BRAUER GROUP OF A MODULI SPACE OF PARABOLIC VECTOR BUNDLES OVER A CURVE

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Abstract. Let $\mathcal{PM}^\alpha_s$ be a moduli space of stable parabolic vector bundles of rank $n \geq 2$ and fixed determinant of degree d over a compact connected Riemann surface X of genus $g(X) \geq 2$. If $g(X) = 2$, then we assume that $n > 2$. Let m denote the greatest common divisor of d, n and the dimensions of all the successive quotients of the quasi–parabolic filtrations. We prove that the Brauer group $\mathrm{Br}(\mathcal{PM}^\alpha_s)$ is isomorphic to the cyclic group $\mathbb{Z}/m\mathbb{Z}$. We also show that $\mathrm{Br}(\mathcal{PM}^\alpha_s)$ is generated by the Brauer class of the Brauer–Severi variety over $\mathcal{PM}^\alpha_s$ obtained by restricting the universal projective bundle over $X \times \mathcal{PM}^\alpha_s$.

1. Introduction

Let Y be a smooth quasi–projective variety over $\mathbb{C}$. The cohomological Brauer group of Y is defined to be $H^2(Y_{\acute{e}tale},\, \mathbb{G}_m)_{\text{torsion}}$, and it is denoted by $\mathrm{Br}'(Y)$. It is known that the group $H^2(Y_{\acute{e}tale},\, \mathbb{G}_m)$ is torsion. The Brauer group of Y, which is denoted by $\mathrm{Br}(Y)$, is defined to be the Morita equivalence classes of Azumaya algebras over Y. Giving an Azumaya algebra over Y is equivalent to giving a Brauer–Severi variety over Y which is also equivalent to giving a principal $\mathrm{PGL}_{\mathbb{C}}$–bundle over Y. Given a principal $\mathrm{PGL}_{\mathbb{C}}(r)$–bundle $E_{\mathrm{PGL}_{\mathbb{C}}(r)} \longrightarrow Y$, using the cohomology exact sequence for the short exact sequence

$$e \longrightarrow \mu_r \longrightarrow \mathrm{SL}_{\mathbb{C}}(r) \longrightarrow \mathrm{PGL}_{\mathbb{C}}(r) \longrightarrow e$$

we get an element of $\mathrm{Br}'(Y)$. The resulting homomorphism $\mathrm{Br}(Y) \longrightarrow \mathrm{Br}'(Y)$ is injective. A theorem due to Gabber says that this homomorphism is surjective. Therefore, the three groups, namely $\mathrm{Br}(Y)$, $\mathrm{Br}'(Y)$, and $H^2(Y_{\acute{e}tale},\, \mathbb{G}_m)$, coincide.

A Brauer–Severi variety over Y is the projectivization of a vector bundle over Y if and only if its class in $\mathrm{Br}(Y)$ vanishes. If U a Zariski open subset of Y such that the codimension of the complement $Y \setminus U$ is at least two, then the natural restriction homomorphism $\mathrm{Br}(Y) \longrightarrow \mathrm{Br}(U)$ is an isomorphism. (See [13], [7], [8], [9] for the above mentioned properties and more.)

Let X be an irreducible smooth complex projective curve of genus $g(X)$, with $g(X) \geq 2$. Fix an integer n, with $n \geq 2$. If $g(X) = 2$, then we assume that $n \geq 3$. Fix distinct points

$$\{p_1\,, \cdots, p_\ell\} \subset X \tag{1.1}$$

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with $\ell \geq 1$. For each $i \in [1, \ell]$, fix positive integers $\{r_{i,1}, \cdots, r_{i,a_i}\}$ such that

$$\sum_{j=1}^{a_i} r_{i,j} = n\,. \tag{1.2}$$

To each pair (i, j), $i \in [1, \ell]$ and $j \in [1, a_i]$, associate a real number $\alpha_{i,j}$ satisfying the following two conditions:

- $0 \leq \alpha_{i,j} < 1$, and
- $\alpha_{i,j} > \alpha_{i,j'}$ whenever $j > j'$.

