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Characterization of Q-property for multiplicative transformations in semidefinite linear complementarity problems

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ABSTRACT

We characterize the Q-property of a multiplicative transformation in semidefinite linear complementarity problems.

1. Introduction

Let $V := \mathcal{S}^{n\times n}$ be the vector space of real symmetric matrices of order n and Σ be the set of all positive semidefinite matrices in V. If $X \in \Sigma$, we will use the notation $X \succeq 0$. Suppose that $L : V \to V$ is a linear transformation. Given an element $Q \in V$, the semidefinite linear complementarity problem SDLCP(L, Q) is to find a matrix $X \in V$ such that

$$X \succeq 0, \quad Y := L(X) + Q \succeq 0 \quad \text{and} \quad XY = 0.$$

SDLCP is a mathematical programming problem introduced in [3]. It has several applications in matrix theory and optimization. We refer to [3] for details. SDLCP is a special case of variational inequality problems (VIPs). A wide literature of VIPs appears in [2]. Focussing specifically to SDLCP has many advantages. In this particular setting, many specialized results can be proved using the extra structure available for matrices. Thus, SDLCP is an useful tool in understanding variational inequality problems.

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Let A be a square matrix of order n. Then the multiplicative transformation $M_A : V \to V$ is defined by $M_A(X) := AXA^T$. It is known from [5] that invertible multiplicative transformations are the only linear transformations on V that satisfy $L(\Sigma) = \Sigma$. The transformation M_A is said to have the Q-property if SDLCP(M_A, Q) has a solution for all $Q \in V$. One of the unsolved problems in SDLCP is to prove the Q-property of M_A. Towards, this we prove the following result:

Theorem 1. *Let $A \in R^{n\times n}$. Then the following are equivalent*:

1. *$A + A^T$ is either positive definite or negative definite*.
2. *For all $Q \in V$*, SDLCP(M_A, Q) *has a unique solution*.
3. SDLCP$(M_A, 0)$ *has a unique solution*.
4. *M_A has the Q-property*.

The proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) in the above theorem is proved in [4]. If A is of order 2, then (4) $\Rightarrow$ (1) is proved in [4]. Our aim in this paper is to establish (4) $\Rightarrow$ (1) for any square matrix $A \in R^{n\times n}$.

2. Preliminaries

We make the following assumption throughout this paper:

$n \geqslant 3$.

The following notations are used in this paper:

- Let $\alpha \subseteq \{1,\ldots,n\}$ and $\beta \subseteq \{1,\ldots,n\}$. Then for a matrix $M \in R^{n\times n}$, $M\langle\alpha,\beta\rangle$ will be the submatrix of M obtained by deleting rows indexed by α and columns indexed by β.
- Let $X \succeq 0, \alpha = \{1, n\}$. Then $X' := X\langle\alpha,\alpha\rangle$. For example, if
$$X = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix},$$
then $X' = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$.
- Set of all solutions to SDLCP(M_A, Q) will be denoted by SOL(M_A, Q).
- Let I_k denote the identity matrix of order k.
- We will use $\widetilde{Q}$ to denote the $n \times n$ matrix
$$\widetilde{Q} := \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We now introduce some definitions.

Definition 1. For a matrix $M \in R^{n\times n}$ with entries m_{ij}, we define the following:

- Let $\alpha = \{2,\ldots,n-2\}$. The corner of M is the principal submatrix $M\langle\alpha,\alpha\rangle$. We denote the corner of M by $\mathrm{cor}(M)$.
- The entry m_{ij} is called a corner entry of M if m_{ij} is an entry in $\mathrm{cor}(M)$. Otherwise we say that m_{ij} is a non-corner entry.
- M is called a corner matrix if all the non-corner entries of M are zero and $\mathrm{cor}(M)$ is a nonzero matrix.
- If M is the sum of identity matrix and a skew-symmetric matrix, then we say that M is a *type*$(*)$ matrix.

- Let $n_1 > 0$ be any positive integer. Then M is called $Form(n_1)$ matrix if M can be partitioned such that

$$M = \begin{bmatrix} W & Q \\ -Q^T & R \end{bmatrix},$$

where W is a skew-symmetric matrix of order m and R is a $type(*)$ matrix of order n_1. Here we assume $m + n_1 = n$ and $m > 0$.

