferromagnets and show that the previously found critical fixed point is stable with respect to the rare region effects. In Sec. V we discuss our results. Various technical points are relegated to three appendices.

## II. AN EFFECTIVE ACTION FOR DISORDERED ANTIFERROMAGNETS

### A. The model

Our starting point is Hertz's action[16] for an itinerant quantum antiferromagnet. It is a $\phi^4$-theory with a $p$-component order parameter field $\boldsymbol{\phi}$ whose expectation value is proportional to the staggered magnetization. The bare two-point vertex function is

$$\Gamma_0(\mathbf{q}, \omega_n) = t_0 + \mathbf{q}^2 + |\omega_n| \quad , \tag{2.1}$$

with $t_0$ the mean distance from the mean-field critical point. $\mathbf{q}$ is the wavevector, and $\omega_n$ denotes a bosonic Matsubara frequency. We measure $\mathbf{q}$ and $\omega_n$ in suitable microscopic units to make them dimensionless. As in Ref. 13, we modify this action by adding disorder in the form of a 'random-mass' or 'random-temperature' term. That is, we add to $t_0$ a random function of position, $\delta t(\mathbf{x})$, which obeys a distribution with zero mean and variance $\Delta$,

$$\langle \delta t(\mathbf{x}) \rangle = 0 \quad , \tag{2.2a}$$

$$\langle \delta t(\mathbf{x})\, \delta t(\mathbf{y}) \rangle = \Delta\, \delta(\mathbf{x} - \mathbf{y}) \quad . \tag{2.2b}$$

For the sake of simplicity, we have taken the distribution to be delta-correlated. The two-point vertex now reads

$$\begin{aligned}\Gamma(\mathbf{x} - \mathbf{y}, \tau - \tau') = \Gamma_0(\mathbf{x} - \mathbf{y}, \tau - \tau')& \\ + \delta(\mathbf{x} - \mathbf{y})\, \delta(\tau - \tau')\, \delta t(\mathbf{x}) \quad .&\end{aligned} \tag{2.3}$$

Here $\tau$ denotes imaginary time, and $\Gamma_0(\mathbf{x}, \tau)$ is the Fourier transform of $\Gamma_0(\mathbf{q}, \omega_n)$ in Eq. (2.1). The action reads

$$S[\boldsymbol{\phi}] = S_{\rm G}[\boldsymbol{\phi}] + u \int dx\ \left(\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x)\right)^2 \quad , \tag{2.4a}$$

with the Gaussian part given by

$$S_{\rm G}[\boldsymbol{\phi}] = \frac{1}{2}\int dx\,dy\ \boldsymbol{\phi}(x)\,\Gamma(x-y)\,\boldsymbol{\phi}(y) \quad . \tag{2.4b}$$

Here we have introduced a four-vector notation, $x \equiv (\mathbf{x},\tau)$, $\int dx \equiv \int d\mathbf{x}\int_0^{1/T} d\tau$, and we use units such that $\hbar = k_{\rm B} = 1$.

At this point, the conventional procedure would be to integrate out the quenched disorder by means of the replica trick.[1] This would lead to an effective action that does not contain the disorder explicitly any longer, and that therefore does not easily allow for saddle-point solutions that are not spatially homogeneous. While the effective action would still be exact, this latter property would make it hard to incorporate the physics we are concentrating on in this paper. We will therefore take a different approach, and consider saddle-point solutions of the model, Eqs. (2.4), *before* integrating out the disorder. Our procedure roughly follows the one by Dotsenko et al.[8] for classical magnets. As we will see, however, there are important differences between the classical and quantum cases.

### B. Saddle-point solutions

Let us consider saddle-point solutions of Eqs. (2.4) that are time independent. For simplicity, we also consider a scalar field, $p=1$, that we denote by $\phi(\mathbf{x})$. It will be obvious how to generalize the following considerations to the case $p>1$. With these restrictions, the saddle-point equation reads

$$\left(t_0 + \delta t(\mathbf{x}) - \partial_{\mathbf{x}}^{\mathbf{2}}\right)\ \phi_{\rm sp}(\mathbf{x}) + 4\,u\,\phi_{\rm sp}^3(\mathbf{x}) = 0 \quad . \tag{2.5}$$

Although $\phi_{\rm sp}^{(1)}(\mathbf{x}) \equiv 0$ is of course always a solution, inhomogeneous solutions also exist provided that $\delta t(\mathbf{x})$ has 'troughs' that are sufficiently wide and deep.[17] In Appendix A we demonstrate this for a one-dimensional toy problem. We have solved Eq. (2.5) numerically for rotationally invariant potentials $\delta t(\mathbf{x}) = f(|\mathbf{x}|)$, and have found behavior that is qualitatively the same as in the one-dimensional model. Thus, if $\delta t(\mathbf{x})$ has one sufficiently deep and wide trough, there will be a solution of Eq. (2.5) that is exponentially small everywhere except within the trough region, where it shows a single hump. There are actually two equivalent single-hump solutions, one positive, and the other negative. We denote the positive one by $\phi_{\rm sp}^{(2)}(\mathbf{x}) \equiv \psi^{(+)}(\mathbf{x})$. As is intuitively obvious, and demonstrated in Appendix A, it leads to a lower free energy than the homogeneous solution $\phi_{\rm sp}^{(1)}(\mathbf{x}) \equiv 0$. Although in principle any saddle point can be used as the starting point for a loop expansion, it is therefore reasonable to assume that the inhomogeneous one will already incorporate physics that would be much harder to obtain if we expanded about the homogeneous saddle point.

Next consider a potential $\delta t(\mathbf{x})$ that contains many troughs that support an essentially non-zero local order parameter field. This will result in a saddle-point solution that contains many regions of local order, which we will refer to as 'islands'. Of course, for an arbitrary potential $\delta t(\mathbf{x})$ it is not possible to solve Eq. (2.5) in closed form. However, as long as the concentration of the islands is low, as will always be the case sufficiently deep in the disordered phase (i.e., for sufficiently large $t_0$), the values of $\phi_{\rm sp}$ outside of the islands will still be exponentially small. If $\delta t$ has troughs leading to $N$ islands, which individually would result in positive saddle-point solutions $\psi_i^{(+)}(\mathbf{x})$, $(i=1,\ldots,N)$, it is therefore a reasonable approximation to write the global saddle-point solution as a linear superposition of the $\psi_i^{(+)}$. For independent islands, there are actually $2^N$ equivalent saddle points, which we write as

$$\begin{aligned}\phi_{\rm sp}^{(a)}(\mathbf{x}) \equiv \Phi^{(a)}(\mathbf{x}) &= \sum_{i=1}^{N} \sigma_i^a\,\psi_i^{(+)}(\mathbf{x}) \\ &\equiv \sum_{i=1}^{N}\psi_i(\mathbf{x}) \quad ,\end{aligned} \tag{2.6}$$

where $a = 1,\ldots,2^N$ numbers the equivalent saddle points, and the $\sigma_i^a$ are random numbers whose values are either $+1$ or $-1$. They thus obey a probability distribution

$$P[\{\sigma_i^a\}] = \prod_i \pi(\sigma_i^a) \quad , \tag{2.7a}$$

with

$$\pi(\sigma) = \frac{1}{2}\,\left[\delta(\sigma-1) + \delta(\sigma+1)\right] \quad . \tag{2.7b}$$

Alternatively, one can consider the $\psi_i(\mathbf{x})$ random functions that are equal to either plus or minus $\psi_i^{(+)}(\mathbf{x})$.

For later reference, let us briefly discuss the effects of a weak interaction between the islands as a result of the exponentially small overlap between the functions $\psi_i(\mathbf{x})$ centered on different islands. One effect will be that the total amount of the order parameter on each island will not necessarily be equal to plus or minus the amount resulting from that island being ideally ordered, but that small deviations from this amount will be possible. If we still assume that the islands are statistically independent, we can model this effect by using a probability distribution for the $\sigma_i$ that is given by Eq. (2.7a) with a distribution $p(\sigma)$ that is a broadended version of the bimodal delta-distribution $\pi(\sigma)$ given in Eq. (2.7b). We thus generalize Eqs. (2.7a,2.7b) to

$$P[\{\sigma_i^a\}] = \prod_i p(\sigma_i^a) \quad . \tag{2.7c}$$

For our purposes we will not need to specify the distribution $p(\sigma)$ explicitly. It will turn out that an interaction between the islands, no matter how small, leads to new physics compared to a model where these interactions are neglected.

We also note that the islands will have some dynamics, both due to interactions between the islands and due to interactions between an island and its immediate neighborhood. In principle, one could try to build this effect into the saddle-point approximation by looking for time dependent saddle points. However, this dynamics is expected to be very slow due to the inertia of the islands. Moreover, the zero frequency component in a frequency expansion is expected to yield the dominant effect. We therefore restrict ourselves to static saddle points.

### C. The partition function for a given disorder realization

Of the $2^N$ saddle-point solutions $\Phi^{(a)}(\mathbf{x})$ discussed in the previous subsection, let us pick one, say $\Phi^{(1)}$, to expand about:

$$\phi(x) = \Phi^{(1)}(\mathbf{x}) + \varphi(x) \quad . \tag{2.8a}$$

Then the partition function can be written

$$Z[\delta t(\mathbf{x})] = \int D[\varphi(x)]\, e^{-S[\Phi^{(1)}(\mathbf{x})+\varphi(x),\delta t(\mathbf{x})]} \quad , \tag{2.8b}$$

where we show $\delta t(\mathbf{x})$ explicitly as an argument to emphasize that we are still working with a fixed realization of the disorder. Equation (2.8b) is exact as long as the integral extends over *all* fluctuations $\varphi(x)$ of the field configuration. However, in practice the integral over $\varphi(x)$ cannot be performed exactly, and in a perturbative treatment one restricts oneself to small deviations $\varphi(x)$ from the chosen saddle point. Typical pairs of saddle points picked from the $2^N$ $\Phi^{(a)}$ represent field configurations that are globally very different. They will thus be far apart in configuration space, with large energy barriers between them. (We will justify this statement in Sec. V A 2 below.) Expanding about one of the saddle points, as in Eqs. (2.8), is therefore not expected to yield a good representation of the partition function if one evaluates the functional integral in Eq. (2.8b) perturbatively. On the other hand, the same argument suggests that we can simply sum the contributions to $Z$ obtained by expanding about *all* of the $2^N$ saddle points, provided that we restrict ourselves to small fluctuations about each saddle point,

$$Z[\delta t(\mathbf{x})] \approx \sum_{a=1}^{2^N} \int_< D[\varphi(x)]\; e^{-S[\Phi^{(a)}(\mathbf{x})+\varphi(x),\delta t(\mathbf{x})]} \quad . \tag{2.9}$$

Here $\int_<$ indicates an integration over small fluctuations only. Apart from a normalization factor, this procedure amounts to an arithmetic average over the perturbative contributions coming from the vicinities of all saddle points. This average is our approximative way of taking into account non-perturbative effects.

