Construction of Near-Capacity Protograph LDPC Code Sequences with Block-Error Thresholds

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Abstract—Density evolution for protograph Low-Density Parity-Check (LDPC) codes is considered, and it is shown that the message-error rate falls double-exponentially with iterations whenever the degree-2 subgraph of the protograph is cyclefree and noise level is below threshold. Conditions for stability of protograph density evolution are established and related to the structure of the protograph. Using large-girth graphs, sequences of protograph LDPC codes with block-error threshold equal to bit-error threshold and block-error rate falling nearexponentially with blocklength are constructed deterministically. Small-sized protographs are optimized to obtain thresholds near capacity for binary erasure and binary-input Gaussian channels.

I. INTRODUCTION

Low-density parity-check (LDPC) codes, which are linear codes with sparse parity-check matrices, are used today in several digital communication system standards. Introduced by Gallager [1] in the 60s, the sparse parity-check matrices of modern LDPC codes are specified using the bit and checknode degree distributions of their Tanner graphs [2]. The set of all Tanner graphs with a given degree distribution defines an ensemble of LDPC codes.

When decoded using the message-passing algorithm over binary-input symmetric-output channels, the expected bit-error rate over the ensemble of LDPC codes shows a threshold phenomenon as blocklength tends to infinity. There is a threshold channel parameter, below which, the expected bit-error rate tends to fall rapidly for large blocklength. The bit-error threshold, which is a function of the degree distribution, is computed using a procedure known as density evolution. The bit-error rate of a code in the ensemble concentrates around the expected value; so, the threshold is an important design parameter in practice. The practical design of LDPC codes involves determining the degree distribution that maximizes the threshold for a fixed rate. Given a degree distribution, a parity-check matrix is sampled from the ensemble with several heuristic criteria to simplify the complexity of implementation and for acceptable performance [3].

The study of protograph LDPC codes, which are a special case of Multi-Edge Type LDPC codes [4], was initiated in [5], and protograph LDPC codes are the most popular codes today in theory (spatially-coupled codes [6], [7]) and practice (included in WiFi and DVB-S2 standards). In [8], protographs are optimized for thresholds nearing capacity, and ensemble-averaged weight distribution is used to establish block-error

A. K. Pradhan and A. Thangaraj are with the Dept. of Electrical Engg., IIT Madras, Chennai, India, Email: asit.pradhan,andrew@ee.iitm.ac.in. threshold for protograph LDPC code ensembles. In [9], conditions on protograph for typical linear growth of minimum distance are derived. There have been numerous other work in the construction of protographs for several applications [10]– [14]. Density evolution for protograph LDPC codes over the Binary Erasure Channel (BEC) was derived in [15], and EXIT charts for protograph design were studied in [16].

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While bit-error threshold is a popular design criterion, block-error thresholds are important both from a theoretical and practical point of view [17]. In spite of the importance, block-error thresholds do not exist for many capacityapproaching degree distributions that have degree-2 bit nodes. Another area of concern is random sampling in the construction of LDPC and other modern codes. While concentration results are useful, deterministic constructions that have a provable block-error performance are the ultimate goal of code design. Finally, the analytical properties of protograph density evolution and optimization of protographs using it are topics that have not received much attention so far in the literature. This work addresses the above shortfalls.

The main contribution of this paper is the design and deterministic construction of a sequence of large-girth, protograph LDPC codes with provable block-error thresholds at rates approaching capacity. The idea of large-girth constructions, pioneered in Gallager's thesis [1], was studied in the context of block-error thresholds in [17] for the standard socket ensemble with minimum bit degree 3. In this work, the crucial property of double-exponential fall of message-error rate with iterations is extended to protograph LDPC ensembles that are allowed to contain degree-2 bit nodes under the condition that the degree-2 subgraph of the protograph is cycle-free. The use of degree-2 bit nodes enables, through a carefully-designed differential evolution algorithm, the design of optimized protographs with block thresholds approaching capacity even at small sizes. To the best of our knowledge, the construction in this work is perhaps the first deterministic LDPC code sequence with guaranteed block-error rate behavior at rates close to capacity. As a specific example, we provide a deterministic rate-1/2protograph LDPC code sequence with a block-error threshold of 0.4953 over the BEC.

The rest of this article is organized as follows. Section II introduces protograph LDPC codes and their notation. The crucial property of double-exponential decay for protograph density evolution and its stability are described in Section III. The construction of large-girth protograph LDPC codes is presented in Section IV. The optimization of protographs and simulation results are given in Sections V and VI, respectively. Concluding remarks are made in Section VII.

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II. PROTOGRAPH LDPC CODES

Following the notation in [5], a protograph $G = (V \cup C, E)$ is a bipartite graph with the bipartition V and C called the set of variable or bit and check nodes, respectively, and E being the set of undirected edges that connect a variable node in V to a check node in C. Multiple parallel edges are allowed between a variable node and a check node. The nodes and edges in the protograph are ordered, and the *i*-th variable node, check node and edge in the protograph are denoted, respectively, v_i , c_i and e_i . The variable and check nodes connected by an edge e_i are denoted $v(e_i)$ and $c(e_i)$, respectively.

A protograph can be represented by a base matrix B of dimension $|C| \times |V|$, whose (i, j)-th element B(i, j) is the number of edges between c_i and v_j . For example, consider a base matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$
 (1)

The protograph corresponding to the above base matrix is shown in Fig. 1. The 9 different edges in this example are

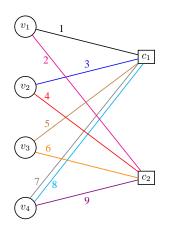


Fig. 1: The protograph for the base matrix in (1).

numbered as shown in the figure.

A. Lifted graphs

A copy and permute operation is applied to a protograph to obtain expanded or lifted graphs of different sizes [5]. A given protograph G is copied, say T times, with the t-th copy having variable nodes denoted (v, t), check nodes denoted (c, t), and edges denoted (e, t) with $v \in V$, $c \in C$ and $e \in E$. Then, for each edge e in the protograph, we assign a permutation π_e of the set $\{1, 2, \ldots, T\}$. In the permute operation, an edge (e, t) connecting (v, t) and (c, t) is permuted so as to connect variable node (v, t) to check node $(c, \pi_e(t))$. An edge (e_i, t) in the lifted graph is said to be of type e_i or, simply, type i.