- Let n_1 and n_2 be positive integers such that $n_1 + n_2 = n$. Then M is called $Form(n_1, n_2)$ matrix if M can be partitioned such that

$$M = \begin{bmatrix} P & Q \\ -Q^T & -R \end{bmatrix},$$

where P and R are $type(*)$ matrices of order n_1 and n_2 respectively.

- Let n_1, n_2 and n_3 be positive integers such that $n_1 + n_2 + n_3 = n$. Then M is called $Form(n_1, n_2, n_3)$ matrix if M has the partitioned form

$$M = \begin{bmatrix} P & E & S \\ -E^T & W & Q \\ -S^T & -Q^T & -R \end{bmatrix},$$

where W is a skew-symmetric matrix of order n_3, P and R are $type(*)$ matrices of order n_1 and n_2 respectively.

- Let $N \in R^{n \times n}$. Then we write $M \sim N$ if and only if there exists a nonsingular matrix P such that $PMP^T = N$.

3. Result

To prove the main result we proceed as follows: Using the Q-property of M_A, we first show that there exists a corner matrix which solves SDLCP($M_A, A\widetilde{Q}A^T$). This lemma is then used to show that if A is either $Form(n_1)$ or $Form(n_1, n_2)$ or $Form(n_1, n_2, n_3)$, then M_A cannot have the Q-property. This will finally imply that A should be either positive definite or negative definite.

We begin with the following lemma.

Lemma 1. *Let $B \succeq 0$. Suppose that P is a $k \times k$ principal submatrix of B. Let $\mathbf{r}_1, \ldots, \mathbf{r}_k$ be the rows of B which contain P. Then $\det P = 0$ if and only if $\mathbf{r}_1, \ldots, \mathbf{r}_k$ are linearly dependent vectors.*

In particular, $\text{rank}(P)$ is the number of linearly independent vectors in $\mathbf{r}_1, \ldots, \mathbf{r}_k$.

Proof. Without loss of generality, assume that P is a leading principal submatrix of B. Let B have the partitioned form

$$B = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}.$$

Observe that Q is of order $k \times (n-k)$. Now, it suffices to prove that $\text{rank}([P\ Q]) = \text{rank}(P)$.

Let $\mathbf{x} = (x_1, \ldots, x_k)^T \in R^k$ be a nonzero vector such that $P\mathbf{x} = 0$. Define $\mathbf{v} \in R^n$ by

$$\mathbf{v} := \begin{bmatrix} \mathbf{x} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It can be verified that $\mathbf{v}^T B\mathbf{v} = \mathbf{x}^T P\mathbf{x}$. Then, $\mathbf{v}^T B\mathbf{v} = \mathbf{x}^T P\mathbf{x} = 0$. Since B is symmetric as well as positive semidefinite, $B\mathbf{v} = 0$ and hence $Q^T\mathbf{x} = 0$. This together with $Px = 0$ implies that

$$\sum_{i=1}^{k} x_i \mathbf{r}_i = 0.$$

Thus the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_k$ are linearly dependent. The converse as well as the rank equality are easily seen. □

Lemma 2. *Let $A \in R^{n\times n}$. Then the following statements are true:*

(i) *If the transformation M_A has the Q-property, then A is nonsingular and M_{PAP^T} will have the Q-property for all P nonsingular.*
(ii) *Let M_A have the Q-property and $X \in \mathrm{SOL}(M_A, A\widetilde{Q}A^T)$. Then X and $X + \widetilde{Q}$ are nonzero positive semidefinite matrices. Further, $\mathrm{rank}(X) < n-1$ or $\mathrm{rank}(X+\widetilde{Q}) < n-1$.*

Proof. We now prove (i). Let $X \in \mathrm{SOL}(M_A, -I)$, where I is the identity matrix. Then $AXA^T - I \succeq 0$. This implies that AXA^T is a positive definite matrix and therefore A is nonsingular. Now, let P be nonsingular and $U := P^{-1}$. Then the following equivalence can be verified for any symmetric matrix Q of order n:

$$X \in \mathrm{SOL}(M_A, Q) \Leftrightarrow UXU^T \in \mathrm{SOL}(M_{P^TAP}, P^TQP).$$

Therefore, M_A has the Q-property if and only if M_{PAP^T} has the Q-property.