Since we are interested in the effects of fluctuations about the saddle points, we next subtract the saddle point action from the exponent in Eq. (2.9).[18] That is, we write

$$Z[\delta t(\mathbf{x})] \approx \sum_a \int_< D[\varphi(x)]\; e^{-\Delta S[\Phi^{(a)}(\mathbf{x}),\varphi(x),\delta t(\mathbf{x})]}, \tag{2.10}$$

where

$$\begin{aligned}\Delta S[\Phi^{(a)},\varphi,\delta t] &\equiv S[\Phi^{(a)}+\varphi,\delta t] - S[\Phi^{(a)},\delta t] \\ &= S[\varphi(x)] + 4u \int dx\, \varphi^3(x)\, \Phi^{(a)}(\mathbf{x}) \\ &\quad +6u \int dx\, \varphi^2(x)\, \Phi^{(a)}(\mathbf{x})^2 \quad .\end{aligned} \tag{2.11}$$

So far we have implicitly assumed that there is no interaction between the islands. In reality, there will be a small interaction, one effect of which will be to replace the bimodal distribution, Eqs. (2.7a,2.7b), by the broadended distribution given in Eq. (2.7c). The sum over $a$ in Eq. (2.10) is then replaced by an integral over the $\sigma_i^a$, weighted by the distribution $p(\sigma)$. The partition function can now be written as

$$Z[\delta t(\mathbf{x})] \approx \int D[\varphi(x)]\; e^{-(S[\varphi(x)]+\delta S[\varphi(x)])} \quad , \tag{2.12a}$$

with the correction to the action, $\delta S$, given by

$$\begin{aligned}e^{-\delta S[\varphi(x)]} = \int \prod_{i=1}^N d\sigma_i\; &\prod_j p(\sigma_j) \\ \times\; &e^{-4u\int dx\,\varphi^3(x)\,\sum_i \sigma_i \psi_i^{(+)}(\mathbf{x})} \\ \times\; &e^{-6u\int dx\,\varphi^2(x)\,\sum_{ij} \sigma_i\sigma_j \psi_i^{(+)}(\mathbf{x})\psi_j^{(+)}(\mathbf{x})} \quad .\end{aligned} \tag{2.12b}$$

As mentioned previously, it is crucial to incorporate a small interaction between the islands. Indeed, if we used the distribution function, Eqs. (2.7), for the $\sigma_i$ that is appropriate for non-interacting islands, then we could do the $\sigma$-integral in Eq. (2.12b) exactly. As we will show in Sec. III, the resulting action would not lead to new physical effects compared to Ref. 13. Apart from the broadening of the distribution, there are other, similar effects of the island-island interaction that we will neglect. For instance, in the second exponent on the right-hand-side of Eq. (2.12b), the absence of any overlap between $\psi_i^{(+)}$ and $\psi_j^{(+)}$ for $i \neq j$ makes the spatial integral in that term vanishes unless $i = j$. This is no longer true for interacting islands. However, we will neglect this effect and still take this term to be proportional to $\delta_{ij}$. Equation (2.12b) can then be written

$$\delta S[\varphi(x)] = -\sum_i \ln \left\langle e^{-4u\int dx\, \varphi^3(x)\, \psi_i^{(+)}(\mathbf{x})\, \sigma_i} \times e^{-6u\int dx\, \varphi^2(x)\, \left(\psi_i^{(+)}(\mathbf{x})\right)^2 \sigma_i^2} \right\rangle \quad . \tag{2.12c}$$

Here $\langle \ldots \rangle$ denotes an average over the $\sigma$ with respect to the broadened bimodal distribution $p(\sigma)$. The $\sigma$-average is carried out by means of a cumulant expansion in powers of our order parameter fluctuations $\varphi(x)$. To the order $\varphi^4$ we obtain

$$\begin{aligned}\delta S[\varphi(x)] &= 6u \int dx\, \varphi^2(x) \sum_i \left(\psi_i^{(+)}(\mathbf{x})\right)^2 \langle \sigma_i^2 \rangle \\ &+4u \int dx\, \varphi^3(x) \sum_i \psi_i^{(+)}(\mathbf{x})\, \langle \sigma_i \rangle \\ &-18u^2 \int dx\, dy\, \varphi^2(x)\, \varphi^2(y) \\ &\times \sum_i \left(\psi_i^{(+)}(\mathbf{x})\right)^2 \left(\psi_i^{(+)}(\mathbf{y})\right)^2 \left(\langle \sigma_i^4 \rangle - \langle \sigma_i^2 \rangle^2\right) \quad .\end{aligned} \tag{2.13}$$

Now $\langle \sigma_i \rangle = 0$, $\langle \sigma_i^2 \rangle > 0$ and $\langle \sigma_i^4 \rangle - \langle \sigma_i^2 \rangle^2 \equiv c_i > 0$. The last relation is only valid for a broadened distribution $p(\sigma)$ which arises from interactions between the islands. For the original distribution $\pi(\sigma)$, $c_i = 0$ and so the $O(\varphi^4)$ term would vanish. If we collect all contributions to the action for one particular disorder configuration, we obtain

$$\begin{aligned}& S[\varphi(x)] + \delta S[\varphi(x)] \\ &= \frac{1}{2} \int dxdy\ \varphi(x)\, \Gamma(x-y)\, \varphi(y) + u \int dx\ \varphi^4(x) \\ &+\ 18u^2 \int dxdy\ \varphi^2(x)\, \varphi^2(y) \sum_i c_i\ \left(\psi_i^{(+)}(\mathbf{x}) \psi_i^{(+)}(\mathbf{y})\right)^2 .\end{aligned} \tag{2.14}$$

Here we have used the fact that the first term in (2.13) only renormalizes the random-mass term in the Gaussian action. We will show in Sec. III that truncating the action at $O(\varphi^4)$ is justified since all higher order terms are irrelevant (in a power-counting sense) with respect to both the Gaussian fixed point and the antiferromagnetic fixed point found in Ref. 13.

### D. The effective action

So far we have considered one particular realization of the disorder. In order to derive an effective action we now need to perform the disorder average. It is important to remember that the Landau functional, Eq. (2.14), depends on the disorder in two different ways: explicitly through the random mass in the Gaussian action, and implicitly through the saddle-point solutions $\psi_i^+(\mathbf{x})$ that depend on $\delta t(\mathbf{x})$.

The quenched disorder average over $\delta t(x)$ is performed via the replica trick,[1] which is based on the identity

$$\{\log Z\}_{\delta t} = \lim_{n \to 0} \frac{\{Z^n\}_{\delta t} - 1}{n} \quad . \tag{2.15}$$

Here $\{\ldots\}_{\delta t}$ denotes the disorder average. This results in an effective action $S_{\rm eff}$ which is defined by

$$\begin{aligned}\{Z^n\}_{\delta t} &= \int \prod_{\alpha=1}^{n} D[\varphi^\alpha(x)] \left\{ e^{-\sum_\alpha (S[\varphi^\alpha(x)] + \delta S[\varphi^\alpha(x)])} \right\}_{\delta t} \\ &\equiv \int \prod_{\alpha=1}^{n} D[\varphi^\alpha(x)]\ e^{-S_{\rm eff}[\varphi^\alpha(x)]} \quad .\end{aligned} \tag{2.16}$$

In the absence of $\delta S$, carrying out the disorder average yields the usual terms that are familiar from the conventional theory. Up to $O(\varphi^4)$ they are:

$$\begin{aligned}&\frac{1}{2} \sum_\alpha \int dx\, dy\ \varphi^\alpha(x)\, \Gamma_0(x-y)\, \varphi^\alpha(y) \\ &+ u \sum_\alpha \int dx (\varphi^\alpha(x))^4 \\ &- \Delta \sum_{\alpha,\beta} \int d\mathbf{x}\, d\tau\, d\tau'\ ((\varphi^\alpha(\mathbf{x},\tau))^2\, (\varphi^\beta(\mathbf{x},\tau'))^2 \ .\end{aligned} \tag{2.17}$$

Taking into account the additional term, $\delta S[\varphi^\alpha(x)]$, is more subtle since the functions $\psi_i(x)$ are implicit functions of $\delta t(x)$. We handle this problem by means of a cumulant expansion. To lowest order, the contribution of $\delta S$ to the effective action is just the disorder average of $\delta S$,

$$\{\delta S\}_{\delta t} = w \int dxdy\ \varphi^2(x)\, \varphi^2(y)\, D_{\rm isl}^{(2)}(\mathbf{x},\mathbf{y}) \quad , \tag{2.18a}$$

where $w \propto u^2$, and the correlation function

$$D_{\rm isl}^{(2)}(\mathbf{x},\mathbf{y}) = \left\{ \sum_i c_i\ \left(\psi_i^{(+)}(\mathbf{x})\, \psi_i^{(+)}(\mathbf{y})\right)^2 \right\}_{\delta t} \tag{2.18b}$$

essentially describes the probability for $\mathbf{x}$ and $\mathbf{y}$ to belong to the same island. The properties of these correlation functions depend on the precise nature of the disorder. If the microscopic disorder $\delta t(\mathbf{x})$ is short-range correlated, as we have assumed in our model, then the island size distribution will generically fall off exponentially for large sizes. In this case the correlation function $D_{\rm isl}^{(2)}$ is also short-ranged in space. Keeping only the leading term in a gradient expansion, we can then replace it by a spatial $\delta$-function. The case of an island size distribution that has a power-law tail (e.g. due to long-range correlations in the microscopic disorder) is discussed in Appendix B.