We will denote a lifted graph as $G(T,\Pi) = (V(T) \cup C(T), E(T,\Pi))$, where $\Pi = \{\pi_e : e \in E\}$, or simply $G' = (V' \cup C', E')$ when the exact T and Π are either clear or not critical. A lifted graph of a protograph can be thought of as a Tanner graph of an LDPC code, which is referred to as a protograph LDPC code. The collection of these lifted graphs is called the protograph ensemble of LDPC codes defined by

G. Protograph LDPC codes are a special class of multi edge type (MET)-LDPC codes [4] with each edge in the protograph being of a different type. The (designed) rate of the protograph LDPC code is given by 1 - |C|/|V|. The degree distribution of check and variable nodes in the lifted graph is the same as that of the protograph, but the protograph LDPC codes have a richer structure than the standard ensemble when we consider computation graphs [2].

B. Tree computation graphs

Consider an edge of type i in the lifted graph G'. The *l*-iteration computation graph for the edge is defined as the subgraph of G' obtained by traversing down to depth 2lalong all adjacent edges at the variable node end [2]. An important observation is that the vertex degrees and edge types in the computation graph are completely determined by the protograph G. Further, let us suppose that the girth of G'is greater than 2l, which makes the *l*-iteration computation graph a tree with no repeated nodes. It is clear that the *l*iteration tree computation graph for an edge of a particular type in the protograph ensemble is *deterministic* and *unique* in the sequence of vertex degrees and edge types encountered. The protograph G completely determines the sequence of degrees and edge types. So, in comparison with the standard socket ensemble [2], no assumption on the distribution of tree computation graphs is needed, and this makes density evolution analysis for large-girth protograph codes precise.

III. DENSITY EVOLUTION AND DOUBLE-EXPONENTIAL FALL PROPERTY

A crucial fact that enables precise block-error rate guarantees from density evolution is the property of doubleexponential fall of message-error rate with iterations [17]. For the standard socket ensemble, double-exponential fall is possible only when the minimum degree is at least 3. In this section, we describe protograph density evolution and show that double-exponential fall is possible even when degree-2 nodes are included in the protograph. We begin with the case of the binary erasure channel (BEC).

A. Binary erasure channel

Let us consider the standard message-passing decoder [2] over a binary erasure channel with erasure probability ϵ , denoted BEC(ϵ), run on a lifted graph G' derived from a protograph $G = (V \cup C, E)$. Since the lifted graphs form an MET ensemble with |E| edge types, density evolution proceeds with |E| erasure probabilities, one for each edge in the protograph [4]. Let $x_t(i)$ be the probability that an erasure is sent from variable node to check node along edge type e_i in the *t*-th iteration. Similarly, let $y_t(j)$ be the probability that an erasure is sent from check node to variable node along edge type e_i in the *t*-th iteration. The protograph density evolution recursion [15] is given by

$$x_0(i) = \epsilon, \tag{2}$$

$$y_{t+1}(j) = 1 - \prod_{i \in E_c(e_j)} (1 - x_t(i)), \tag{3}$$

$$x_{t+1}(i) = \epsilon \prod_{j \in E_v(e_i)} y_{t+1}(j), \tag{4}$$

for $t \ge 0$ and $1 \le i, j \le |E|$, where $E_c(e) = \{i : c(e) = c(e_i), e \ne e_i\}$ and $E_v(e) = \{i : v(e) = v(e_i), e \ne e_i\}$ are the sets of other edge types incident to the same check node and variable node, respectively, as the edge e. The density evolution threshold, denoted $\epsilon_{\rm th}$, for the protograph-based LDPC code ensemble is defined as the supremum of the set of ϵ for which erasure probability on each edge of the protograph tends to zero, as $t \rightarrow \infty$, i.e. $\epsilon_{\rm th} = \sup\{\epsilon : \max_i x_t(i) \rightarrow 0\}$. All protographs in this work have minimum bit-node degree 2 ensuring that $\epsilon_{\rm th}$ is the threshold for variable node erasure probability as well.

1) Double-exponential fall: Consider a protograph G with density evolution recursion as defined in (2) - (4). Because the recursion steps for $x_{t+1}(i)$ and $y_{t+1}(j)$ involve the neighbors of the edges e_i and e_j , it is useful to visualize (2) - (4) as iterative message passing on G with bit-to-check messages $x_t(i)$ and check-to-bit messages $y_t(j)$ in iteration t. So, it is easy to see that all variable nodes in walks of length 2t + 1 starting with e_i are visited in the computation of $x_t(i)$. Every time a variable node of degree at least 3 is traversed, multiplication of two or more terms occurs in the recursion as per (4) resulting in a squaring or higher power effect. Variable nodes of degree 2 result in a linear term with no squaring. It turns out that ensuring at least a squaring effect at regular intervals in every walk is sufficient for double-exponential fall, and this is made precise in the following theorem.

Theorem 1. Let a protograph G be such that (1) there are no loops involving only degree-2 variable nodes, and (2) every degree-2 variable node is connected to a variable node of degree at least 3. Then, for $\epsilon < \epsilon_{th}$,

$$x_t(i) = O(\exp(-\beta 2^{\alpha t})) \tag{5}$$

for sufficiently large t, where α , β are positive constants.

Proof: Let $\bar{x}_t = \max_i x_t(i)$ and let $|v_2|$ be the number of degree-two variable nodes in G. We will show that there exists a positive integer R such that, for $t \ge R$,

$$\bar{x}_{t+|v_2|+1} \le A(\bar{x}_t)^2,$$
 (6)

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where A is a constant independent of t, and $A\bar{x}_R < 1$. By repeatedly applying (6), we can readily show that

$$\bar{x}_{R+i(|v_2|+1)} \le A^{-1} (A\bar{x}_R)^{2^\circ},$$
(7)

for a positive integer i, which implies (5).

Suppose we have the upper bounds $x_{t+l}(i) \leq C_l \bar{x}_t^{m(l,i)}$, where $l \in \{0, 1, \dots, |v_2|\}$, $m(l, i) \in \{1, 2\}$ and C_l is a positive constant independent of t. We will propagate the bounds through one round of (3) - (4) to obtain bounds $x_{t+l+1}(i) \leq C_{l+1} \bar{x}_t^{m(l+1,i)}$. For l = 0, we have $C_0 = 1$ and m(0,i) = 1. We will show that $m(|v_2| + 1, i) = 2$ for all *i*, which proves (6).