We now prove (ii). Since $X \in \mathrm{SOL}(M_A, A\widetilde{Q}A^T)$, we have

$$X \succeq 0, \quad \widetilde{Y} := AXA^T + A\widetilde{Q}A^T \succeq 0 \quad \text{and} \quad X\widetilde{Y} = 0. \tag{1}$$

Since M_A has the Q-property, by (i) A must be nonsingular. Let $B := A^{-1}$. Then $B\widetilde{Y}B^T \succeq 0$. This means that $X + \widetilde{Q} \succeq 0$. From (1), we see that

$$X \succeq 0, \quad Y := X + \widetilde{Q} \succeq 0 \quad \text{and} \quad XAY = 0.$$

Since $\widetilde{Q}$ is an indefinite matrix, from the conditions $X \succeq 0$ and $Y \succeq 0$, we see that X and Y are nonzero. If $\mathrm{rank}(X) = n$ or $\mathrm{rank}(Y) = n$, then $XAY = 0$ implies that $Y = 0$ or $X = 0$ which is not true. So, $\mathrm{rank}(X) < n$ and $\mathrm{rank}(Y) < n$.

If possible, suppose $\mathrm{rank}(X) = n-1$ and $\mathrm{rank}(Y) = n-1$. As A is nonsingular, $\mathrm{rank}(XA) = \mathrm{rank}(X) = n-1$. Now, by Frobenius inequality,

$$2(n-1) = \mathrm{rank}(XA) + \mathrm{rank}(Y) \leqslant \mathrm{rank}(XAY) + n = n,$$

which does not hold as $n \geqslant 3$. Therefore either $\mathrm{rank}(X) < n-1$ or $\mathrm{rank}(Y) < n-1$. This completes the proof. □

Lemma 3. *Let the transformation M_A have the Q-property. If $X \in \mathrm{SOL}(M_A, A\widetilde{Q}A^T)$ and $\mathrm{rank}(X') = k$, then*

(1) $\mathrm{rank}(X) > k$,
(2) $\mathrm{rank}(X + \widetilde{Q}) > k$,
(3) $\det X' = 0$.

Proof. We prove (1). By (ii) in Lemma 2, it follows that

$$X \succeq 0, \quad Y := X + \widetilde{Q} \succeq 0, \quad X \neq 0, \ Y \neq 0.$$

As $\mathrm{rank}(X') \leqslant \mathrm{rank}(X)$, suppose if possible, $\mathrm{rank}(X') = \mathrm{rank}(X)$. Since X is nonzero, it suffices to assume that $k > 0$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be the rows of X and x_{ij} be the (i,j)-entry of X.

Since $\mathrm{rank}(X') = k$, X' has k linearly independent row vectors. Without any loss of generality, assume that $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}$ are linearly independent. Then by Lemma 1, the leading principal submatrix of X' with order k must be nonsingular. This means that the matrix

$$G := \begin{bmatrix} x_{22} & x_{23} & \cdots & x_{2k+1} \\ x_{32} & x_{33} & \cdots & x_{3k+1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{k+12} & x_{k+13} & \cdots & x_{k+1k+1} \end{bmatrix}$$

is nonsingular.

Let H be the $(k+1) \times (k+1)$ leading principal submatrix of X. As we have $k = \text{rank}(X)$, $\det H = 0$ and the vectors in the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ must be linearly dependent. Observe that H is also the leading $(k+1) \times (k+1)$ principal submatrix of Y. Suppose that $\mathbf{e}_n \in R^n$ is the vector $\mathbf{e}_n := (0, \ldots, 0, 1)^T$. Now $\mathbf{u}_1 + \mathbf{e}_n, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}$ are the rows of Y which contain H and $\det H = 0$. Further $Y \succeq 0$. Using Lemma 1, we now deduce that $\mathbf{u}_1 + \mathbf{e}_n, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}$ must be linearly dependent.