Collecting all contributions to the effective action $S_{\rm eff}$ up to $O(\varphi^4)$, absorbing a constant into $w$, and restoring the vector nature of the order parameter field, we finally obtain

$$S_{\rm eff}[\boldsymbol{\varphi}^\alpha(x)] = \frac{1}{2}\sum_\alpha \int dx\,dy\,\Gamma_0(x-y)\,\boldsymbol{\varphi}^\alpha(x)\cdot\boldsymbol{\varphi}^\alpha(y)$$
$$+u\sum_\alpha \int d\mathbf{x}\,d\tau\,(\boldsymbol{\varphi}^\alpha(\mathbf{x},\tau)\cdot\boldsymbol{\varphi}^\alpha(\mathbf{x},\tau))^2$$
$$-\sum_{\alpha,\beta}(\Delta + w\,\delta_{\alpha\beta})\int d\mathbf{x}\,d\tau\,d\tau'$$
$$\times(\boldsymbol{\varphi}^\alpha(\mathbf{x},\tau)\cdot\boldsymbol{\varphi}^\alpha(\mathbf{x},\tau))\,(\boldsymbol{\varphi}^\beta(\mathbf{x},\tau')\cdot\boldsymbol{\varphi}^\beta(\mathbf{x},\tau'))\,. \tag{2.19}$$

The $w$-term is generated by taking into account the inhomogeneous saddle points. A perturbative expansion about the homogeneous saddle point, as was performed in Ref. 13, misses this term. It has the time structure of the random-mass or $\Delta$-term, and the replica structure of the quantum fluctuation or $u$-term. We will further discuss its physical meaning in Sec. V A. In the following section we will show that the critical fixed point found in Ref. 13 is unstable with respect to this new term in the action.

## III. RENORMALIZATION GROUP ANALYSIS

### A. Tree-level analysis

Let us first justify our truncation of the Landau expansion in Sec. II by showing that all terms of higher than quartic order in $\boldsymbol{\varphi}$ are irrelevant (in the renormalization group sense) by power counting with respect to the critical fixed point of Ref. 13. To this end, we analyze the effective action, $S_{\rm eff}$, Eq. (2.19), at tree level.

Let us denote the scale dimension of any quantity $Q$ by $[Q]$, and define the scale dimension of a length $L$ to be $[L] = -1$. The scale dimension of the imaginary time is $[\tau] = -z$, which defines the dynamical critical exponent $z$. We first analyze the Gaussian fixed point. From the structure of the two-point vertex function $\Gamma_0$ given in Eq. (2.1), we see that $\omega_n$ scales like $q^2$. This implies $z = 2$. The scale dimension of the field $\boldsymbol{\varphi}$ is found (from the requirement that the action must be dimensionless) to be $[\boldsymbol{\varphi}(x)] = D/2$. The scale dimensions of the coefficients of the terms of $O(\boldsymbol{\varphi}^4)$ in Eq. (2.19) are found to be $[u] = 2 - D$, and $[\Delta] = [w] = 4 - D$. Thus, $u$ is irrelevant with respect to the Gaussian fixed point as long as $D > 2$, while $\Delta$ and $w$ are relevant for $D < 4$. The Gaussian fixed point is therefore unstable, and we will have to perform a loop expansion close to $D = 4$ in the next subsection.

We now show that all terms of $O(\boldsymbol{\varphi}^6)$ and higher are irrelevant with respect to the Gaussian fixed point. First of all, there are the conventional terms of the schematic form

$$u_{2m}\int dx\,\varphi^{2m}(x) \quad , \tag{3.1}$$

with coupling constants $u_{2m}$ ($u_4 \equiv u$). These are irrelevant since $[u_{2m}] = 2 - (m-1)D < 0$. In addition to these terms, the cumulant expansion of (2.12c) generates higher order terms with more time integrations than the conventional terms for a given power of $\varphi$. For instance, at $O(\varphi^6)$ we have two terms,

$$8u^2\int dx\,dy\;\varphi^3(x)\,\varphi^3(y)\sum_i \psi_i^{(+)}(\mathbf{x})\,\psi_i^{(+)}(\mathbf{y})\,\langle\sigma_i^2\rangle \quad , \tag{3.2a}$$

and

$$-36u^3\int dx\,dy\,dz\;\varphi^2(x)\,\varphi^2(y)\,\varphi^2(z)\sum_i\left(\psi_i^{(+)}(\mathbf{x})\right)^2$$
$$\times\left(\psi_i^{(+)}(\mathbf{y})\right)^2\,\left(\psi_i^{(+)}(\mathbf{z})\right)^2$$
$$\times\left(\langle\sigma_i^6\rangle - 3\langle\sigma_i^4\rangle\langle\sigma_i^2\rangle + 6\langle\sigma_i^2\rangle^3\right) \quad . \tag{3.2b}$$

Upon averaging over the disorder these terms become

$$v_6\int dx_1\,dx_2\,\varphi^3(x_1)\,\varphi^3(x_2)\,C^{(2)}_{\rm isl}(\mathbf{x}_1,\mathbf{x}_2) \quad , \tag{3.3a}$$

and

$$w_6\int dx_1\,dx_2\,dx_3\,\varphi^2(x_1)\,\varphi^2(x_2)\,\varphi^2(x_3)$$
$$\times D^{(3)}_{\rm isl}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \quad , \tag{3.3b}$$

respectively, with $v_6 \propto u^2$, $w_6 \propto u^3$. The correlation functions $C^{(2)}_{\rm isl}(\mathbf{x},\mathbf{y})$ and $D^{(3)}_{\rm isl}(\mathbf{x},\mathbf{y},\mathbf{z})$ are defined analogously to $D^{(2)}_{\rm isl}(\mathbf{x},\mathbf{y})$ in Eq. (2.18b), and are related to the probability for $\mathbf{x},\mathbf{y}$ and $\mathbf{x},\mathbf{y},\mathbf{z}$, respectively, to belong to the same island. We again concentrate on the generic case where the island size distribution falls of exponentially for large islands (for a discussion of other cases, see Appendix B). In this case both correlation functions are short-ranged and can be localized for power-counting purposes. This effectively leaves only one spatial integral in Eqs. (3.3a) and (3.3b). Therefore, the scale dimensions of the coefficients are $[v_6] = 2(2-D)$ and $[w_6] = 2(3-D)$. Consequently, both terms are irrelevant with respect to the Gaussian fixed point near $D = 4$.

More generally, we obtain from Eq. (2.12c) terms that contain powers of $\varphi^3$, terms that contain powers of $\varphi^2$, and mixed terms that contain both $\varphi^3$ and $\varphi^2$. For power counting purposes, the most relevant term for a fixed power of $\varphi$ is the one with the most time integrations. For even powers of $\varphi$, these are the terms

$$w_{2m}\int dx_1\,\ldots dx_m\,\varphi^2(x_1)\ldots\varphi^2(x_m)\,D^{(m)}_{\rm isl}(\mathbf{x}_1,\ldots,\mathbf{x}_m). \tag{3.4}$$

Localizing the correlation function $D^{(m)}$ we find for the scale dimension of the coupling constant $[w_{2m}] = 2m - (m-1)D$. Terms with odd powers of $\varphi$ are always less relevant than the preceding term of even order. We

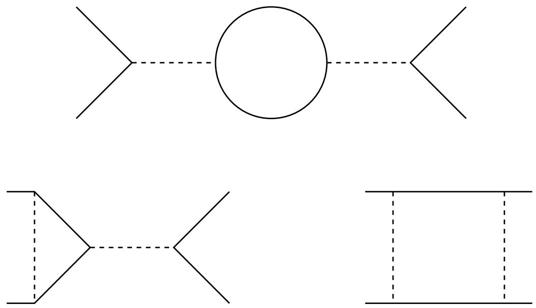

FIG. 1. The three diagram structures that contribute to the flow equations shown in Eqs. (3.5). The dashed lines stand for any of the vertices whose coupling constants are $u$, $\Delta$, or $w$.

conclude that all terms of higher than quartic order are irrelevant with respect to the Gaussian fixed point near $D = 4$.

So far we have determined the scale dimensions with respect to the Gaussian fixed point. At the non-trivial critical fixed point discussed in Ref. 13 the anomalous dimension of the field $\varphi$ is $\eta = 0 + O(\epsilon^2)$ since, as in the ordinary $\phi^4$ theory, there is no wavefunction renormalization at 1-loop order. This implies that all results on the irrelevancy of the terms of order $\varphi^6$ and higher carry over from the Gaussian fixed point to the non-trivial critical fixed point found in Ref. 13.

### B. Perturbation theory, and flow equations

In the last subsection we have shown that the Gaussian fixed point is unstable for $D < 4$. We must therefore carry out a loop expansion for the effective action, Eq. (2.19). To control the perturbation theory we consider $D = 4 - \epsilon$ spatial dimension and $\epsilon_\tau$ time dimensions.[19] This leads to the replacement of $\int d\tau$ by $\int d\tau\, \tau^{\epsilon_\tau - 1}$. At the Gaussian fixed point the scale dimension of the field $\varphi$ is now $[\varphi] = (d + z\epsilon_\tau - 2)/2$. In the same vein the scale dimensions of $u$, $\Delta$, and $w$ are $[u] = \epsilon - z\epsilon_\tau$, $[\Delta] = \epsilon$, and $[w] = \epsilon$, respectively. The perturbation theory becomes a double expansion in $\epsilon$ and $\epsilon_\tau$.