We will use the following inequality. For any $x \in [0, 1]$ and a positive integer d,

$$(d-1)x \ge 1 - (1-x)^{d-1}$$
. (8)

First consider the RHS of (3) for a fixed j. Let

$$n(l,j) = \min_{i \in E_c(e_j)} m(l,i).$$
 (9)

Since $\bar{x}_t \leq 1$, we have $x_{t+l}(i) \leq C_l \bar{x}_t^{n(l,j)}$. So,

$$1 - x_{t+l}(i) \ge 1 - C_l \bar{x}_t^{n(l,j)}, \quad i \in E_c(e_j),$$

$$\Rightarrow \prod_{i \in E_c(e_j)} (1 - x_{t+l}(i)) \ge \left(1 - C_l \bar{x}_t^{n(l,j)}\right)^{r(j)-1},$$

where r(j) is the degree of $c(e_j)$. Since $\bar{x}_t \to 0$, for large enough t, we have $C_l \bar{x}_t^{n(l,j)} < 1$. So, using (3) and (8), we get, for large enough t,

$$y_{t+l+1}(j) \le (r(j)-1)C_l \bar{x}_t^{n(l,j)}.$$
 (10)

Now consider (4) for a fixed i. We get

$$x_{t+l+1}(i) = \epsilon \prod_{j \in E_v(e_i)} y_{t+l+1}(j)$$

$$\leq \epsilon ((r_{\max} - 1)C_l)^{l(i)-1} \prod_{j \in E_v(e_i)} \bar{x}_t^{n(l,j)}, \quad (11)$$

$$\leq \epsilon ((r_{\max} - 1)C_l)^{l(i) - 1} \bar{x}_t^{m(l+1,i)}, \tag{12}$$

$$\leq C_{l+1}\bar{x}_{t}^{m(l+1,i)},$$
(13)

where l(i) is the degree of $v(e_i)$, r_{\max} is the maximum checknode degree $(r_{\max} \ge 2)$, $C_{l+1} = \epsilon \max_i ((r_{\max} - 1)C_l)^{l(i)-1}$ and we set

$$m(l+1,i) = \begin{cases} 1, & \text{if } \sum_{j \in E_v(e_i)} n(l,j) = 1, \\ 2, & \text{if } \sum_{j \in E_v(e_i)} n(l,j) \ge 2. \end{cases}$$
(14)

Note that $\sum_{j \in E_v(e_i)} n(l, j) = 1$ only when $v(e_i)$ is a degree-2 variable node and n(l, j) = 1 for the single edge $e_j \in E_v(e_i)$.

We now claim that $m(|v_2| + 1, i) = 2$. The proof for the claim is by contradiction. Suppose that $m(|v_2| + 1, i) = 1$ for some e_i . Then, for the single edge $e_j \in E_v(e_i)$, $n(|v_2|, j) = 1$, which in turn implies $m(|v_2|, i') = 1$ for some $e_{i'} \in E_c(e_j)$. Proceeding in this manner, there exists a walk in G of length $|v_2| + 1$ containing only degree-2 variable nodes. This is a contradiction because G has exactly $|v_2|$ degree-2 variable nodes, and by the assumptions of the theorem, G has no loops involving degree-2 variable nodes, and every degree-2 variable node in G is connected to at least one variable node of degree at least 3.

Now, if there is a cycle involving degree-2 nodes in the protograph, we can show, using a method similar to the proof above (after setting $x_t(i) = 0$ when $v(e_i)$ has degree at least 3), that $x_t(i)$ for an edge e_i in the degree-2 cycle falls at most exponentially with t. Therefore, the degree-2 subgraph being cycle-free is a necessary and sufficient condition for double-exponential fall of message error probability in protograph density evolution. We remark that the condition of degree-2 subgraph being loopfree has been used before in the context of typical linear growth of minimum distance [9] [18].

2) large-girth lifted graph sequences and block-error *threshold*: Consider a protograph $G = (V \cup C, E)$ satisfying the conditions of Theorem 1 and a lifted graph $G' = G(T, \Pi)$. Let n = T|V| denote the blocklength of the LDPC code defined by G', and consider message-passing decoding over BEC(ϵ). When G' has girth g, the probability of erasure on an edge of type i from bit node to check node in iteration tis exactly equal to $x_t(i)$ if $t \leq g/2 - 1$. So, for $t \leq g/2 - 1$, the probability of erasure from bit to check on any edge is upper bounded by \bar{x}_t , and by the union bound, the probability of block error, denoted $P_B(n)$, is upper bounded as $P_B(n) = O(n\bar{x}_t)$. In Section IV, for a given protograph G, we provide constructions of lifted graphs with large girth or girth growing as $\Theta(\log n)$. So, for large-girth lifted graphs, the girth can be increased arbitrarily by increasing n, and we have

$$P_B(n) = O(n\bar{x}_t) = O(n\exp(-\beta 2^{\alpha t}))$$
(15)

for $\epsilon < \epsilon_{\text{th}}$ and sufficiently large *n* using Theorem 1. Now, setting $t = c \log n$, c > 0, in (15), we get

$$P_B(n) = O(n \exp(-\beta n^{c\alpha})), \tag{16}$$

for $\epsilon < \epsilon_{\text{th}}$. By noting that $\lim_{n\to\infty} n^k n \exp(-\beta n^{c\alpha}) \to 0$, we can say that the block-error probability $P_B(n)$ falls faster than $1/n^k$ for a positive integer k.

Therefore, for large-girth protograph LDPC code sequences, if the protograph satisfies the conditions of Theorem 1, blockerror threshold is equal to the bit error threshold ϵ_{th} .

3) Stability of protograph density evolution: Let $\mathbf{x}_t = [x_t(1) \ x_t(2) \ \cdots \ x_t(|E|)]$ denote the vector of bit-to-check erasure probabilities in iteration t as per the protograph density evolution of (2) - (4) for a protograph G. The density evolution recursion can be represented as $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \epsilon)$, where the *i*-th coordinate of the vector function f is

$$f_i(\mathbf{x}_t, \epsilon) = \epsilon \prod_{j \in E_v(e_i)} \left(1 - \prod_{i' \in E_c(e_j)} (1 - x_t(i')) \right). \quad (17)$$

The monotonicity of f_i with $x_t(i')$ and ϵ is easy to establish [4]. We concern ourselves with the stability of the recursion. Approximating f using Taylor series around origin, we get

$$\mathbf{x}_{t+1} = \nabla_f \, \mathbf{x}_t + e_f(\mathbf{x}_t),\tag{18}$$

where ∇_f is the $|E| \times |E|$ gradient matrix of f with (i, i')-th element, denoted $[\nabla_f]_{ii'}$, defined as the partial derivative of $f_i(\mathbf{x}_t, \epsilon)$ with respect to $x_t(i')$ evaluated at the origin $\mathbf{x}_t = \mathbf{0}$, and $e_f(\mathbf{x}_t)$ is a length-|E| vector satisfying $||e_f(\mathbf{x}_t)||^2 = O(||\mathbf{x}_t||^2)$ ($||\cdot||$ denotes Euclidean norm). Letting l(i) denote the degree of bit node $v(e_i)$, we readily see from (17) that

$$[\nabla_f]_{ii'} = \frac{\partial f_i(\mathbf{x}_t, \epsilon)}{\partial x_t(i')} \Big|_{\mathbf{x}_t = \mathbf{0}} = \begin{cases} 0, \text{ if } l(i) \neq 2, \\ \epsilon, \text{ if } l(i) = 2, i' \in E_c(e_j), \end{cases}$$
(19)

where, for the case l(i) = 2, e_i and e_j are the two edges connected to the degree-2 node $v(e_i)$.