We also fix a line bundle L over X. The degree of L will be denoted by d.

We consider parabolic vector bundles over X of the following type.

Let E be a vector bundle of rank n with $\det(E) := \bigwedge^n E = L$. The parabolic points are $\{p_1, \cdots, p_\ell\}$ (see (1.1)). The *quasi–parabolic filtration* of E_{p_i} is of the form

$$E_{p_i} =: F_{i,1} \supsetneq \cdots \supsetneq F_{i,a_i} \neq 0 \tag{1.3}$$

with $\dim F_{i,j} = \sum_{k=j}^{a_i} r_{i,k}$. The parabolic weight of $F_{i,j}$ is $\alpha_{i,j}$. (See [12], [11] for more details on parabolic bundles.)

Let $\mathcal{PM}_s^\alpha$ denote the moduli space of stable parabolic vector bundles of the above type. This moduli space $\mathcal{PM}_s^\alpha$ is a smooth quasi–projective variety. (See [12], [11] for the construction of $\mathcal{PM}_s^\alpha$.)

Over $X \times \mathcal{PM}_s^\alpha$, there is a unique universal projective bundle, which we will denote by $\mathbb{P}$. For any $E_* \in \mathcal{PM}_s^\alpha$, the restriction of $\mathbb{P}$ to $X \times \{E_*\}$ is identified with the projectivization $P(E)$, where E is the underlying vector bundle for the parabolic vector bundle E_*. Fix a point $x_0 \in X$. Let

$$\mathbb{P}_{x_0} := \mathbb{P}|_{\{x_0\}\times\mathcal{PM}_s^\alpha} \longrightarrow \mathcal{PM}_s^\alpha \tag{1.4}$$

be the projective bundle over $\mathcal{PM}_s^\alpha$.

We prove the following theorem:

Theorem 1.1. *The Brauer group* $\mathrm{Br}(\mathcal{PM}_s^\alpha)$ *is isomorphic to the cyclic group* $\mathbb{Z}/m\mathbb{Z}$*, where*

$$m = \text{g.c.d.}(d, n, r_{1,1}, \cdots, r_{1,a_1}, \cdots, r_{\ell,1}, \cdots, r_{\ell,a_\ell})$$

(as before, $d = \text{degee}(L)$*).*

The cyclic group $\mathrm{Br}(\mathcal{PM}_s^\alpha)$ *is generated by the Brauer class of the Brauer–Severi variety* $\mathbb{P}_{x_0}$ *in* (1.4)*.*

The Brauer group of a smooth complex quasi–projective variety Y parametrizes the equivalence classes of principal $\mathrm{PGL}_r(\mathbb{C})$–bundles, $r \geq 1$, over Y. We recall that a principal $\mathrm{PGL}_r(\mathbb{C})$–bundle P is equivalent to a principal $\mathrm{PGL}_{r'}(\mathbb{C})$–bundle P' if there are vector bundles F and F' of ranks r and r' respectively, such that the two principal $\mathrm{PGL}_{rr'}(\mathbb{C})$–bundles $P \bigotimes P(F')$ and $P(F) \bigotimes P'$ are isomorphic. As mentioned in the introduction, the Brauer group of Y coincides with the cohomological Brauer group $\mathrm{Br}'(Y)$.

2. Variation of moduli space of parabolic bundles

In this section we recall a result of Thaddeus which will be crucially used here.

As before, X is an irreducible smooth complex projective curve of genus $g(X)$, with $g(X) \geq 2$. We will compare two different moduli spaces of stable parabolic vector bundles on X. For that, fix a rank, a determinant line bundle (top exterior product), a nonempty set of parabolic points of X and quasi–parabolic filtration types over the parabolic points; if $g(X) = 2$, then fix the rank to be at least three. Take two different sets of parabolic weights, say Λ_1 and Λ_2. Let $\mathcal{PM}^{\Lambda_1}$ and $\mathcal{PM}^{\Lambda_2}$ be the corresponding moduli spaces of stable parabolic vector bundles.