To obtain the renormalization group flow equations, we perform a frequency-momentum shell RG procedure.[16] The diagrams that contribute to the renormalization of the coupling constants $u$, $\Delta$, and $w$ are shown in Fig. 1. To one-loop order, we obtain the following flow equations,

$$\frac{du}{dl} = (\epsilon - 2\epsilon_\tau)\, u - 4\,(p+8)\, u^2 + 48\, u\, \Delta \quad , \tag{3.5a}$$

$$\frac{d\Delta}{dl} = \epsilon\Delta + 32\,\Delta^2 - 8\,(p+2)\, u\,\Delta + \frac{4p}{T^{\epsilon_\tau}}\,(2\,\Delta - w)\, w \ , \tag{3.5b}$$

$$\frac{dw}{dl} = \epsilon\, w + \frac{4p}{T^{\epsilon_\tau}}\, w^2 - 8\,(p+2)\, u\, w + 24\,(2\,\Delta - w)\, w + 8\, w^2 \quad . \tag{3.5c}$$

The mass $t$ of the two-point vertex, which describes the distance from the critical point, is of course also renormalized. However, since we are interested in the stability of a critical fixed point, it suffices to consider the flow on the critical surface. The factors of $T^{-\epsilon_\tau}$ in Eqs. (3.5) arise from the fact that some diagrams that contain the $w$-vertex lead to Matsubara frequency sums without an accompanying temperature factor. Since the critical surface for the quantum phase transition is defined by $T = 0$ in addition to $t = 0$, the natural coupling constant for the $T = 0$ flow is $\bar{w} = w\, T^{-\epsilon_\tau}$. Putting $T = 0$,[20] the flow equations can then be rewritten in the form

$$\frac{du}{dl} = (\epsilon - 2\epsilon_\tau)\, u - 4\,(p+8)\, u^2 + 48\, u\, \Delta \quad , \tag{3.6a}$$

$$\frac{d\Delta}{dl} = \epsilon\,\Delta + 32\,\Delta^2 - 8\,(p+2)\, u\,\Delta + 8\, p\,\Delta\,\bar{w} \quad , \tag{3.6b}$$

$$\frac{d\bar{w}}{dl} = (\epsilon - 2\epsilon_\tau)\,\bar{w} + 4\, p\,\bar{w}^2 - 8\,(p+2)\, u\,\bar{w} + 48\,\Delta\,\bar{w} \ . \tag{3.6c}$$

### C. Fixed points and their stability

The flow equations, Eqs. (3.6), possess eight fixed points. The fixed-point values of the coupling constants, and the corresponding eigenvalues of the linearized RG transformation are listed in Table I. Four of the fixed points (Nos. 1–4 in Table I) have a zero fixed-point value of $\bar{w}$, $\bar{w}^* = 0$. These are the fixed points studied before in Ref. 13. The other four fixed points have $\bar{w}^* \neq 0$.

Let us first consider fixed point No. 4. This is the critical fixed point that was found within the conventional approach.[13] We find that the local moments, represented by the $w$-term, render this fixed point unstable for $p < 4$, since in this case the third eigenvalue, $\lambda_w$, is positive. However, for $p > 4$ the $w$-term is irrelevant with respect to this fixed point, and the fixed point is stable for $4 < p < p_c$. To one-loop order, and for $\epsilon = \epsilon_\tau$, $p_c = 16$.[13]

A stability analysis for the new fixed points shows that they are all unstable for $p < 4$ with the exception of No. 8. At this fixed point, $w$ has a negative value, while Eq. (2.13), with reasonable assumptions about the distribution $p(\sigma)$, yields a positive value for the bare value of $w$. Since the structure of the flow equations does not allow $w$ to change sign, we conclude that this fixed point is unphysical for generic realizations of the disorder. It is interesting to note, however, that fixed point No. 8 is

| FP No. | FP values | | | eigenvalues | | |
|---|---|---|---|---|---|---|
| | $u^*$ | $\Delta^*$ | $w^*$ | $\lambda_1$ | $\lambda_2$ | $\lambda_w$ |
| 1 | 0 | 0 | 0 | $\epsilon - 2\epsilon_\tau$ | $\epsilon$ | $\epsilon - 2\epsilon_\tau$ |
| 2 | $\frac{\epsilon-2\epsilon_\tau}{4(p+8)}$ | 0 | 0 | $-(\epsilon - 2\epsilon_\tau)$ | $\frac{8(p+4)\epsilon_\tau-(p-4)\epsilon}{p+8}$ | $\frac{(p-4)(\epsilon-2\epsilon_\tau)}{p+8}$ |
| 3 | 0 | $-\epsilon/32$ | 0 | $2\epsilon_\tau - \epsilon/2$ | $-\epsilon$ | $2\epsilon_\tau - \epsilon/2$ |
| 4 | $\frac{(\epsilon+\epsilon_\tau)}{16\,(p-1)}$ | $\frac{(4-p)\epsilon+4(p+2)\epsilon_\tau}{64\,(p-1)}$ | 0 | $\frac{-A+\sqrt{A^2-B}}{p-1}$ | $\frac{-A-\sqrt{A^2-B}}{p-1}$ | $\frac{(4-p)\epsilon+4(4-p)\epsilon_\tau)}{4\,(p-1)}$ |
| 5 | 0 | 0 | $-(\epsilon-2\epsilon_\tau)/4p$ | $\epsilon/4 - \epsilon_\tau/2$ | $4\epsilon_\tau - \epsilon$ | $-(\epsilon - 2\epsilon_\tau)$ |
| 6 | $\frac{\epsilon-2\epsilon_\tau}{4\,(p+8)}$ | 0 | $\frac{(p-4)\,(\epsilon-2\epsilon_\tau)}{4p\,(p+8)}$ | $-(\epsilon - 2\epsilon_\tau)$ | $\frac{2\,(p+4)\,2\,\epsilon_\tau-(p-4)\epsilon}{p+8}$ | $\frac{(p-4)\,(\epsilon-2\,\epsilon_\tau)}{p+8}$ |
| 7 | 0 | $(4\epsilon_\tau - \epsilon)/64$ | $-(4\epsilon_\tau+\epsilon)/16p$ | $\epsilon_\tau + \epsilon/4$ | $-\epsilon$ | $2\epsilon_\tau - \epsilon/2$ |
| 8 | $\frac{\epsilon+4\epsilon_\tau}{8\,(10-p)}$ | $\frac{((p-4)\epsilon+24\epsilon_\tau)}{32\,(10-p)}$ | $\frac{(p-4)(\epsilon+4\epsilon_\tau)}{8p\,(10-p)}$ | $\frac{-C+\sqrt{C^2-D}}{p-1}$ | $\frac{-C-\sqrt{C^2-D}}{2\,(10-p)}$ | $\frac{(p-4)(\epsilon+4\epsilon_\tau)}{2\,(10-p)}$ |

TABLE I. Fixed points of the flow equations, Eqs. (3.6) and the eigenvalues of the corresponding linearized RG transformation. $p$ is the number of order parameter components. $\lambda_{1,2}$ are the eigenvalues in the absence of $w$, and $\lambda_w$ is the additional eigenvalue. $A$, $B$, $C$, and $D$ are defined as $A = (3\epsilon - 4\epsilon_\tau)p + 16\epsilon_\tau$, $B = 16(p-1)\,(\epsilon + 4\epsilon_\tau)\,[(4-p)\epsilon + 4(p+2)\epsilon_\tau]$, $C = (16-p)\epsilon + 4(p-4)\epsilon_\tau$, and $D = (10-p)\,(p+8)\,(\epsilon + 4\epsilon_\tau)\,[(p-4)\epsilon + 24)\epsilon_\tau]$.

stable against replica symmetry breaking (see Appendix C).

For $p < 4$, and to one-loop order in our double expansion in powers of $\epsilon$ and $\epsilon_\tau$, there is thus no stable fixed point. Consistent with this, a numerical solution of the flow equations, Eqs. (3.6), shows runaway flow in all of physical parameter space. We will discuss the physical meaning of this result in Sec. V below.

## IV. THE CASE OF ITINERANT FERROMAGNETS

In Ref. 15 a generalized LGW functional for the ferromagnetic transition in a disordered itinerant electron system was derived starting from a fermionic description. The effects of rare regions were not explicitly considered in this work. Here we show that although the rare regions were neglected in the explicity calculations in that paper, the effective field theory derived in Ref. 15 still contains these effects. We will further show that taking them into account does not change the previous conclusions.

We first briefly recall the effective action that was derived in Ref. 15. In the long-wavelength and low-frequency limit, the replicated action is given by

$$S_{\text{eff},1} = \frac{1}{2}\sum_\alpha \int dx_1\,dx_2\;\Gamma_0(x_1-x_2)\,\mathbf{M}^\alpha(x_1)\cdot\mathbf{M}^\alpha(x_2)$$
$$+\sum_\alpha \int dx_1\,dx_2\,dx_3\,dx_4\;u_4(x_1,x_2,x_3,x_4)$$
$$\times(\mathbf{M}^\alpha(x_1)\cdot\mathbf{M}^\alpha(x_2))\;(\mathbf{M}^\alpha(x_3)\cdot\mathbf{M}^\alpha(x_4))$$
$$-\Delta\sum_{\alpha,\beta}\int d\mathbf{x}\,d\tau\,d\tau'\;(\mathbf{M}^\alpha(\mathbf{x},\tau))^2\,(\mathbf{M}^\beta(\mathbf{x},\tau'))^2 \quad . \tag{4.1}$$

Here $\mathbf{M}$ is the order parameter field whose expectation value is the magnetization, and $u$ and $\Delta$ are coupling constants that in general are wavenumber and frequency dependent. An important point is that these coupling constants in general do not exist in the limit of zero frequencies and wavenumbers, i.e. the effective action describes a non-local field theory. This is because in the process of deriving a LGW functional that depends only on the order parameter field, soft (viz., diffusive) fermionic degrees of freedom have been integrated out. In writing Eq. (4.1) we have used that the coupling constant $\Delta$ is finite in the long-wavelength limit, so that it can be treated as a number. $u_4$, on the other hand, is singular in this limit, see Eq. (4.3) below. For small wavenumbers the Fourier transform of the two-point vertex $\Gamma_0$ is given by

$$\Gamma_0(\mathbf{q},\omega_n) = t_0 + u_2(\mathbf{q}) + |\omega_n|/\mathbf{q}^2 \quad , \tag{4.2a}$$

with,

$$u_2(\mathbf{q}) = u_2^{(D-2)}|\mathbf{q}|^{D-2} + u_2^{(0)}\mathbf{q}^2 \quad . \tag{4.2b}$$

Here $u_2^{(D-2)}$ and $u_2^{(0)}$ are finite numbers. Note that for $D < 4$ (in particular, in the physical dimension $D = 3$), the first term in Eq. (4.2b) dominates the second one as $\mathbf{q} \to 0$. $u_4$, in wavenumber space at zero frequency, is schematically given by

$$u_4(\mathbf{q} \to 0) = u_4^{(D-6)} |\mathbf{q}|^{D-6} + O(|\mathbf{q}|^{D-4}) \quad , \tag{4.3}$$

i.e., $u_4$ diverges for $D < 6$. The singularities in the wavenumber dependences of $u_2$ and $u_4$ mean that the field theory is non-local. As mentioned above, their physical origin are diffusive fermionic particle-hole excitations that were integrated out in deriving Eq. (4.1).