For sufficiently small \mathbf{x}_t , the convergence of $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \epsilon)$ to **0** depends on the eigenvalues of ∇_f being less than one [19]. To study the eigenvalues of ∇_f , we use Perron-Frobenius theory on eigenvalues of non-negative matrices

following [20]. For this purpose, we introduce some notation and definitions.

A directed graph $D(\mathbf{A})$ is associated with a nonnegative $n \times n$ matrix \mathbf{A} . The vertex set of $D(\mathbf{A})$ is $\{1, 2, \ldots, n\}$ with a directed edge from i to j if and only if the (i, j)-th element of \mathbf{A} , is nonzero. A directed graph D is said to be *strongly connected* if there is a directed path between any two vertices of D. A nonnegative square matrix \mathbf{A} is said to be *irreducible* if $D(\mathbf{A})$ is strongly connected. For a non-negative square matrix \mathbf{A} , there exists a permutation matrix \mathbf{P} such that

$$\mathbf{P}\mathbf{A}\mathbf{P}^{T} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \cdots & \mathbf{A}_{1s} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{A}_{23} & \cdots & \mathbf{A}_{2s} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \cdots & \mathbf{A}_{2s} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_{ss} \end{bmatrix}, \quad (20)$$

where \mathbf{A}_{ii} is either a square irreducible matrix or a 1×1 zero matrix. The block upper-triangular form of (20) is called the *Frobenius normal form* of \mathbf{A} . Note that $D(\mathbf{P}\mathbf{A}\mathbf{P}^T)$ is isomorphic to $D(\mathbf{A})$ with vertices permuted by \mathbf{P} , and the eigenvalues of $\mathbf{P}\mathbf{A}\mathbf{P}^T$ are the same as that of \mathbf{A} . So, for the purposes of stability, we will assume that the gradient matrix ∇_f is in Frobenius normal form with the diagonal blocks denoted as ∇_{ii} , $1 \le i \le s_f$, where s_f denotes the number of diagonal blocks. The subgraphs $D(\nabla_{ii})$ are called the strongly connected components of $D(\nabla_f)$.

The next two lemmas connect edges and cycles in $D(\nabla_f)$ to the structure of the protograph G.

Lemma 1. The directed graph $D(\nabla_f)$ has an edge from i to i'if and only if l(i) = 2 and $i' \in E_c(e_j)$, where e_j is the single edge in $E_v(e_i)$. This implies the following: (1) Vertex i is in a strongly-connected component of $D(\nabla_f)$ only if l(i) = 2; (2) for edge (i, i') in $D(\nabla_f)$, there exists a path $(e_i, e_j, e_{i'})$ in the protograph with l(i) = l(j) = 2.

Proof: The lemma is a restating of (19). Claim (1) follows because an edge needs to originate out of a vertex in a strongly-connected component. Claim (2) follows from (19).

Lemma 2. There is a length-l cycle in $D(\nabla_f)$ if and only if there is a length-2l cycle in the subgraph of the protograph induced by degree-2 bit nodes.

Proof: Let $(e_{i_1}, e_{i_2}, \ldots, e_{i_l}, e_{i_1})$ be a cycle in $D(\nabla_f)$. This implies directed edges $(e_{i_m}, e_{i_{m+1}})$, $1 \leq m \leq l-1$, and (e_{i_l}, e_{i_1}) in $D(\nabla_f)$. By Lemma 1, there are edges $e_{j_1}, e_{j_2}, \ldots, e_{j_l}$ such that $(e_{i_1}, e_{j_1}, e_{i_2}, e_{j_2}, \ldots, e_{i_l}, e_{j_l}, e_{i_1})$ is a cycle in the protograph and $l(i_m) = l(j_m) = 2$ for $1 \leq m \leq l$.

Structural stability conditions on G following from the above two lemmas are collected in the next theorem.

Theorem 2. Consider a protograph $G = (V \cup C, E)$ with gradient matrix ∇_f , whose Frobenius normal form has diagonal blocks ∇_{ii} $(1 \le i \le s_f)$. Let G_2 denote the subgraph of Ginduced by degree-two bit nodes. Let E_2 denote set of edges of G incident on degree-two bit nodes. Protograph density evolution over $BEC(\epsilon)$ is stable in each of the following cases:

1) for all ϵ , if G_2 is cycle-free.

- 2) for $\epsilon < 1$, if no two cycles of G_2 overlap in an edge.
- 3) for $\epsilon < 1/r_{\text{max}}$, where $r_{\text{max}} = \max_{e \in E_2} |E_c(e) \cap E_2|$.

Proof: 1) If the subgraph of G induced by degree-2 bit nodes is cycle-free, we get, by Lemma 2, that there are no cycles in $D(\nabla_f)$. So, there are no strongly-connected subgraphs in $D(\nabla_f)$, which implies that $\nabla_{ii} = 0$ for all i in the Frobenius normal form of ∇_f . Therefore, the eigenvalues of ∇_f are all 0, implying stability for all ϵ .

2) If no two cycles of G_2 overlap in an edge, the cycles of $D(\nabla_f)$ do not overlap in a vertex or an edge by Lemma 2. So, the strongly-connected components of $D(\nabla_f)$ are cycles. Since $D(\nabla_{ii})$ is a cycle, the eigenvalues of ∇_{ii} have absolute value equal to ϵ [21], implying stability for $\epsilon < 1$.

3) The result follows because the maximum eigenvalue of ∇_{ii} is upper bounded by its maximum row sum [19], and ϵr_{\max} is an upper bound on the maximum row sum of the matrices ∇_{ii} , $1 \le i \le s_f$.

From Theorems 1 and 2, the degree-2 subgraph of the protograph being cycle-free emerges as an important design condition. Next, we provide an example to illustrate the stability conditions for protographs.