Lemma 2.1. *There is a smooth quasi–projective complex variety $\mathcal{U}$ and open embeddings*

$$\varphi_i : \mathcal{U} \hookrightarrow \mathcal{PM}^{\Lambda_i},$$

$i = 1, 2$, *such that the codimension of the complement*

$$\mathcal{PM}^{\Lambda_i} \setminus \varphi_i(\mathcal{U}) \subset \mathcal{PM}^{\Lambda_i}$$

is at least two.

Proof. For $i = 1, 2$, let $\overline{\mathcal{PM}}^{\Lambda_i}$ be the moduli space of semistable parabolic vector bundles of the above type with parabolic weights Λ_i. The moduli space $\overline{\mathcal{PM}}^{\Lambda_i}$ is normal, and $\mathcal{PM}^{\Lambda_i}$ is the smooth locus of $\overline{\mathcal{PM}}^{\Lambda_i}$. Thaddeus proved that there are subschemes $\mathcal{S}_1 \subset \overline{\mathcal{PM}}^{\Lambda_1}$ and $\mathcal{S}_2 \subset \overline{\mathcal{PM}}^{\Lambda_2}$, of codimension at–least two, such that the blow–up of $\overline{\mathcal{PM}}^{\Lambda_1}$ along $\mathcal{S}_1$ is isomorphic to the blow–up of $\overline{\mathcal{PM}}^{\Lambda_2}$ along $\mathcal{S}_2$ [14, Section 7] (see also [15]). The lemma follows immediately from this result. $\square$

3. One parabolic point and small parabolic weights

In this section we assume the following:

- there is only one parabolic point, so $\ell = 1$, and
- all the parabolic weights are sufficiently small (smaller than $1/n^2$).

If $a_1 = 1$ (see (1.2)), then $\mathcal{PM}^{\alpha}_s$ coincides with the moduli space of stable vector bundles of rank n and determinant L, and the Brauer group of it is already computed in [3]. Hence we will assume that $a_1 > 1$.

Let $\overline{\mathcal{N}}$ denote the moduli space of semistable vector bundles E of rank n over X with $\det(E) := \bigwedge^n E = L$. Let

$$\mathcal{N} \subset \overline{\mathcal{N}} \tag{3.1}$$

be the nonempty Zariski open subset that parametrizes the stable ones.

Since the parabolic weights are sufficiently small we know the following:

- For a stable parabolic bundle, the underlying vector bundle is semistable.
- For any quasi–parabolic structure on a stable vector bundle of rank n and determinant L, the corresponding parabolic vector bundle is stable.

Therefore, we have a forgetful morphism

$$\pi_0 \,:\, \mathcal{PM}^{\alpha}_s \,\longrightarrow\, \overline{\mathcal{N}} \tag{3.2}$$

that sends any stable parabolic bundle to its underlying vector bundle. Let

$$M \,:=\, \pi_0^{-1}(\mathcal{N}) \,\subset\, \mathcal{PM}^{\alpha}_s \tag{3.3}$$

be the inverse image, where $\mathcal{N}$ is defined in (3.1). Note that M is a Zariski open dense subset of $\mathcal{PM}^{\alpha}_s$. Let

$$\pi \,:=\, \pi_0\vert_M \,:\, M \,\longrightarrow\, \mathcal{N} \tag{3.4}$$

be the restriction of π_0 constructed in (3.2).

Let

$$P \,\subset\, \mathrm{SL}(n,\mathbb{C}) \tag{3.5}$$

be the parabolic subgroup that preserves a fixed filtration of subspaces

$$\mathbb{C}^n \,=\, C_1 \,\supsetneq\, \cdots \,\supsetneq\, C_{a_1} \tag{3.6}$$

such that $\dim C_j \,=\, \sum_{k=j}^{a_1} r_{1,k}$ (see (1.3)). We noted earlier that any quasi–parabolic structure on a stable vector bundle of rank n and determinant L lies in $\mathcal{PM}^{\alpha}_s$. Therefore, the projection π in (3.4) defines a fiber bundle over $\mathcal{N}$ with fiber $\mathrm{SL}(n,\mathbb{C})/P$, where P is the subgroup in (3.5).