Next we argue that in at least one well defined physical situation, it is easy to uncover the effects of rare regions that are implicit in Eq. (4.1). The basic argument is that on length scales small compared to the elastic mean free path $\ell$, the field theory is effectively local. This implies that as long as the local moments or instantons decay on a scale $\lambda < \ell$, the techniques discussed in Sec. II can be used to include the effects of these inhomogeneous saddle points on the final long-wavelength theory. Further we will show that the quenched randomness that leads to these local moments is implicitly contained already in the last term in Eq. (4.1). To this end we first note, cf. Ref. 15 and below, that the *exact* critical behavior near the ferromagnetic transition can be determined from Eq. (4.1). Second, we partially undo the replica trick in Eq. (4.1) by writing the logarithm of the partition function, or the free energy, as Eq. (2.15) with $Z$ on the right hand side given by,

$$Z = \int D[\mathbf{M}(x)] \ \exp\left(-S[\mathbf{M}(x), \delta t(\mathbf{x})]\right) \quad , \tag{4.4a}$$

with,

$$\begin{aligned} S[\mathbf{M}(x), \delta t(\mathbf{x})] = \frac{1}{2} \int dx_1\, dx_2\ \Gamma(x_1 - x_2) \\ \times \mathbf{M}(x_1) \cdot \mathbf{M}(x_2) \\ + \int dx_1 \ \ldots \ dx_4 \ u_4(x_1, x_2, x_3, x_4) \\ \times (\mathbf{M}(x_1) \cdot \mathbf{M}(x_2))\, (\mathbf{M}(x_3) \cdot \mathbf{M}(x_4)) \quad . \end{aligned} \tag{4.4b}$$

$\Gamma(x)$ is given by

$$\Gamma(x) = \Gamma_0(x) + \delta t(\mathbf{x}) \quad , \tag{4.5}$$

with $\Gamma_0$ from Eq. (4.2a). $\delta t(\mathbf{x})$ in Eqs. (4.4) is a random function of position that is Gaussian distributed with the first two moments given by Eqs. (2.2). It trivally follows that Eqs. (2.15), (4.3) - (4.5), and (2.2) are equivalent to Eq. (4.1).

The arguments given in Sec. II for the AFM case apply equally well to the action given by Eq. (4.4b). To make this more precise let us consider the non-localities. In terms of scaled variables, Eq. (4.2b) for $u_2(q)$ can be written to lowest order in the disorder as,

$$u_2(\mathbf{q}) = u_2^{(0)} \left[ \left( \frac{\mathbf{q}}{k_F} \right)^2 + \frac{c}{k_F \ell} \left( \frac{|\mathbf{q}|}{k_F} \right)^{D-2} \right] \quad , \tag{4.6}$$

with $k_F$ the Fermi wavenumber, $\ell$ the elastic mean free path, and $c$ an interaction dependent constant that is at most of order unity. In general the expansion implied by Eq. (4.6) assumes that $|\mathbf{q}| \ll k_F$. In terms of length scales let $\lambda \sim 1/|\mathbf{q}|$ be the scale over which the order parameter varies, and $\lambda_F \sim k_F^{-1}$ the Fermi wavelength. Further, to be specific we consider the physical dimension $D = 3$. The analytic, square gradient, term in Eq. (4.6) then dominates the second term when $\lambda \ll \ell$. That is, the non-locality in Eq. (4.6) is irrelevant when spatial scales shorter than a mean free path are considered. For $\lambda_F \ll \lambda \ll \ell$ we then have,

$$u_2(\mathbf{q}) \simeq u_2^{(0)} \left( \frac{\mathbf{q}}{k_F} \right)^2 \quad . \tag{4.7}$$

Note that for this argument to be valid we need $\lambda_F/\ell \approx 1/k_F\ell \ll 1$, i.e., weak disorder is required. Similarly, $u_4$ in Eq. (4.3) can be replaced by a constant when $\lambda_F \ll \lambda \ll \ell$. The net result is that when the local moments vary on a length scale smaller than $\ell$, they can be described by a local field theory analogous to the one discussed in Sec. II even though the long-wavelength theory is non-local. If we assume, as we did in the antiferromagnetic case in the previous section, that the island distribution falls off exponentially for large island sizes, this will always be true for sufficiently small disorder.

With the above ideas and the techniques developed in Sec. II, the final long wavelength theory to descibe the ferromagnetic phase transition, explicitly including the effects of rare regions, is determined by the action,

$$S_{\text{eff}} = S_{\text{eff},1} + \delta S_{\text{eff}}, \tag{4.8a}$$

with $S_{\text{eff},1}$ given by Eq. (4.1) and

$$\begin{aligned} \delta S_{\text{eff}} = -w \sum_\alpha \int d\mathbf{x}\, d\tau\, d\tau' \ (\mathbf{M}^\alpha(x,\tau) \cdot \mathbf{M}^\alpha(x,\tau)) \\ \times (\mathbf{M}^\alpha(x,\tau') \cdot \mathbf{M}^\alpha(\mathbf{x},\tau')) \quad , \end{aligned} \tag{4.8b}$$

where $w$ is a finite constant.

Power counting immediately reveals that the coupling constant $w$, just like $\Delta$, is an irrelevant operator with respect to the Gaussian fixed point discussed in Ref. 15. The rare regions therefore do not change the critical behavior in this case. The physical reason for this is the effective long-range interaction between the order parameter fluctuations that is described by the $|\mathbf{q}|^{D-2}$-term in Eq. (4.2b). This stabilizes the Gaussian fixed point by suppressing all fluctuations, including the static disorder fluctuations resonsible for the local moments.

## V. DISCUSSION AND CONCLUSION

In this section, we conclude by discussing the results obtained in the previous sections.

### A. General considerations

We begin our discussion by considering the physical underpinnings of some general aspects of our technical procedure.

#### 1. Local moments and annealed disorder

Let us first of all give a simple physical interpretation of the $w$-term in the effective action, Eq. (2.19), which is the most important of the contributions that reflect the existence of rare regions and local moments. Since the local moments are self-generated by the electronic system, in response to the potential created by the quenched disorder, they are an integral part of the system and in equilibrium with all other degrees of freedom. In our approximation, which takes into account only the static local moment fluctuations, the effect of the rare regions therefore amounts to the existence of static, annealed disorder. Indeed, a straightforward generalization of Eq. (2.10) is to integrate over a manifold of saddle points $\mathbf{\Phi}(\mathbf{x})$, weighted with an appropriate distribution $P[\mathbf{\Phi}(\mathbf{x})]$,

$$Z \approx \int D[\mathbf{\Phi}(\mathbf{x})]\, P[\mathbf{\Phi}(\mathbf{x})] \int_< D[\boldsymbol{\varphi}(x)]\; e^{-\Delta S[\Phi(\mathbf{x}),\boldsymbol{\varphi}(x),\delta t(\mathbf{x})]}, \tag{5.1}$$

which makes obvious the annealed-disorder character of the average over the saddle points. The detailed result of the integration over the saddle points will of course depend on the distribution $P$, which in turn depends on the microscopic details of the disorder realization in the system. However, any physically reasonable distribution will lead in particular to a term in the effective action that has the structure of the $w$-term in Eq. (2.19). Since the saddle points are separated by large energy barriers in configuration space (see Sec. V A 2 below), this term clearly cannot be obtained by perturbatively expanding about the trivial homogeneous saddle point as is done in the conventional theory. Thus, our method approximately takes into account what one would call 'non-perturbative' effects in the usual approach.

It is important to note that the new term in the action, Eq. (2.19), differs from the usual quantum fluctuation or $u$-term only in its time structure. In the classical limit, therefore, $w$ just renormalizes $u$, decreasing its bare value. This is indeed well known to be the only effect of static annealed disorder in classical systems. In their analysis of classical magnets, the authors of Ref. 8 therefore considered a more elaborate scheme for doing the sum over saddle points in Eq. (2.10), or the integral in Eq. (5.1), that leads to a term that breaks the replica symmetry. Our way of approximating that integral can be considered as a zeroth order step in the approximation scheme of Ref. 8. In the quantum case, the time structure results in this zeroth step already giving a nontrivial result, and in this sense quantum systems are more sensitive to rare region effects than classical ones. The physical meaning of replica symmetry breaking in this context is not quite clear. However, in the quantum case it is not necessary to enter into this discussion. The AFM fixed point is unstable already under the effects considered above, and no new fixed point exists. Considering replica symmetry breaking in addition to our effect would not change this conclusion. In the FM case, it turns out that the previously found Gaussian fixed point is stable against replica symmetry breaking as well as against the quantum effect, as we will discuss in more detail below.