Example 1. Consider a protograph whose subgraph induced by degree-two bit nodes, denoted G_2 , is as shown in Fig. 2. For such a protograph, the gradient graph has one non-trivial

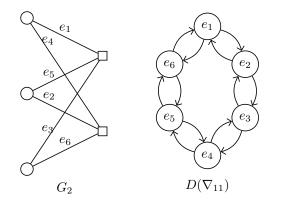


Fig. 2: Illustration of stability for protograph density evolution.

strongly-connected component $D(\nabla_{11})$ as shown. Clearly, cases (1) and (2) of Theorem 2 do not apply. A quick calculation shows $r_{\max} = 2$, which results in stability for $\epsilon < 0.5$. In this case, an exact eigenvalue calculation matches with the bound based on r_{\max} .

B. Binary-input symmetric channel

The extension to binary-input symmetric channels uses the method of Bhattacharya parameters, and we will be brief in our description referring to [2] and [17] for details. A binary input channel $X \rightarrow Y$ with $X \in \{-1, +1\}$ is said to be symmetric if the transition probability p(Y|X) satisfies p(Y = y|X = +1) = p(Y = -y|X = -1). The standard message-passing

decoder uses the log-likelihood ratio (LLR) $l_0 = \log \frac{p(y|+1)}{p(y|-1)}$ as input, and the message passed from a bit node to a check node in iteration t is an LLR l_t for the corresponding bit. The bit node operation is simply addition, while the check node operation uses the standard tanh rule. Assuming that the all-+1s codeword is transmitted and that the computation graph is a tree, the LLR l_t is of the form $\log \frac{p_t(y|+1)}{p_t(y|-1)}$, where $p_t(Y|X)$ is the transition probability of a symmetric channel. Density evolution computes the transition probability $p_t(Y|X = +1)$ using $p_{t-1}(Y|X = +1)$ and p(Y|X).

For many symmetric channels of practical interest such as the Binary-Input Additive White Gaussian Noise (BIAWGN) channel, the transition probability p(Y|X) is nonzero over the real line, which makes density evolution analysis cumbersome. However, probability of message error in each iteration can be upper bounded by using the Bhattacharyya parameter following [17]. The method in [17] readily extends to protograph density evolution for a binary-input symmetric channel as described next.

The Bhattacharyya parameter for the channel corresponding to the bit-to-check message in the t-th iteration is defined as follows:

$$B_t = \int_{-\infty}^{\infty} \sqrt{p_t(y|+1)p_t(y|-1)} dy.$$
 (21)

The probability of message error in the *t*-th iteration is bounded above by the Bhattacharya parameter B_t .

Consider the standard message-passing decoder over a binary-input symmetric channel p(Y|X) run on a lifted graph G' derived from a protograph $G = (V \cup C, E)$. Protograph density evolution for this situation involves |E| densities $p_t^{(i)}(Y|X = +1), 1 \le i \le |E|$, with corresponding Bhattacharya parameters $B_t(i)$. The evolution of Bhattacharya parameters satisfies a set of inequalities given in the next lemma.

Lemma 3. The Bhattacharyya parameters $B_t(i)$ satisfy

$$B_{t+1}(i) \le B_0 \prod_{j \in E_v(e_i)} \sum_{i' \in E_c(e_j)} B_t(i')$$
(22)

for $1 \le i \le |E|$, where $B_0 = \int_{-\infty}^{\infty} \sqrt{p(y|+1)p(y|-1)} dy$ is the Bhattacharya parameter of the channel p(Y|X), and E_v , E_c are as defined earlier.

Proof: The proof follows the proof of Lemma 1 in [17] closely, and we skip the details.

Let $ch(\sigma)$ be a family of binary-input symmetric output channels, where σ denotes the channel parameter with $ch(\sigma)$ being a degraded version of $ch(\sigma')$ whenever $\sigma > \sigma'$. Let σ_{th} be the threshold below which the maximum probability of error in protograph density evolution for a protograph Gtends to zero as $t \to \infty$. Since probability of error tending to zero implies that Bhattacharya parameter tends to zero, we have that, for $\sigma < \sigma_{th}$, the maximum Bhattacharya parameter $\max_i B_t(i) \to 0$ as $t \to \infty$.

Now, using ideas similar to those used for the binary erasure channel, we can show that the Bhattacharya parameter, and, hence, the probability of message error, exhibits a double exponential fall with iterations if the degree-2 subgraph of G is cycle-free. This result is stated as a theorem for reference.

Theorem 3. Let a protograph G be such that (1) there are no loops involving only degree-2 variable nodes, and (2) every degree-2 variable node is connected to a variable node of degree at least 3. Then, for $\sigma < \sigma_{th}$,

$$B_t(i) = O(\exp(-\beta 2^{\alpha t})) \tag{23}$$

for sufficiently large t, where α , β are positive constants.

Proof: The proof is similar to the proof of Theorem 1.

The statements about large-girth constructions in Section III-A2 for binary erasure channels carry over for the binaryinput symmetric channel case as well. In particular, a sequence of large-girth protograph LDPC codes over binary-input symmetric channels will have bit-error threshold equal to blockerror threshold, and block-error rate falling near-exponentially with blocklength for noise levels below threshold, whenever the degree-2 subgraph of the protograph is cycle-free.

IV. LARGE-GIRTH PROTOGRAPH LDPC CODES

We have seen that a sequence (in *n*) of length-*n* protograph LDPC codes with girth increasing as $c \log n$, c > 0, results in block-error rate falling as $O(n \exp(-\beta n^{c\alpha}))$ (where $\alpha, \beta > 0$) below the message or bit-error threshold of the protograph.

The construction of large-girth regular graphs is a classic problem in graph theory [22]. For a recent construction and survey of latest results, see [23]. For applications of large-girth graphs in the construction of LDPC codes, see [24], [25], [26]. In this section, we show how sequences of large-girth protograph LDPC codes can be constructed starting from sequences of regular large-girth graphs. We also discuss explicit deterministic constructions. Parts of this construction were presented earlier in [27].

A. Construction of large-girth protograph LDPC codes

Let $G = (V \cup C, E)$ be a protograph. The starting point for the construction is a sequence of |E|-regular bipartite graphs $B_{n_i} = (V_i \cup C_i, E_i), i = 1, 2, ...,$ with $|V_i| = |C_i| = n_i$. The existence of such sequences is well-known in graph theory, and we provide explicit examples later on in this section. For now, we assume that such a sequence is available.