Consider the moduli space $\mathcal{N}$ in (3.1). There is a universal projective bundle $\mathbb{P}^0$ on $X \times \mathcal{N}$ [2, p. 6, Theorem 2.7]. For the very general point $E \in \mathcal{N}$, the projective bundle $P(E)$ does not admit any nontrivial automorphism. Hence $\mathbb{P}^0$ is unique. Fix a point $x_0 \in X$. Let

$$\mathbb{P}^0_{x_0} \,\longrightarrow\, \mathcal{N} \tag{3.7}$$

be the projective bundle obtained by restricting $\mathbb{P}^0$ to $\{x_0\} \times \mathcal{N}$. Let

$$\mathbb{P}' \,:=\, \pi^*\mathbb{P}^0_{x_0} \tag{3.8}$$

be the projective bundle over M, where π is the projection in (3.4).

Define

$$m \,:=\, \text{g.c.d.}(d\,,n\,,r_{1,1}\,,\cdots\,,r_{1,a_1})\,, \tag{3.9}$$

where $r_{1,j}$ are as in (1.2) and $d \,=\, \text{degree}(L)$.

Lemma 3.1. *The Brauer group* $\mathrm{Br}(M)$ *of the smooth quasi–projective variety* M *in* (3.3) *is the cyclic group* $\mathbb{Z}/m\mathbb{Z}$*, where* m *is defined in* (3.9)*. The group* $\mathrm{Br}(M)$ *is generated by the the Brauer class of the Brauer–Severi variety* $\mathbb{P}'$ *constructed in* (3.8)*.*

Proof. Applying the Leray spectral sequence (see [13, p. 89, Theorem 1.18]) to the projection π in (3.4),

$$E_2^{p,q} \,:=\, H^p(\mathcal{N}_{\acute{e}tale},\, R^q\pi_*\mathbb{G}_m) \,\Longrightarrow\, H^{p+q}(M_{\acute{e}tale},\, \mathbb{G}_m)\,, \tag{3.10}$$

where $\mathbb{G}_m$ denotes the sheaf of regular invertible function. Consequently, we have a long exact sequence

$$(3.11)\qquad \longrightarrow E_2^{0,1} \xrightarrow{\theta} E_2^{2,0} \xrightarrow{\theta_1} H^2(M_{\acute{e}tale},\, \mathbb{G}_m) \longrightarrow E_2^{0,2} \longrightarrow .$$

We will investigate the first term in (3.11).

The variety $\mathcal{N}$ is simply connected [3, p. 266, Proposition 1.2(b)], and

$$\mathrm{Pic}(\mathrm{SL}(n,\mathbb{C})/P) \,=\, \mathbb{Z}^{\oplus(a_1-1)}$$

(see (3.5)), where a_1 is the integer in (3.6). Hence

$$R^1\pi_*\mathbb{G}_m \longrightarrow \mathcal{N}$$

is the constant sheaf with stalk isomorphic to $\mathbb{Z}^{\oplus(a_1-1)}$. From the connectedness of $\mathrm{SL}(n,\mathbb{C})$ it follows that the automorphisms of $\mathrm{SL}(n,\mathbb{C})/P$ given by the left translation action of $\mathrm{SL}(n,\mathbb{C})$ on it act trivially on $\mathrm{Pic}(\mathrm{SL}(n,\mathbb{C})/P)$. Consequently, the direct image

$$R^1\pi_*\mathbb{G}_m \longrightarrow \mathcal{N}$$

is canonically identified with the constant sheaf with stalk $\mathrm{Pic}(\mathrm{SL}(n,\mathbb{C})/P)$.