Let L be the rectangular matrix whose rows are $\mathbf{u}_1 + \mathbf{e}_n, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}$. Then $\text{rank}(L) < k+1$. Now we define

$$\widetilde{L} := \begin{bmatrix} x_{12} & x_{13} & \cdots & x_{1k+1} & x_{1n}+1 \\ x_{22} & x_{23} & \cdots & x_{2k+1} & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{k+12} & x_{k+13} & \cdots & x_{k+1k+1} & x_{k+1n} \end{bmatrix}.$$

It can be verified that $\widetilde{L}$ is a $(k+1) \times (k+1)$ submatrix of L and $\widetilde{L}\langle\{1\}, \{n\}\rangle = G$. If $\det \widetilde{L} \neq 0$, then $\text{rank}(L) \geqslant k+1$ which will be a contradiction. Thus $\det \widetilde{L} = 0$.

Also,

$$\widehat{L} := \begin{bmatrix} x_{12} & x_{13} & \cdots & x_{1k+1} & x_{1n} \\ x_{22} & x_{23} & \cdots & x_{2k+1} & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{k+12} & x_{k+13} & \cdots & x_{k+1k+1} & x_{k+1n} \end{bmatrix}$$

must be singular, as $\text{rank}(X) = k$. Now, it follows that

$$0 = \det \widetilde{L} = \det \widehat{L} + \det \widetilde{L}\langle\{1\}, \{n\}\rangle = \det G.$$

This contradicts that G is nonsingular. This completes the proof of (1).

By repeating the same argument as above, we get (2).

We now prove (3). Suppose $\det X' \neq 0$. This implies $\text{rank}(X') = n-2$. Now, by (1) and (2), we have $\text{rank}(X) > n-2$ and $\text{rank}(Y) > n-2$, which is a contradiction to item (ii) in Lemma 2. Hence the proof. □

Lemma 4. *Let* $P \succeq 0$, $\det P' = 0$ *and* $\text{rank}(P') < \text{rank}(P)$. *Then there is a corner matrix T such that $P = S + T$, where $S \succeq 0$ and $T \succeq 0$. Further S has the following properties*:

(a) *Non-corner entries of S and P are equal*.
(b) $\text{rank}(S) = \text{rank}(P')$.

Proof. Let U be a permutation matrix such that P' is the $(n-2) \times (n-2)$ leading principal submatrix of UPU^T. Define $Y := UPU^T$. Let Y have the partitioned form

$$Y = \begin{bmatrix} P' & B \\ B^T & C \end{bmatrix}.$$

To prove the result, we will show that

$$Y = \begin{bmatrix} P' & B \\ B^T & N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix},$$

where

$$\text{rank}\left(\begin{bmatrix} P' & B \\ B^T & N \end{bmatrix}\right) = \text{rank}(P'), \quad \begin{bmatrix} P' & B \\ B^T & N \end{bmatrix} \succeq 0 \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix} \succeq 0, \quad \text{and} \quad L \neq 0.$$

Put $k := \text{rank}(P')$. Since $\det P' = 0$, $k < n-2$. Since $P' \succeq 0$, P' is the sum of k rank one positive semidefinite matrices. Let

$$P' = \sum_{\nu=1}^{k} [x_i^\nu x_j^\nu], \quad i = 1, \ldots, n-2 \text{ and } j = 1, \ldots, n-2.$$

In view of Lemma 1, $\text{rank}([P'\ B]) = k$. Therefore

$$[P'\ B] = \sum_{\nu=1}^{k}[x_i^{\nu}x_j^{\nu}] \quad i = 1, \ldots, n-2 \text{ and } j = 1, 2, \ldots, n.$$

Let

$$\widetilde{S} := \sum_{\nu=1}^{k}[x_i^{\nu}x_j^{\nu}], \quad i = 1, \ldots, n \text{ and } j = 1, \ldots, n.$$