#### 2. Energy barriers between saddle points

A question that arises in connection with Eq. (2.9) or (5.1) is whether it is really true that there are large energy barriers between the various saddle-point configurations, as our approximation for the partition crucially depends on this assumption. Let us first consider the case of the Ising model ($p = 1$), for which we performed the explicit derivation in Sec. II. Suppose we have two saddle points that differ only by the sign of the order parameter on one particular island. In order to turn one of these spin configurations into the other, we need to flip all of the spins on that island. (For simplicity, we refer to the order parameter field as 'spins'.) In order to do so, one must go through an intermediate state with a domain wall across the island. The energy of that domain wall can be estimated from the squared gradient term in the free energy, integrated over the island, $J\int d\mathbf{x}\; (\nabla\phi(\mathbf{x}))^2$, with $J$ the coupling between the spins. The thickness of the domain wall is a microscopic length $a$, and hence the energy of the domain wall, or the energy barrier between the two saddle points, is proportional to $J\,L^{D-1}a/a^2 = J\,L^{D-1}/a$, with $L$ the linear size of the island. In the case of a continuous spin model ($p > 1$) an analogous argument holds, except that now all length scales are of order $L$.[1] This leads to $J\,L^{D-2}$ for the energy of a domain wall. For $D > 2$, there is thus only a quantitative difference between the Ising case and the continuous spin case.

In either model, the domain wall energy will have to be multiplied by the number of islands by which two typical saddle points differ. For the Ising case, let us consider $N$ islands, with $2^N$ saddle points and $2^{N-1}(2^N-1)$ pairs of saddle points. The probability distribution $\{p_N(n)\}$, for a pair of saddle points to have $n$ islands that are different is easily found to be

$$p_N(n) = \frac{1}{2^N-1}\binom{N}{n} \quad . \tag{5.2a}$$

For large $N$, this becomes a Gaussian distribution with mean $N/2$ and variance $\sqrt{N}/2$,

$$p_{N\to\infty}(n) = \frac{2}{\sqrt{2\pi N}}\, e^{-2(n-N/2)^2/N} \quad . \tag{5.2b}$$

One expects this to be true for the continuous spin case as well, although the statistical analysis becomes much more involved in that case. The miscroscopic energy of a domain wall thus gets multiplied by a macroscopic number, leading to energy barriers between almost all pairs of saddle points that go to infinity in the thermodynamic limit. This justifies our approximation.

Finally, we note that our considerations maximize, and probably overestimate, the effects of local moments or disorder induced instantons. The discussion above seems to imply an exponential number of saddle point solutions that are unrelated by symmetries, with barriers between them that approach infinity in the bulk limit. This in turn implies an exponential number of thermodynamic states, or a finite complexity. Such a proposition is controversial in other contexts, e.g. for spin glasses. However, in our considerations we have effectively neglected the interactions between the local moments. One anticipates these interaction to correlate and weaken the rare regions. Indeed, in Ref. 23 it was argued that long-range interactions that arise from the itinerant nature of the electrons quench most of the local moments. If this happens in the systems we consider, then we likely overestimate the number of distinct thermodynamic states. It is also possible that our theory is valid only in an intermediate time region, and that the interactions between the local moments must be taken into account in the limit of asymptotically long times.

#### *3. Nature of the local-moment phase*

Another point we have not yet addressed is the physical nature of the phase that is induced by the presence of the local moments. In order to show that we are dealing with a Griffiths phase, let us consider the local moment contribution to the order parameter susceptibility, $\chi_{\rm LM}$. Let us adopt a ferromagnetic language for simplicity, and denote the magnetic moment on the island number $i$ by $M_i$. Then we have

$$\chi_{\rm LM} = \left\{ \frac{1}{\sum_i V_i} \int_0^{1/T} d\tau \sum_{ij} \left( \langle M_i(\tau)\, M_j(0)\rangle - \langle M_i\rangle\, \langle M_j\rangle \right) \right\}_{\delta t} \quad , \qquad (5.3)$$

where $\langle\ldots\rangle$ denotes a thermodynamic average. Since there is no overall magnetization, $\sum_i \langle M_i\rangle = 0$, and in our saddle-point approximation the island magnetization is static. This yields

$$\chi_{\rm LM} = \left\{ \frac{1}{\sum_i V_i} \frac{1}{T} \sum_{ij} \langle M_i\, M_j\rangle \right\}_{\delta t} = \frac{\rm const}{T} \quad , \qquad (5.4)$$

where the constant is given by $\{\sum_i \langle M_i^2\rangle / \sum_i V_i\}_{\delta t}$. We see that the order parameter susceptibility diverges for $T \to 0$ whenever there are islands, and in our simple saddle-point approximation the divergence takes the form of a Curie law. Our saddle point thus really describes a Griffiths phase.

#### *4. Finiteness of the free energy*

As a final general point, let us again consider the effective action, Eq. (2.19), which determines the free energy. Since the $w$-term and the $u$-term have the same structure except for an extra time integral in the former, it seems as if the $w$-term contributes a term to the free energy that diverges as the temperature goes to zero. One has to keep in mind, however, that Eq. (2.19) represents a Landau expansion that has been truncated at $O(\boldsymbol{\varphi}^4)$. It is easy to see that higher order terms in the Landau expansion lead to even more strongly divergent contributions to the free energy, see Eq. (3.4). This simply means that the loop expansion for the free energy of a quantum system with static annealed disorder is singular, and a resummation to all orders would be necessary to obtain a finite result. From a RG point of view, which holds that the higher order terms in the Landau expansion are irrelevant, the solution of this paradox lies in the fact that, if a fixed point existed, it would be $\bar w$ that has a finite fixed-point value, not $w$. Since $w = \bar w\, T$ (for the physical case $\epsilon_\tau = 1$), this ensures that the fixed-point Hamiltonian has a finite free energy.

### B. Results for the AFM case

As we have shown in Sec. III, and reiterated above, taking into account the rare regions in the AFM case destroys the stability of the fixed point found in Ref. 13, and one finds runaway flow in all of the physically accessible parameter space. Three possible interpretations of this result are, (i) there is no transition to a state with long-range order, (ii) there is a transition, but the corresponding fixed point is inaccessible by perturbative RG techniques, or (iii) there is a fluctuation-induced first order transition (which causes the runaway flow). The last conjecture can be checked by calculating the free energy to one-loop order and then explicitly verifying whether it has a double minimum structure as a function of the order parameter. We have performed such a calculation,[21] and found that this is not the case. This rules out scenario (iii).

On the basis of our results, we cannot decide between scenarios (i) and (ii). Scenario (i) would imply that arbitrarily weak disorder necessarily destroys quantum AFM long-range order. This is an unlikely proposition, but it cannot be ruled out at present. The alternative is scenario (ii), i.e. the existence of a non-perturbative fixed point. The nature of such a fixed point, if it exists, is a priori unclear. The analogies with 1-$D$ systems mentioned in the Introduction, as well as Ref. 12, suggest that an unconventional infinite disorder fixed point with activated scaling is a possible interpretation of the runaway flow. However, there also could be a conventional fixed point that is not accessible by our methods. In this context it is interesting to note that the case $p > 4$ discussed in Sec. III C provides an example of a stable conventional fixed point that describes a transition with power-law critical behavior in the presence of rare regions.

Let us also come back to the fact that in Sec. III we found a stable fixed point (No. 8 in Table I) with $\bar{w}^* \neq 0$. As was pointed out in Sec. III C, for generic realizations of the disorder, which lead to a positive bare value of $w$, this fixed point is unphysical since it has $\bar{w}^* < 0$. However, mathematically one can have $w < 0$ for certain choices of the distribution $P[\{\sigma_i^a\}]$ in Sec. II that are more general than Eq. (2.7c). This leaves open the possibility that at least in some systems there is a stable, conventional critical fixed point that is accessible with our method. We note that this fixed point is stable against replica symmetry breaking, see Appendix C. This is in contrast to the case of classical magnets,[8] where all fixed points are unstable against replica symmetry breaking, and reminiscent of the result of Read et al.[22] on quantum spin glasses, where the quantum model was also found to be more stable against replica symmetry breaking than its classical counterpart. The technical reason for this enhanced stability is very similar to the point discussed at the end of Sec. V A, namely that the parameter that would induce replica symmetry breaking appears as $T$ times a finite fixed point value, and hence vanishes at the quantum critical point.

### C. Results for the FM case

For itinerant quantum ferromagnets, we have found that the rare regions do not affect our previous results.[15] The physical reason for this is the long-range interactions between the spin fluctuations in these systems. They are induced by soft modes in the itinerant electron system and stabilize the Gaussian critical behavior against fluctuations, including the static disorder fluctuations that lead to local-moment formation. A crucial point for our conclusion is the survival of these long-range interactions in the presence of local moments, so it is worth discussing this in some detail.

The derivation of the long-range interaction[15] shows that its origin is soft spin-triplet particle-hole excitations in the electron system. An obvious question is whether local moments act effectively as magnetic impurities that give these soft modes a mass. If this were the case, then the singular wavenumber dependences $|\mathbf{q}|^{d-2}$ and $|\omega_n|/\mathbf{q}^2$ in the Gaussian vertex, Eq. (4.2a), would be cut off and the ferromagnetic effective action would have the same structure as the antiferromagnetic one. The answer to this question is not obvious since the local moments are self-generated, and thinking about them as analogous to externally introduced magnetic moments can be misleading. This is underscored by the fact that the rare-regions/local-moment physics enters the theory in the form of *annealed* disorder, as we have seen in Sec. V A above. In our derivation of the effective action, Sec. IV, the wavenumber singularities are not cut off. The physics behind this is that both singularities are consequences of spin diffusion, which in turn is a consequence of the spin conservation law. The rare regions ultimately derive from a spin-independent disorder potential, which clearly cannot destroy spin conservation. The long-range interactions between the spin fluctutations are therefore still present in the bare effective action, and hence the Gaussian fixed point is stable in our tree-level analysis. We note, however, that at present we cannot rule out the possibility that loop corrections might lead to qualitatively new terms in the action. If such new terms included a RG-generated spin dependent potential, then this might change our conclusions. This would not necessarily violate the spin conservation argument given above, since an effective spin-dependent potential acting only on the itinerant electrons, which are not taking part in the local moment formation, could change the critical behavior while preserving spin conservation for the system as a whole.