The homogeneous variety $\mathrm{SL}(n,\mathbb{C})/P$ (see (3.5)) is smooth complete and rational [6]. Hence

$$(3.12)\qquad H^2((\mathrm{SL}(n,\mathbb{C})/P)_{\acute{e}tale},\, \mathbb{G}_m) \,=\, 0\,.$$

We will show that

$$(3.13)\qquad (R^2\pi_*\mathbb{G}_m)_{\mathrm{torsion}} \,=\, 0\,.$$

Take any integer $\delta \geq 2$. From the Kummer sequence,

$$(3.14)\qquad R^1\pi_*\mathbb{G}_m \longrightarrow R^2\pi_*\mu_\delta \longrightarrow (R^2\pi_*\mathbb{G}_m)[\delta] \longrightarrow 0\,,$$

where $(R^2\pi_*\mathbb{G}_m)[\delta] \,\subset\, R^2\pi_*\mathbb{G}_m$ is the subgroup generated by δ–torsion elements. Take any geometric point $z \longrightarrow \mathcal{N}$, and fix a Henselization of $\mathcal{N}$ at z. Now pull back (3.14) to z. From the above description of $R^1\pi_*\mathbb{G}_m$ as the constant sheaf with stalk isomorphic to $\mathbb{Z}^{\oplus(a_1-1)}$, we know that $(R^1\pi_*\mathbb{G}_m)_z \,=\, \mathrm{Pic}(\pi^{-1}(z))$. From the proper base change theorem [13, p. 223, Theorem 2.1],

$$(R^2\pi_*\mu_\delta)_z \,=\, H^2(\pi^{-1}(z),\, \mu_\delta)\,.$$

The homomorphism $\mathrm{Pic}(\pi^{-1}(z)) \longrightarrow H^2(\pi^{-1}(z),\, \mu_\delta)$ is surjective due to (3.12) in the long exact sequence of cohomologies for the Kummer sequence. Hence $((R^2\pi_*\mathbb{G}_m)[\delta])_z \,=\, 0$ for all z. This implies that (3.13) holds.

Note that $R^0\pi_*\mathbb{G}_m \,=\, \mathbb{G}_m$. Hence $E_2^{2,0} \,=\, H^2(\mathcal{N}_{\acute{e}tale},\, \mathbb{G}_m)$ (see (3.10)). The cohomological Brauer group $H^2(\mathcal{N}_{\acute{e}tale},\, \mathbb{G}_m)_{\mathrm{torsion}}$ is generated by the Brauer class of the projective bundle $\mathbb{P}^0_{x_0}$ in (3.7) [3, p. 266, Proposition 1.2(a)]. From the construction of $\mathbb{P}'$ in (3.8) it follows that the Brauer class of $\mathbb{P}'$ coincides with $\theta_1(\beta)$, where θ_1 is the homomorphism in (3.11), and $\beta \in H^2(\mathcal{N}_{\acute{e}tale},\, \mathbb{G}_m)$ is the Brauer class of $\mathbb{P}^0_{x_0}$. Therefore, from (3.11) and (3.13) we conclude that the Brauer class of $\mathbb{P}'$ generates $H^2(M_{\acute{e}tale},\, \mathbb{G}_m)_{\mathrm{torsion}}$.

Therefore, from (3.11) and (3.13),

$$\longrightarrow \mathrm{Pic}(\mathrm{SL}(n,\mathbb{C})/P) \xrightarrow{\theta} \mathrm{Br}(\mathcal{N}) \xrightarrow{\theta_1} \mathrm{Br}(M) \longrightarrow 0\,. \tag{3.15}$$

We will describe generators of $\mathrm{Pic}(\mathrm{SL}(n,\mathbb{C})/P)$. For each $j \in [2\,, a_1]$, define

$$c_j := \sum_{k=j}^{a_1} r_{1,k} \tag{3.16}$$

(see (1.2)). Let

$$f_j : \mathrm{SL}(n,\mathbb{C})/P \longrightarrow P(\bigwedge\nolimits^{c_j} \mathbb{C}^n) \tag{3.17}$$

be the morphism that sends any filtration

$$\mathbb{C}^n = V_1 \supsetneq \cdots \supsetneq V_{a_1}$$

to the line in $\bigwedge^{c_j} \mathbb{C}^n$ defined by $\bigwedge^{c_j} V_j$. Let