Then $\widetilde{S} \succeq 0$ and $\text{rank}(\widetilde{S}) = k$. As $\widetilde{S} \succeq 0$, Lemma 1 implies that at least one $k \times k$ principal submatrix of $\widetilde{S}$ must be nonsingular. Without any loss of generality, we assume that the $k \times k$ leading principal submatrix of $\widetilde{S}$ is nonsingular. Suppose the $k \times k$ leading principal submatrix of $\widetilde{S}$ is denoted by $\widehat{S}$. Then $\det \widehat{S} > 0$. It can be noted that $\widetilde{S}$ has the partitioned form

$$\widetilde{S} = \begin{bmatrix} P' & B \\ B^T & N \end{bmatrix}.$$

Define

$$\widetilde{T} := Y - \widetilde{S}.$$

Suppose $\widetilde{T} = 0$. Then $\text{rank}(Y) = \text{rank}(\widetilde{S})$. This means that $\text{rank}(Y) = k$ and hence $\text{rank}(P) = k$ which is a contradiction to our assumption $\text{rank}(P) > \text{rank}(P')$. Therefore $\widetilde{T}$ is nonzero. Apparently, $\widetilde{T}$ has the partitioned form

$$\widetilde{T} = \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix},$$

where $L = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

It remains to show that $\widetilde{T} \succeq 0$. We claim $a \geqslant 0$, $c \geqslant 0$ and $\det L \geqslant 0$. Let $E = [e_{ij}]$ be the $(k+1) \times (k+1)$ matrix defined by

$$e_{ij} = \begin{cases} 1 & (i,j) = (k+1, k+1), \\ 0 & \text{else}. \end{cases}$$

Let

$$V := \widetilde{S}\langle \alpha, \alpha \rangle, \quad \alpha = \{k+1, \ldots, n-2, n\}.$$

Then $V + aE$ is a principal submatrix of Y. Put $\beta = \{k+1\}$. Then $V\langle \beta, \beta \rangle = \widehat{S}$. Since $Y \succeq 0$, $\det(V + aE) \geqslant 0$. As $\text{rank}(\widetilde{S}) = k$, $\det V = 0$.

Now we have

$$\det(V + aE) = \det V + a \det V\langle \beta, \beta \rangle = a \det \widehat{S} \geqslant 0.$$

Since $\det \widehat{S} > 0$, $a \geqslant 0$. Similarly it can be proved that $c \geqslant 0$.

Let G be the $(k+2) \times (k+2)$ principal submatrix of $\widetilde{S}$ defined by

$$G = \widetilde{S}\langle \alpha, \alpha \rangle, \quad \alpha = \{k+1, \ldots, n-2\}.$$

Suppose that F is the $(k+2) \times (k+2)$ matrix defined by

$$F := \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix}.$$

Now $G + F$ is a principal submatrix of Y and therefore $\det(G + F) \geqslant 0$. By an easy calculation we find that

$$\det(G + F) = \det \widehat{S} \det L,$$

and so $\det L \geqslant 0$. Thus $\widetilde{T} \succeq 0$. This completes the proof. $\square$

Lemma 5. *Let $R \succeq 0$, $S \succeq 0$,* $\text{rank}(R) = \text{rank}(S)$ *and* $\text{rank}(R') = \text{rank}(R)$. *Assume that the non-corner entries of R and S are same. Then $R = S$.*

Proof. Let $R := [r_{ij}]$, $S := [s_{ij}]$ and $k := \text{rank}(R)$. We need to prove that $r_{11} = s_{11}$, $r_{nn} = s_{nn}$ and $r_{1n} = s_{1n}$.