A more detailed investigation of this point is not feasible within the existing framework of the ferromagnetic theory.[15] This is because in the existing theory all degrees of freedom other than the order parameter, including various soft modes, have been integrated out. This leads to a non-local field theory which is unsuitable for an explicit loop expansion. A remedy would be to derive an effective theory that keeps all soft modes explicitly and treats them on equal footing, leading to well-behaved vertices that allow for explicit calculations. This project is left for future work. We also note that at such a level of the analysis one should also include effects due to interactions between the rare regions, which we have mostly neglected. Such interactions are known to weaken the effects of the rare region,[23] but in general it is not known by how much.

### D. Summary, and Outlook

In summary, we have studied the effects of rare disorder fluctuations, and the resulting local moments, on itinerant ferromagnets and antiferromagnets. Technically, this has been achieved by considering non-trivial saddle-point solutions before performing the disorder average. A perturbative RG analysis of the resulting effective field theory incorporates effects that would require non-perturbative methods within a more standard procedure. In the ferromagnetic case we have found that, at least within our level of analysis, the previously found quantum critical behavior[15] is stable with respect to local moment physics. The reason is an effective long-range interaction between the spin fluctuations that strongly suppresses fluctuations, stabilizing a Gaussian critical fixed point. In the antiferromagnetic case, however, we have found that the local moments destroy the previously found critical fixed point.[13] To one-loop order, and for order parameter dimensionalities less than 4, no new fixed point exists and one finds runaway flow in all of physical parameter space. This may indicate either the absence of long-range order, or a transition that is not perturbatively accessible within our theory.

An important technical conclusion is that for quantum phase transitions, and within the framework of a replicated theory, rare regions can have a qualitative effect already at the level of a replica-symmetric theory, in contrast to the case of classical magnets.[8] The ferromagnetic fixed point, which was found to be stable against the replica-symmetric quantum effects induced by the rare regions, is also stable against replica-symmetry breaking.

We have concentrated on the role of fluctuations about a non-trivial, but fairly crudely constructed, saddle-point solution of the field theory. It would also be interesting to study a somewhat more sophisticated saddle-point theory in more detail, and to determine the detailed properties of the Griffiths phase in such an approximation.

Finally, we mention that our methods are not specific to magnets, and can be applied to other quantum phase transitions as well. For instance, it is believed that for a complete understanding of the properties of doped semiconductors, and of the metal-insulator transitions observed in such systems, it is necessary to consider the effects of local moments.[24,23,25] This can be studied with the methods developed in this paper.

## ACKNOWLEDGMENTS

We are indebted to Ferdinand Evers for a collaboration in the early stages of this work. We also gratefully acknowledge helpful discussions with John Toner. This work was supported by the NSF under grant Nos. DMR-95-10185, DMR-96-32978, and DMR-98-70597, and by the DFG under grant No. SFB 393/C2.

## APPENDIX A: A ONE-DIMENSIONAL SADDLE-POINT EQUATION

In this Appendix we discuss the saddle-point equation, Eq. (2.5), for a particular realization of the disorder potential $\delta t(\mathbf{x})$. In particular, we aim to show that the existence of a non-zero solution requires the width and depth of the potential well to be above a threshold, and that the non-zero solution lowers the free energy with respect to the zero one.

We first consider the one-dimensional counterpart of Eq. (2.5),

$$\left(t_0 + \delta t(x) - \partial_x^2\right)\, \phi(x) + g\phi^3(x) = 0 \quad , \tag{A1}$$

with a simple square well potential,

$$\delta t(x) = \begin{cases} -V_0 & \text{for } 0 \le x < a \\ 0 & \text{elsewhere} \end{cases} \quad . \tag{A2}$$

Standard methods lead to a solution inside the well ($0 \le x < a$)

$$\phi_{\text{in}}(x) = \sqrt{\frac{v_1}{g}}\, \frac{\text{cn}\sqrt{v_2\, x/2}}{\text{dn}\sqrt{v_2\, x/2}} \quad , \tag{A3a}$$

where cn an dn are elliptic functions, and a solution outside of the well,

$$\phi_{\text{out}}(x) = \sqrt{\frac{t_0}{g}}\, \frac{2\sqrt{2}\, c\, e^{-\sqrt{t_0}\, x}}{c^2\, e^{-2\sqrt{t_0}\, x} - 1} \tag{A3b}$$

Here

$$v_{2,1} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\,\beta}}{2} \quad , \tag{A4a}$$

with

$$\alpha = 2(t_0 - V_0) \quad . \tag{A4b}$$

$c$ and $\beta$ are constants of integration that are determined by the requirement that the solution and its derivative be continuous at $x = a$. Other solutions exist, but the one given is the only one that satisfies physical boundary conditions. Furthermore, the physical solution exists only for

$$0 < \beta < \alpha^2/4 \quad . \tag{A5}$$

To demonstrate the existence of a threshold, we expand the above solution for small values of $v_2 a$. To leading order in this small parameter, we obtain for the constants of integration

$$c = \frac{1}{4}\sqrt{\frac{2}{t_0}}\, \sqrt{v_1}\, e^{\sqrt{t_0}\, a} \quad , \tag{A6a}$$

and

$$\beta = \frac{\alpha^2}{4} - \frac{t_0}{2\, a^2} \quad . \tag{A6b}$$

From the condition for the existence of the physical solution, Eq. (A5), we see that $\alpha^2 a^2 > 2t_0$ is a necessary and sufficient condition for the solution to exist, which is the desired threshold property.

The free energy in saddle-point approximation is simply given by the saddle-point action. It is physically plausible that the non-homogeneous solution constructed above leads to a negative free energy, and is thus energetically favorable compared to the zero solution. We have

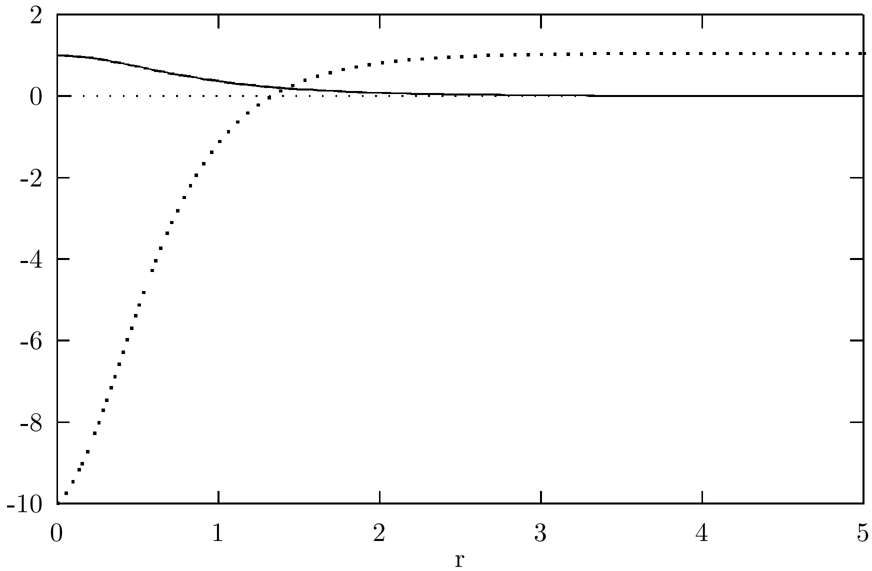


FIG. 2. $t(r)$ (dotted line) and $\phi(r)$ (solid line) for $a = 1, b = 1, c = 1$.

ascertained this numerically for a large variety of well parameter values, and have found the inhomogeneous solution to lead to a negative free energy whenever it exists.

To solve the three-dimensional saddle-point equation, Eq. (2.5), is much harder. For sperically symmetric wells, $t_0 + \delta t(\mathbf{x}) = t(r)$, with $r = |\mathbf{x}|$, analytic solutions can still be found in closed form for special forms of the potential. By scaling $\phi$ and $t$, the equation can be written

$$\nabla^2 + t(r)\,\phi(r) + \phi^3(r) = 0 \quad . \tag{A7}$$

It is easy to show that for

$$\begin{aligned} t(r) = {} & \frac{-2b}{(1+b\,r)} + b^2 + \frac{4\,b\,c\,r}{(1+c\,r^2)} - \frac{2\,c}{(1+c\,r^2)} \\ & - \frac{4\,b\,c\,r}{(1+c\,r^2)\,(1+b\,r)} + \frac{8\,c^2\,r^2}{(1+c\,r^2)^2} - \frac{2\,b^2}{(1+b\,r)} \\ & - \frac{4\,c}{(1+b\,r^2)} - \frac{a^2\,(1+b\,r)^2\,e^{-2\,b\,r}}{(1+c\,r^2)^2} \quad , \end{aligned} \tag{A8a}$$

the physical solution is given by

$$\phi(r) = \frac{a\,(1+b\,r)\,\exp(-b\,r)}{(1+c\,r^2)} \quad . \tag{A8b}$$

Here $a$, $b$, and $c$ are parameters that determine the shape of the well. In this case, the physical solution exists for all real values of the three parameters, but the form of the potential is such that the volume of the well cannot be smaller than some minimum value. This is the three-dimensional analog of the threshold behavior demonstrated above for the one-dimensional case. We have also solved the ODE, Eq. (A7), numerically for more general potential wells, and have found the same type of threshold behavior.

As in the 1-$D$ case, physical arguments suggest, and numerical integration confirms, that the inhomogeneous solution leads to a lower free energy than the homogeneous one whenever the former exists. In Fig. 2 we show the solution and the corresponding potential well, Eqs. (A8), as a representative example of a locally ordered region in a 3-$D$ system.

## APPENDIX B: ISLAND SIZE DISTRIBUTIONS WITH A POWER-LAW TAIL

In Secs. III and IV we have assumed that the island correlation functions $D^{(m)}_{\text{isl}}$ and $C^{(m)}_{\text{isl}}$ are short-range correlated, i.e. have a scale dimension of $(m-1)D$. Here we briefly discuss the extent to which one can relax this condition without changing our results.