$$\zeta_j := f_j^*\mathcal{O}_{P(\wedge^{c_j}\mathbb{C}^n)}(1) \longrightarrow \mathrm{SL}(n,\mathbb{C})/P \tag{3.18}$$

be the line bundle, where f_j is the morphism in (3.17). It is known that the homomorphism

$$\eta : \mathbb{Z}^{\oplus(a_1-1)} \longrightarrow \mathrm{Pic}(\mathrm{SL}(n,\mathbb{C})/P) \tag{3.19}$$

defined by

$$(z_1\,, \cdots, z_{a_1-1}) \longmapsto \bigotimes_{j=1}^{a_1-1} \zeta_{j+1}^{\otimes z_j}$$

is an isomorphism.

Let $\beta \in H^2(\mathcal{N}_{\acute{e}tale}, \mathbb{G}_m)$ be the Brauer class of the projective bundle $\mathbb{P}^0_{x_0}$ defined in (3.7). The order of the generator β of $H^2(\mathcal{N}_{\acute{e}tale}, \mathbb{G}_m)$ is g.c.d.(n, d) [3, p. 267, Theorem 1.8]. Consider the homomorphism θ in (3.15). For each $j \in [2\,, a_1]$, we have

$$\theta(\zeta_j) = c_j \cdot \beta\,, \tag{3.20}$$

where c_j is defined in (3.16) [1, p. 203, Proposition 4.4(ii)] (see also [3, p. 267, Lemma 1.5]). From (3.20) it follows that the order of the generator $\theta_1(\beta) \in \mathrm{Br}(M)$ is

$$m' := \text{g.c.d.}(\text{g.c.d.}(n, d)\,, c_2\,, \cdots, c_{a_1})\,, \tag{3.21}$$

where θ_1 is the homomorphism in (3.15).

Using (1.2) it follows that $m = m'$, where m is defined in (3.9), and m' is defined in (3.21). Therefore, the order of the generator $\theta_1(\beta) \in \mathrm{Br}(M)$ is m. We noted earlier that the Brauer class of $\mathbb{P}'$ (defined in (3.8)) coincides with $\theta_1(\beta)$. This completes the proof of the lemma. $\square$

Lemma 3.2. *Consider the Zariski open subset $M \subset \mathcal{PM}^\alpha_s$ defined in* (3.3)*. The codimension of its complement*

$$\mathcal{PM}^\alpha_s \setminus M \subset \mathcal{PM}^\alpha_s$$

is at least two.

Proof. We use the notation of [4, p. 246, Proposition 1.2]. The codimension of $\mathcal{PM}^{\alpha}_s \setminus M$ is bounded by that of $R^{\text{ss}} \setminus R^{\text{s}} \subset R^{\text{ss}}$. Hence the lemma follows from [4, p. 246, Proposition 1.2(3)]. □

There is a universal projective bundle $\mathbb{P}$ over $X \times \mathcal{PM}^{\alpha}_s$. It is universal in the sense that each parabolic vector bundle $E_* \in \mathcal{PM}^{\alpha}_s$, the restriction of $\mathbb{P}$ to $X \times \{E_*\}$ coincides with the projectivization $\mathbb{P}(E)$ of the underlying vector bundle. To construct $\mathbb{P}$, we recall that $\mathcal{PM}^{\alpha}_s$ is constructed as a geometric invariant theoretic quotient of a variety $\mathcal{R}$ (see [11], [12]). There is a canonical universal projective bundle over $X \times \mathcal{R}$ admitting a lift of the action of the group. The projective bundle $\mathbb{P}$ is the corresponding quotient. (See also [2, p. 6, Theorem 2.7].) The universal projective bundle $\mathbb{P}$ is unique.

Let $\mathbb{P}_{x_0} \longrightarrow \mathcal{PM}^{\alpha}_s$ be the projective bundle obtained by restricting the above projective bundle to $\{x_0\} \times \mathcal{PM}^{\alpha}_s$, where x_0 is the fixed point of X (see (3.7)). The restriction of $\mathbb{P}_{x_0}$ to the open subset M (see (3.3)) is isomorphic to the projective bundle $\mathbb{P}'$ defined in (3.8).