Since $R' \succeq 0$, by Lemma 1, at least one $k \times k$ principal submatrix of R' is nonsingular. Without any loss of generality, let us assume that the leading $k \times k$ principal submatrix of R', say F, is nonsingular. Let $E_{11} := [e_{ij}]$ be the $(k+1) \times (k+1)$ matrix defined as follows:

$$e_{ij} = \begin{cases} 1 & (i,j) = (1,1), \\ 0 & \text{else.} \end{cases}$$

Now the $(k+1) \times (k+1)$ leading principal submatrix of S can be written as

$$V := [s_{ij}] = [r_{ij}] + \alpha E_{11}, \quad i,j = 1, \ldots, k+1.$$

Set

$$X := [r_{ij}], \quad i,j = 1, \ldots, k+1.$$

Let the columns of X be $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}$ and $\mathbf{f} := (\alpha, 0, \ldots, 0)^T$. It can be noted that $\det X = \det V = 0$ and therefore we have

$$\begin{aligned} 0 = \det V &= \det[\mathbf{u}_1 + \mathbf{f}, \mathbf{u}_2, \ldots, \mathbf{u}_{k+1}] \\ &= \det[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k+1}] + \det[\mathbf{f}, \mathbf{u}_2, \ldots, \mathbf{u}_{k+1}] \\ &= \det[\mathbf{f}, \mathbf{u}_2, \ldots, \mathbf{u}_{k+1}] \\ &= \alpha \det F. \end{aligned}$$

Since $\det F > 0$, $\alpha = 0$. Thus, $s_{11} = r_{11}$. By a similar argument it can be proved that $s_{nn} = r_{nn}$ and $s_{1n} = r_{1n}$. $\square$

Lemma 6. *Assume that M_A has the Q-property. Then there exists $T \in \text{SOL}(M_A, A\widetilde{Q}A^T)$ such that T is a corner matrix.*

Proof. Let $X \in \text{SOL}(M_A, A\widetilde{Q}A^T)$. Then $X \succeq 0$ and $Y := X + \widetilde{Q} \succeq 0$. From Lemmas 3 and 4,

$$X = S + T \quad \text{and} \quad Y = R + T_1, \tag{2}$$

where S, R, T and T_1 satisfy all the properties stated in Lemma 4. In particular T and T_1 are corner.

Since $Y' = X'$, it follows from (b) of Lemma 4 that $\text{rank}(R) = \text{rank}(S)$. Put $k := \text{rank}(S)$. Now the non-corner entries of R and S are same. Thus R and S satisfy all the conditions of Lemma 5. Hence $R = S$. Equations in (2) thus imply $Y = X + \widetilde{Q} = S + T + \widetilde{Q} = R + T + \widetilde{Q} = R + T_1$ and therefore $T + \widetilde{Q} = T_1$. Hence $T + \widetilde{Q} \succeq 0$. As $X \in \text{SOL}(M_A, A\widetilde{Q}A^T)$, we have

$$X(AXA^T + A\widetilde{Q}A^T) = (S + T)(AXA^T + A\widetilde{Q}A^T) = 0. \tag{3}$$

Setting $P = AXA^T + A\widetilde{Q}A^T$, we have

$$(S + T)P = 0. \tag{4}$$

Since $P \succeq 0$, $S \succeq 0$ and $T \succeq 0$, $\text{trace}(SP) \geqslant 0$ and $\text{trace}(TP) \geqslant 0$. Taking trace on both the sides in (4), we obtain

$$\text{trace}(TP) = 0 \quad \text{and} \quad \text{trace}(SP) = 0.$$

Therefore $TP = 0$ and $SP = 0$. Thus we see that

$$T(AXA^T + A\widetilde{Q}A^T) = 0. \tag{5}$$

Put $X = S + T$ in (5). Now $A(T + \widetilde{Q})A^T \succeq 0$ and $S \succeq 0$. Using a similar argument as above, it follows that

$$T(ATA^T + A\widetilde{Q}A^T) = 0. \tag{6}$$

Thus, the corner matrix T solves SDLCP$(M_A, A\widetilde{Q}A^T)$. This completes the proof. □

The proof of the following lemma is a direct verification and hence omitted.

Lemma 7. *Suppose that* $X \in \mathrm{SOL}(M_A, A\widetilde{Q}A^T)$. *If* X *is a corner matrix, then* $\mathrm{cor}(X) \in \mathrm{SOL}(M_{\mathrm{cor}(A)}, \mathrm{cor}(A)\,\mathrm{cor}(\widetilde{Q})\mathrm{cor}(A)^T)$.