1) Edge coloring: According to König's theorem [28], the graph $B_{n_i} = (V_i \cup C_i, E_i)$ can be edge-colored with |E| colors numbered from 1 to |E|. We fix such a coloring. For a vertex $v \in V_i$, let $e_j(v)$, $j = 1, 2, \ldots, |E|$, denote the edge of color j incident on v. Similarly, let $e_j(c)$ denote the edge of color j incident on $c \in C_i$.

2) Node splitting: Let us number the left vertices of the protograph G as 1, 2, ..., |V|, the right vertices as 1, 2, ..., |C|, and the edges as 1, 2, ..., |E|. Let l(j) and r(j) denote the left and right vertex indices of G connected by the edge j.

From the graph B_{n_i} , we will construct a bipartite graph $G' = (V' \cup C', E')$, where $|V'| = n_i |V|$, $|C'| = n_i |C|$ and $|E'| = |E_i| = n_i |E|$, by operations that we call node splitting followed by edge reconnecting. Every vertex $v \in V_i$ is split

into |V| vertices and denoted, say, as $v_1, v_2, \ldots, v_{|V|} \in V'$. Every vertex $c \in C_i$ is split into |C| vertices denoted $c_1, c_2, \ldots, c_{|C|} \in C'$. Now, we connect the edge $e_j(v)$ originally incident on $v \in V_i$ to the new vertex $v_{l(j)} \in V'$. Similarly, we connect the edge $e_j(c)$ incident on $c \in C_i$ to the new vertex $c_{r(j)} \in C'$.

The node splitting step is illustrated in Fig. 3.

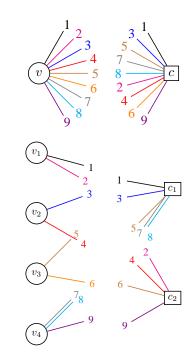


Fig. 3: Illustration of node splitting with the protograph of Fig. 1.

3) Properties: Two main properties are quite easy to prove. The first is that the graph G' obtained by node splitting is a lifted version of the protograph G, and can be generated by a copy-permute operation on G. In the copy-permute operation, the protograph G is copied n_i times, and the permutation of the edge type j in G is determined precisely by the matching M = $\{e_j(v) : v \in V_i\}$ in B_{n_i} . Numbering the left/right vertices of B_{n_i} from 1 to n_i , let M map the left vertex t to the right vertex M(t) in B_{n_i} . In the t-th copy of the protograph G, the edge (e, t) connecting (v, t) to (c, t) is permuted to connect (v, t) to (c, M(t)) in the lifted graph.

The second property is that the girth of G' is at least as large as the girth of B_{n_i} . This is easy to see because a cycle in G' readily maps to a cycle of the same length in B_{n_i} .

In summary, we see that, given a sequence of |E|-regular bipartite graphs B_{n_i} with n_i nodes in each bipartition and girth at least $c \log n_i$, we can construct a sequence of liftings of a protograph with |E| edges such that the girth of the lifted graphs grows at least as $c \log n_i$.

B. Deterministic constructions

The construction method described above can use any sequence of regular large-girth graphs. For completeness and to give deterministic constructions, we describe the parameters of two large-girth graph sequences called LPS graphs [29] and D(m,q) graphs [30], which we have used in simulations.

1) LPS Graphs $X^{p,q}$: Let p and q be distinct, odd primes with $q > 2\sqrt{p}$. The LPS graph, denoted $X^{p,q}$ [29], is a connected, (p + 1)-regular graph and has the following properties:

- If p is a quadratic residue mod q, then $X^{p,q}$ is a non-bipartite graph with $q(q^2 1)/2$ vertices and girth $g(X^{p,q}) \ge 2\log_p q$.
- If p is a quadratic non-residue mod q, then $X^{p,q}$ is a bipartite graph with $q(q^2 1)$ vertices and girth $g(X^{p,q}) \ge 4 \log_p q \log_p 4$.

When $X^{p,q}$ is non-bipartite, we can convert it to a bipartite graph using the following algorithm [22] [26]:

- Given a graph G with vertices V(G) and edges E(G), construct a copy G' with a new vertex set V(G') and a new edge set E(G'). Let f : V(G) → V(G') be the 1-1 mapping from a vertex in G to its copy in G'.
- Create a bipartite graph H with vertex set V(G) ∪ V(G') and edge set E(H) = {(x, f(y)) : (x, y) ∈ E(G)}.

Following [22], it was shown in [26] that $g(H) \ge g(G)$. For constructing a sequence of *d*-regular large-girth graphs for an arbitrary *d* using the LPS graphs, we use the following trick from [26]. There exists an infinite number of primes *p* such that *d* divides (p+1), i.e., d|(p+1). For each such prime *p* and a suitable *q*, we construct $X^{p,q}$ and split each (p+1)-degree node into (p+1)/d nodes of degree *d*. As shown in [26], node splitting does not reduce girth and we have a large-girth graph of the required degree *d*.

2) D(m,q) graph: The D(m,q) graphs satisfy the following properties ([30] and [31]):

(a) For a prime power q and an integer $m \ge 2$, the girth of D(m,q) satisfies

$$g(D(m,q) \ge \begin{cases} m+5, \ m \text{ odd,} \\ m+4, \ m \text{ even.} \end{cases}$$
(24)

- (b) For $q \ge 5$ and $2 \le m \le 5$, D(m,q) is a connected bipartite graph with $2q^m$ vertices.
- (c) For m ≥ 6, the graph D(m,q) is disconnected. Because of edge transitivity all connected components are isomorphic. There are q^{t-1} components of D(m,q), where t = \[m+2 \/ 4 \]. Each component of D(m,q) has 2q^{m-t+1} vertices and has girth equal to g(D(m,q)) defined in (24). Thus, for m ≥ 6, any connected component of D(m,q) can be used for constructing LDPC codes.

3) Comparison between $X^{p,q}$ and D(m,q): In the LPS construction $X^{p,q}$, to guarantee a minimum girth g, a careful calculation shows that we must have blocklength $n \sim p^{\frac{3g}{2}}$ or $n \sim p^{\frac{3g}{4}}$.

For $m \geq 6$, to guarantee girth g in the D(m,q) construction, the blocklength grows as $n \sim q^{\frac{3g-13}{4}}$, which is smaller than that of $X^{p,q}$. Hence, we can generate graphs of smaller block length by using the D(m,q) graph with the node-splitting algorithm. But unlike $X^{p,q}$, in D(m,q), the vertex degree is always a power of a prime, which implies that the number of edge types in the protograph needs to be a prime power. The constructions in [23] work directly for an arbitrary degree, and could be used as well.