Suppose that the island-size distribution is power-law correlated, leading to scale dimensions of the above correlation functions that are given by $(m-1)(D-\alpha)$ with $\alpha > 0$. Let us consider the FM case first. The least irrelevant term, viz. Eq. (4.8b), remains irrelevant with respect to the Gaussian fixed point of Ref. 15 as long as $\alpha < 4 - D$ (for $2 < D < 4$). The $D$-dependence of this result reflects the fact that for $D \geq 4$ the effective interaction ceases to be long-ranged, and an ordinary mean-field fixed point is stable. All higher order terms in the action are less relevant than the $w$-term.

In the AFM case, the $w$-term is relevant with respect to the conventional fixed point even for $\alpha = 0$. By power counting, we find the condition that none of the higher order terms become relevant as well, viz. $\alpha < D - 3$ for $D$ close to 4. Here the $D$-dependence reflects the fact that the coupling constant $w_6$ is marginal in $D = 3$ even for $\alpha = 0$, see Sec. III A.

## APPENDIX C: STABILITY UNDER REPLICA SYMMETRY BREAKING

In this Appendix we briefly consider the effects of replica symmetry breaking (RSB). A generalization of our action, Eq. (2.19), analogous to Ref. 8 that allows for RSB is

$$
\begin{aligned}
S_{\rm eff}[\boldsymbol{\varphi}^\alpha(x)] &= \frac{1}{2}\sum_\alpha \int dx\, dy\, \boldsymbol{\varphi}^\alpha(x)\cdot \Gamma_0(x-y)\,\boldsymbol{\varphi}^\alpha(y) \\
&+u\sum_\alpha \int d\mathbf{x}\, d\tau\, (\boldsymbol{\varphi}^\alpha(\mathbf{x},\tau)\cdot(\boldsymbol{\varphi}^\alpha(\mathbf{x},\tau))^2 \\
&-\sum_{\alpha,\beta}(\Delta + w_{\alpha\beta})\int d\mathbf{x}\, d\tau\, d\tau' \\
&\quad \times(\boldsymbol{\varphi}^\alpha(\mathbf{x},\tau)\cdot\boldsymbol{\varphi}^\alpha(\mathbf{x},\tau))\,(\boldsymbol{\varphi}^\beta(\mathbf{x},\tau')\cdot(\boldsymbol{\varphi}^\beta(\mathbf{x},\tau'))\, .
\end{aligned}
\tag{C1}
$$

In Sec. II we had $w_{\alpha\beta}\equiv w$, which resulted in a replica symmetric theory. Now we allow for 1-step RSB in Parisi's hierarchical scheme,[26] where $w_{\alpha\beta}$ in the replica limit is parameterized by means of a step function with a parameter $x_0$,

$$
w(x) = \begin{cases} w & \text{for } 0\le x\le 1 \\ w_1 & \text{for } x_0 < x \le 1\end{cases}\quad . \tag{C2}
$$

Defining $\bar w = w\, T^{\epsilon_\tau}$ as before, and $\tilde w = w_1, T^{\epsilon_\tau}$, we obtain the 1-loop flow equations

$$
\frac{du}{dl} = (\epsilon - 2\epsilon_\tau)\,u - 4\,(p+8)\,u^2 + 48\,u\,\Delta \quad , \tag{C3a}
$$

$$
\begin{aligned}
\frac{d\Delta}{dl} &= \epsilon\,\Delta + 32\,\Delta^2 - 8\,(p+2)\,u\,\Delta + 8p\,\Delta\,\bar w \\
&\quad - 8p\,x_0\,\Delta\,\tilde w + 8p\,\Delta\,\tilde w \quad ,
\end{aligned}
\tag{C3b}
$$

$$
\begin{aligned}
\frac{d\bar w}{dl} &= (\epsilon - 2\,\epsilon_\tau)\,\bar w + 4\,p\,\bar w^2 - 8\,(p+2)\,u\,\bar w + 48\,\Delta\,\bar w \\
&\quad -4p\,(1-x_0)\,\tilde w^2 \quad ,
\end{aligned}
\tag{C3c}
$$

$$
\begin{aligned}
\frac{d\tilde w}{dl} &= (\epsilon - 2\,\epsilon_\tau)\,\tilde w + 48\,\Delta\,\tilde w - 8\,(p+2)\,u\,\tilde w \\
&\quad + 8p\,\bar w\,\tilde w + 8p\,(1-x_0)\,\tilde w^2 \quad .
\end{aligned}
\tag{C3d}
$$

The replica symmetric case is recovered by putting $x_0 = 1$. We now perform a linear stability analysis of fixed point No. 8, with fixed-point values of $u$, $\Delta$ and $w$ as given in Table I, and $\tilde w^* = 0$. The first three eigenvalues are as shown in Table I, and the fourth one is $\lambda_{\tilde w} = \lambda_w = (p-4)(\epsilon + 4\epsilon_\tau)/2(10-p)$. Fixed point No. 8 is therefore stable against 1-step RSB. Although the fixed point is unphysical for generic realizations of the disorder, as discussed in Sec. III, this is an interesting contrast to the classical case,[8] where all fixed points are unstable against successive terms in the hierarchical RSB scheme.


[1] For a pedagogical discussion, see, G. Grinstein in *Fundamental Problems in Statictical Mechanics VI*, E.G.D. Cohen (ed.), Elsevier (New York 19850), p.147, and references therein.

[2] A.B. Harris, J. Phys. C **7**, 1671 (1974); J. Chayes, L. Chayes, D.S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).

[3] For a recent discussion concerning the validity of the Harris criterion, see, F. Pazamandi, R.T. Scalettar, and G. Zimanyi, Phys. Rev. Lett. **79**, 5130 (1997); S. Wiseman and E. Domany, Phys. Rev. Lett. **81**, 22 (1998); A. Aharony, A.B. Harris, and S. Wiseman, Phys. Rev. Lett. **81**, 252 (1998).

[4] J. Villain, J. Phys. (Paris) **46**, 1843 (1985); D. S. Fisher, Phys. Rev. Lett. **56**, 416 (1986); A. E. Nash, A. R. King, and V. Jaccarino, Phys. Rev. B **43**, 1272 (1991).

[5] B.M. McCoy and T.T. Wu, Phys. Rev. **176**, 631 (1968); **188**, 982 (1969); B.M. McCoy, Phys. Rev. **188**, 1014 (1969). See also R. Shankar and G. Murthy, Phys. Rev. B **36**, 536 (1987).

[6] D.S. Fisher, Phys. Rev. B **51**, 6411 (1995).

[7] R.B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).

[8] Viktor Dotsenko, A.B. Harris, D. Sherrington, and R.B. Stinchcombe, J. Phys. A **28**, 3093 (1995); Viktor Dotsenko and D.E. Feldman, J. Phys. A **28**, 5183 (1995).

[9] For recent reviews of quantum phase transitions, see, e.g., S.L. Sondhi, S.M. Girvin, J.P. Carini, and D. Shahar, Rev. Mod. Phys. **69**, 315 (1997); T.R. Kirkpatrick and D. Belitz, cond-mat/9707001; D. Belitz and T.R. Kirkpatrick, cond-mat/9811058.

[10] A Griffiths region for a quantum critical point has recently been discussed by A.H. Castro Neto, G. Castilla, and B.A. Jonse, Phys. Rev. Lett. **81**, 3531 (1998); M.C. de Andrade, R. Chau, R.P. Dickey, N.R. Dilley, E.J. Freeman, D.A. Gajewski, M.B. Maple, R. Movshovich, A.H. Castro Neto, and G. Castilla, Phys. Rev. Lett. **81**, 5620 (1998).

[11] J. Kisker and A.P. Young, cond-mat/9807025.

[12] C. Pich, A. P. Young, H. Rieger, and N. Kawashima, Phys. Rev. Lett. **81**, 5916 (1998).

[13] T.R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. **76**, 2571 (1996); *ibid.* **78**, 1197 (1997).

[14] R. Narayanan, Thomas Vojta, D. Belitz, and T.R. Kirkpatrick, cond-mat/9903194.

[15] T.R. Kirkpatrick and D. Belitz, Phys. Rev. B **53**, 14364 (1996). See also T. Vojta, D. Belitz, R. Narayanan, and T.R. Kirkpatrick, Z. Phys. B **103**, 451 (1997) which corrected some minor mistakes in the earlier paper.

[16] J. Hertz, Phys. Rev. B **14**, 1165 (1976).

[17] We will concentrate on rare fluctuations that lead to isolated regions of order within the disordered phase. Therefore, we take $t_0$ to be positive, and consider potentials with regions where $t_0 + \delta t(\mathbf{x}) < 0$.

[18] This procedure is not necessary, but it is convenient. We have derived the same final results without using the subtraction of the saddle-point action.

[19] S.N. Dorogovtsev, Phys. Lett. **76A**, 169 (1980); D. Boyanovsky and J.L. Cardy, Phys. Rev. B **26**, 154 (1982).

[20] Notice that the limit $T\to 0$ does not commute with the classical limit $\epsilon_\tau \to 0$. Therefore, the flow equations for the classical model can be recovered from Eqs. (3.5), but not

from Eqs. (3.6).
[21] Rajesh Narayanan, Ph.D. Thesis, University of Oregon (1999), unpublished.
[22] N. Read, S. Sachdev, and J. Ye, Phys. Rev. B **52**, 384 (1995).
[23] R.N. Bhatt and D.S. Fisher, Phys. Rev. Lett. **68**, 3072 (1992).
[24] R.N. Bhatt and P.A. Lee, Phys. Rev. Lett. **48**, 344 (1982); M. Milovanovic, S. Sachdev, and R.N. Bhatt, Phys. Rev. Lett. **63**, 82 (1989).
[25] For a review, see, D. Belitz and T.R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).
[26] M. Mezard, G. Parisi, and M.A. Virasoro, *Spin Glass Theory and Beyond*, World Scientific (Singapore, 1987).