Lemma 8. *If* A *is a Form*(n_1, n_2) *matrix or Form*(n_1, n_2, n_3) *matrix, then* M_A *does not have the* Q*-property*.

Proof. Suppose that A is a *Form*(n_1, n_2) matrix. Then A has the partitioned form

$$A = \begin{bmatrix} B & C \\ -C^T & -D \end{bmatrix},$$

where B and D are *type*$(*)$ matrices of order n_1 and n_2 respectively. Let $\mathbf{c}$ be the last column of C.

As $n \geqslant 3$, it follows that either $n_1 > 1$ or $n_2 > 1$. Without any loss of generality, assume $n_1 > 1$. As $\mathbf{c} \in R^{n_1}$ and $n_1 > 1$, there exists a unit vector $\mathbf{u}$ orthogonal to $\mathbf{c}$. Now construct an orthogonal matrix U of order n_1 whose first row is $\mathbf{u}^T$.

Define

$$V := \begin{bmatrix} U & 0 \\ 0 & I_{n_2} \end{bmatrix}.$$

Then V is orthogonal and

$$K := VAV^T = \begin{bmatrix} UBU^T & UC \\ -C^TU^T & -D \end{bmatrix}.$$

Since B is a *type*$(*)$ matrix, so is UBU^T. Thus, $\mathrm{cor}(K) = \mathrm{diag}[1, -1]$.

If M_A has the Q-property, then by item (i) in Lemma 2, M_K will have the Q-property. By Lemma 6, there exists $X \in \mathrm{SOL}(M_K, K\widetilde{Q}K^T)$ such that X is corner. Setting $S := \mathrm{diag}[1, -1]$ it follows from Lemma 7, that

$$\mathrm{cor}(X) \in \mathrm{SOL}(M_S, S\,\mathrm{cor}(\widetilde{Q})S).$$

This contradicts Lemma 11 (see Appendix). Thus, M_A does not have the Q-property.

If A is a *Form*(n_1, n_2, n_3) matrix, a similar argument can be repeated. □

Lemma 9. *If* A *is a Form*(n_1) *matrix or a skew-symmetric matrix, then* M_A *does not have the* Q*-property*.

Proof. If A is skew-symmetric (or more generally, normal), the result will follow from Lemma 2.15 in [1].

Assume that A is a *Form*(n_1) matrix. Let M_A have the Q-property. Suppose A has the partitioned form

$$A = \begin{bmatrix} W & G \\ -G^T & D \end{bmatrix},$$

where D is a *type*$(*)$ matrix of order n_1 and W is skew-symmetric of order m. It can be verified that A is normal if and only if $G = 0$ and hence to prove the lemma we can assume that $G \neq 0$. Then there exists a permutation matrix

$$U = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},$$

where P_1 and P_2 are permutation matrices of order m and n_1 respectively such that $B := UAU^T$ is a *Form*(n_1) matrix and $\det(\mathrm{cor}(B)) \neq 0$. Without any loss of generality we can assume that

$$\operatorname{cor}(B) = \begin{bmatrix} 0 & -b \\ b & 1 \end{bmatrix}, \quad b > 0.$$

By Lemma 2, M_B will have the Q-property. By Lemma 6, there is a corner matrix which is a solution to SDLCP($M_B, B\widetilde{Q}B^T$). Hence from Lemma 7, there is a solution to SDLCP($M_{\operatorname{cor}(B)}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) which is a contradiction to Lemma 11 (see Appendix). This completes the proof. □

The next lemma is a consequence of the following well known theorem for symmetric matrices.

Theorem 2 (Sylvester's inertia theorem)**.** *Let Q and R be symmetric matrices of order n with ν_1 zero eigenvalues, ν_2 positive eigenvalues and ν_3 negative eigenvalues. Then there is a nonsingular matrix P such that $PQP^T = R$.*

Lemma 10. *Let $A \in R^{n\times n}$. Assume that A is neither positive definite nor negative definite. Then we have the following.*

1. *If $A + A^T$ is a nonsingular matrix, then there is a Form (n_1, n_2) matrix B such that $A \sim B$.*
2. *Suppose $A + A^T$ is singular and nonzero. Then either there is a Form(n_1, n_2, n_3) matrix B such that $A \sim B$ or there is a Form(n_1) matrix C such that $A \sim \pm C$.*