V. OPTIMIZATION OF PROTOGRAPHS

In this section, we describe the search procedure used for generating optimized protographs. The erasure channel version was partly presented in [27].

A. Differential evolution

We have optimized protographs using differential evolution [32] [33], where we use the threshold given by density evolution as the cost function. The salient steps of the differential evolution algorithm are described briefly in the following:

- 1. Initialization: For generation G = 0, we randomly choose N_P base matrices $B_{k,G}$, with $0 \le k \le N_P 1$, of size $|C| \times |V|$, where $N_P = 10|C||V|$. Each entry of $B_{k,G}$ is binary, chosen independently and uniformly.
- 2. Mutation: Protographs of a particular generation are interpolated as follows.

$$M_{k,G} = [B_{r_1,G} + 0.5(B_{r_2,G} - B_{r_3,G})], \quad (25)$$

where r_1 , r_2 , r_3 are randomly-chosen distinct values in the range $[0, N_P - 1]$, and [x] denotes the absolute value of x rounded to the nearest integer.

- 3. Crossover: A candidate protograph $B'_{k,G}$ is chosen as follows. The (i, j)-th entry of $B'_{k,G}$ is set as the (i, j)-th entry of $M_{k,G}$ with probability p_c , or as the (i, j)-th entry of $B_{k,G}$ with probability $1 p_c$. We use $p_c = 0.88$ in our optimization runs. In $B'_{k,G}$, if any cycle of degree-2 nodes emerges, edges are reassigned.
- Selection: For generation G+1, protographs are selected as follows. If the threshold of B_{k,G} is greater than that of B'_{k,G}, set B_{k,G+1} = B_{k,G}; else, set B_{k,G+1} = B'_{k,G}.
- 5. Termination: Steps 2–4 are run for several generations (we run up to G = 6000) and the protograph that gives the best threshold is chosen as the optimized protograph.

In the crossover step, we ensure that the subgraph induced by the degree-2 nodes of the protograph is a tree. This ensures that the block-error threshold equals the bit-error threshold. If this condition is not enforced, better thresholds might result from the optimization, but with no guarantee of a block-error threshold.

The value of p_c is the crossover step has been taken as 0.88 based on trial and error. The optimization can be run with other values of p_c , but we have obtained acceptable results with this value.

B. Optimized protographs for BEC

A few optimized protographs obtained from the above optimization process are as follows. An optimized 3×12 , rate-3/4 protograph with threshold 0.238 is given by the following base matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 7 & 4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 7 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 1 & 5 & 5 & 3 & 4 & 0 & 1 & 2 & 0 & 1 & 3 & 3 \end{bmatrix}$$
(26)

An optimized 4×12 , rate-2/3 protograph with threshold 0.32 is given by the following base matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 5 & 3 & 1 & 0 & 2 & 3 & 1 & 1 & 1 \\ 0 & 1 & 0 & 6 & 0 & 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 & 1 & 2 & 4 & 0 & 4 & 1 & 1 \end{bmatrix}$$
(27)

An optimized 4×8 , rate-1/2 protograph with threshold 0.479 is given by the following base matrix:

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 3 & 0 & 4 & 1 \\ 1 & 0 & 1 & 0 & 6 & 1 & 0 & 0 \end{bmatrix}$$
(28)

We observe that high thresholds are obtained even with smallsized protographs. As the size increases, the thresholds get close to capacity bounds.

An optimized 8×16 protograph with threshold 0.486 is given by the following base matrix:

[1	2	0	0	1	0	0	4	0	0	0	0	0	0	0	1]
0	1	0	0	0	1	0	0	2	2	1	0	0	0	1	1
0	3	1	2	1	0	0	0	4	0	0	3	2	2	0	3
0	5	0	0	0	0	1	1	0	0	1	0	0	1	0	0
1	3	1	1	1	2	0	0	1	0	0	0	0	0	0	0
1	5	0	0	0	3	1	0	0	0	1	0	0	0	0	0
0	4	0	0	0	0	0	1	1	0	0	0	0	0	0	1
0	5	0	0	0	0	0	0	0	1	0	0	1	0	1	0
															(29)

A 16×32 protograph with threshold 0.4952 is given in (30). The above protographs from our optimization runs are compared against other protographs in Table I. We see that the optimized protographs give better thresholds than irregular standard ensemble codes with minimum degree 3 [26] and other construction such as AR4JA [34, Figure 7], and standards such as WIMAX [35] and DVB-S2 [36].

Code type	Rate	Size	Threshold	Gap
WIMAX	0.5	12×24	0.448	0.052
DVB-S2	0.444	25×45	0.516	0.040
Standard $(l_{\min} = 3)$	0.5	Not applicable	0.461	0.039
AR4JA	0.5	4×8	0.468	0.032
protograph in (28)	0.5	4×8	0.479	0.021
protograph in (29)	0.5	8×16	0.486	0.014
protograph in (30)	0.5	16×32	0.4953	0.0047
AR4JA	0.67	2×6	0.291	0.039
WIMAX	0.67	8×24	0.292	0.038
DVB-S2	0.67	15×45	0.305	0.028
protograph in (27)	0.67	4×12	0.32	0.01
WIMAX	0.75	6×24	0.212	0.038
DVB-S2	0.73	12×45	0.232	0.018
protograph in (26)	0.75	3×12	0.238	0.012

TABLE I: Comparison of protograph thresholds for BEC.

C. Optimized protographs for BIAWGN channel

For BIAWGN channel, the threshold of protograph density evolution is computed using the EXIT chart method described in [16]. A few optimized protographs are given below, and their SNR thresholds (denoted SNR_{th}) are compared against the capacity-achieving SNR (denoted SNR_{cap}) and other protographs such as AR4JA [34, Figure 7], and those from the DVB-S2 and WIMAX standards in Table II.