Proof. Define $\widetilde{A} := A + A^T$. If $\widetilde{A}$ is nonsingular, then $\widetilde{A}$ will have n_1 positive eigenvalues and n_2 negative eigenvalues. Now by Theorem 2, there exists a nonsingular matrix P such that

$$P\widetilde{A}P^T = \begin{bmatrix} 2I_{n_1} & 0 \\ 0 & -2I_{n_2} \end{bmatrix}.$$

Put $B := PAP^T$. We then see that $A \sim B$, where B is a *Form*(n_1, n_2) matrix.

Let $\widetilde{A}$ be singular and nonzero. Now at least one of the eigenvalues of $\widetilde{A}$ must be zero. Suppose that $\widetilde{A}$ has n_1 positive eigenvalues and n_2 negative eigenvalues. Then by the above theorem, there exists a nonsingular matrix P such that

$$P\widetilde{A}P^T = \begin{bmatrix} 2I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2I_{n_2} \end{bmatrix}.$$

Therefore PAP^T must be a *Form*(n_1, n_2, n_3) matrix.

Suppose that $\widetilde{A}$ is singular, nonzero and has n_1 positive eigenvalues. Now we can find a nonsingular matrix P such that

$$P\widetilde{A}P^T = \begin{bmatrix} 0 & 0 \\ 0 & 2I_{n_1} \end{bmatrix}.$$

This implies that PAP^T must be a *Form*(n_1) matrix.

If $\widetilde{A}$ is singular, nonzero and has n_1 negative eigenvalues then $-A \sim B$, where B is a *Form*(n_1) matrix. Thus, $A \sim -B$. This completes the proof. □

As a consequence of the above lemmas, we have the following theorem.

Theorem 3. *Let $A \in R^{n\times n}$. Then the following are equivalent.*

1. *$A + A^T$ is either positive definite or negative definite.*
2. *If Q is a symmetric matrix, then* SDLCP(M_A, Q) *has a solution.*

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Appendix

We now prove a result which is used in Lemmas 8 and 9. As we have assumed that $n \geqslant 3$ throughout the paper, we present this result here.

Lemma 11. *Let* $Q := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. *Let S denote either* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & -b \\ b & 1 \end{bmatrix}$, *where* $b > 0$. *Then* $\mathrm{SDLCP}(M_S, SQS)$ *has no solution*.

Proof. In both cases, S is nonsingular and SQS is indefinite. Let $X := \begin{bmatrix} d & e \\ e & r \end{bmatrix}$ be a solution to $\mathrm{SDLCP}(M_S, SQS)$. Then

$$X \succeq 0, \quad Y := SXS + SQS \succeq 0 \quad \text{and} \quad XY = 0.$$

Since S is nonsingular, the condition $X(SXS + SQS) = 0$ implies $XS(X + Q) = 0$. Suppose $X = 0$. Then the condition $Y \succeq 0$ will mean that $SQS \succeq 0$ which is a contradiction as $\det(SQS) < 0$. So, $X \neq 0$. Suppose $Y = 0$. Then, $Y \succeq 0$ implies that $-SQS = SXS$. Since $X \succeq 0$, $SXS \succeq 0$ and therefore, $-SQS \succeq 0$ which is again a contradiction. Hence X and Y are nonzero. Suppose that $\mathrm{rank}(X) = 2$. Then from the condition $XY = 0$, we see that $Y = 0$. This is not possible. So, $\mathrm{rank}(X) = 1$. Similarly, $\mathrm{rank}(Y) = 1$. Now $\mathrm{rank}(S^{-1}YS^{-1}) = 1$ and therefore, $\mathrm{rank}(X + Q) = 1$. Hence $\det X = 0$ and $\det(X + Q) = 0$. Using these equations, we obtain $e = -\frac{1}{2}$. Now putting this in $XS(X + Q) = 0$, and noting $d \geqslant 0$ and $r \geqslant 0$, we get a contradiction in both the instances of S. This completes the proof. □

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