$\begin{bmatrix} 2\\0\\4\\5 \end{bmatrix}$	${0 \\ 0 \\ 1 \\ 1 \end{array}$	$egin{array}{c} 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \$	$ \begin{array}{c} 1 \\ 1 \\ 1 \\ 6 \end{array} $	$7 \\ 7 \\ 5 \\ 1$	${0 \\ 0 \\ 0 \\ 3 }$	$egin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array}$	$egin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array}$	$\begin{array}{c} 0 \\ 2 \\ 0 \\ 1 \end{array}$	$\begin{array}{c} 0 \\ 0 \\ 0 \\ 3 \end{array}$	$ \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array} $	$\begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$	(31)
$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$	3	7	2	2	3	1	3	2	3	5	3	(32)

Code type	Rate	Size	SNR _{th} (dB)	Gap (dB)
DVB-S2	0.444	25×45	0.474	1.042
WIMAX	0.5	12×24	0.812	0.625
AR4JA	0.5	4×8	0.496	0.309
protograph in (33)	0.5	16×32	0.3	0.113
WIMAX	0.67	8×24	2.799	0.491
DVB-S2	0.67	15×45	2.749	0.441
AR4JA	0.67	2×6	1.338	0.279
protograph in (31)	0.67	4×12	2.429	0.121
DVB-S2	0.73	12×45	3.62	0.498
WIMAX	0.75	6×24	3.83	0.443
protograph in (32)	0.75	3×12	3.551	0.164

TABLE II: Comparison of protograph thresholds for BIAWGN channel.

Note that the optimized protographs presented in this section satisfy the conditions of Theorem 1 and have block-error threshold same as the bit-error threshold. Also, the block-error rate falls inverse-polynomially (or better) in blocklength under the large-girth construction as described in Sections III-A2 and IV. Moreover, even for small sizes of protographs such as 4×8 , 8×16 or 16×32 , the optimization results in thresholds that are near capacity.

VI. SIMULATION RESULTS

We now present simulation results that confirm the predicted threshold behavior for both the BEC and the BIAWGN channel.

A. BEC

Protographs in V-B can be lifted using D(m,q) graphs (*m*: positive integer, *q*: prime power) from Section IV since they have a prime number of edges. The parameters of the constructed LDPC codes are as given in Table III.

Code type	Rate	\mathbf{m}, \mathbf{q}	Blocklength
16×32 protograph in (30)	0.5	2, 173	957728
4×12 protograph in (27)	0.66	2, 61	44652
3×12 protograph in (26)	0.75	2, 61	44652

TABLE III: Parameters of protograph LDPC codes used for simulation over BEC.

The standard message passing decoder is simulated over the BEC, and the bit and block-error rate curves are shown in Fig. 4. Error rates of AR4JA, DVB-S2 and WIMAX codes of the same rate are shown for comparison. These codes were lifted

to blocklengths comparable to those in Table III of the same rate. The rate-1/2 code was lifted to a length of 90000, while the rate-2/3 and rate-3/4 codes were lifted to a length of around 45000. As seen from the figure, the optimized protographs perform better than other comparable codes.

B. BIAWGN channel

The protographs in in Table II are lifted using D(m,q) graphs to obtained protograph LDPC codes with parameters given in Table IV. The standard message-passing decoder is

Code type	Rate	\mathbf{m}, \mathbf{q}	Blocklength
16×32 protograph in (33)	0.5	2, 173	957728
4×12 protograph in (31)	0.66	2, 67	53868
3×12 protograph in (32)	0.75	2, 71	60492

TABLE IV: Parameters of protograph LDPC codes used for simulation over BIAWGN channel.

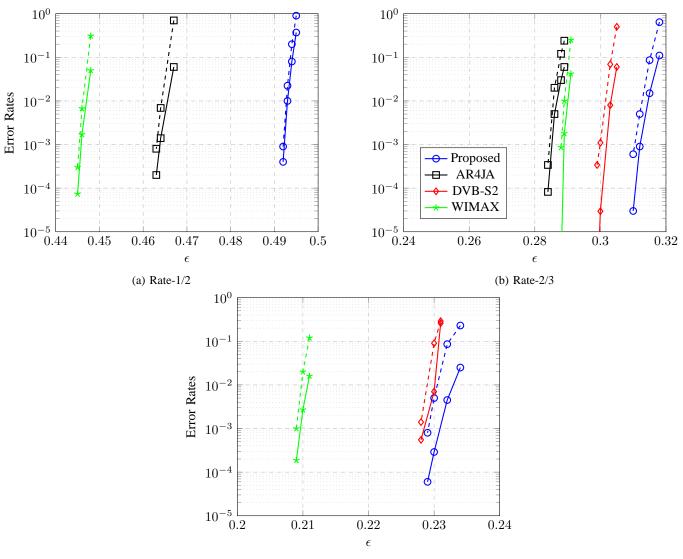
simulated over the BIAWGN channel The bit and block-error rates are compared with codes such as AR4JA and those from

DVB-S2 and WIMAX standard in Fig. 5. As seen from the figure, the optimized protographs perform better than other comparable codes. Further, we note that the waterfall region for both block-error and bit-error rate curves are the same for both the BEC and BIAWGN channels.

VII. CONCLUSION

In this work, we studied protograph density evolution and derived conditions under which the bit and block-error thresholds coincide. Using large-girth graphs, we presented a deterministic construction for a sequence of LDPC codes with block-error rate falling faster than any inverse polynomial in blocklength. We described methods to optimize protographs and presented small-sized protographs with thresholds close to capacity.

As part of future work, characterizing the gap to capacity of finite-length protographs theoretically appears to be an interesting problem for study, particularly because the thresholds can be close to capacity.



(c) Rate-3/4

Fig. 4: Error rates over a BEC. Solid: bit-error rate, Dashed: Block-error rate.

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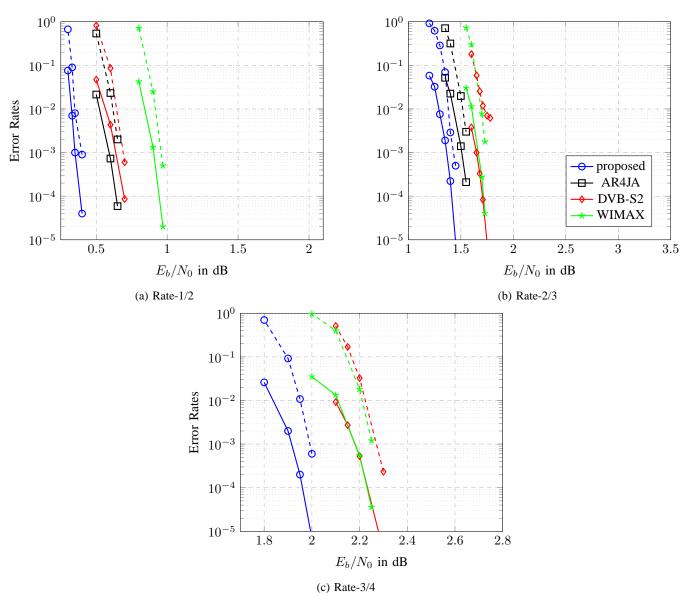


Fig. 5: Error rates over a BIAWGN channel. Solid: bit-error rate, Dashed: Block-error rate.

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