As mentioned in the introduction, any moduli space of stable parabolic bundles is smooth (the obstruction to smoothness of a stable point is H^2 of the sheaf of parabolic endomorphisms, and this H^2 vanishes because the base is a curve). Therefore, we conclude that in the special case under consideration, Theorem 1.1 follows from Lemma 3.1 and Lemma 3.2.

4. One parabolic point and arbitrary weights

We continue with the assumption that there is exactly one parabolic point. But the earlier condition on parabolic weights is now removed. As before, we assume that $a_1 \geq 2$.

As before, fix a rank, a determinant line bundle and a quasi–parabolic filtration type. Let α and β be two sets of parabolic weights for this quasi–parabolic type with α being sufficiently small. Let $\mathcal{PM}^{\beta}$ (respectively, $\mathcal{PM}^{\alpha}$) be the moduli space of stable parabolic vector bundles with parabolic weights β (respectively, α).

In Section 3 we have proved Theorem 1.1 for $\mathcal{PM}^{\alpha}_s$. Therefore, from Lemma 2.1 we conclude that Theorem 1.1 holds also for $\mathcal{PM}^{\beta}_s$.

5. Multiple parabolic points

In this section we drop the assumption in Section 4 that there is only one parabolic point.

Fix ℓ parabolic points as in (1.1). At each parabolic point p_i, fix the quasi–parabolic structure of type $\{r_{i,j}\}_{j=1}^{a_i}$ together with the parabolic weights $\alpha := \{\alpha_{i,1}, \cdots, \alpha_{i,a_i}\}$ (see (1.2) and (1.3)).

The proof of Theorem 1.1 for multiple parabolic points is similar to that for the case of one point.

First assume that all the parabolic weights are sufficiently small. More precisely, assume that all the parabolic weights are smaller than $(n^2\ell)^{-1}$. This condition ensures that the underlying vector bundle of a stable parabolic vector bundle is semistable.

We use the notation of Section 3. Let $\mathcal{N} \subset \overline{\mathcal{N}}$ be the moduli space of stable vector bundles over X of rank n and determinant L (see (3.1)). Define

$$M := \pi_0^{-1}(\mathcal{N}), \tag{5.1}$$

where $\pi_0 : \mathcal{PM}_s^\alpha \longrightarrow \overline{N}$ is the morphism that sends a parabolic vector bundle to its underlying vector bundle. Define

$$\pi := \pi_0|_M : M := \pi_0^{-1}(\mathcal{N}) \longrightarrow \mathcal{N} \tag{5.2}$$

(see (3.4)).

For each $i \in [1, \ell]$, let

$$P_i \subset \mathrm{SL}(n, \mathbb{C}) \tag{5.3}$$

be the parabolic subgroup that preserves a fixed filtration of subspaces

$$\mathbb{C}^n = C_1^i \supsetneq \cdots \supsetneq C_{a_i}^i$$

such that $\dim C_j^i = \sum_{k=j}^{a_i} r_{i,k}$.

Since the parabolic weights are sufficiently small, any quasi–parabolic structure on a stable vector bundle of rank n and determinant L is parabolic stable. Consequently, the projection π in (5.2) makes M a fiber bundle over $\mathcal{N}$ with fiber

$$\mathbb{F} := \prod_{i=1}^{\ell} \mathrm{SL}(n, \mathbb{C})/P_i\,. \tag{5.4}$$

Our first aim is to determine the Brauer group $\mathrm{Br}(M)$ of M using the fibration π.

Consider the long exact sequence constructed as in (3.11) for the projection π in (5.2). Since $\mathcal{N}$ is simply connected [3, p. 266, Proposition 1.2(b)], the direct image

$$R^1\pi_*\mathbb{G}_m \longrightarrow \mathcal{N}$$

is canonically identified with the constant sheaf with stalk

$$\mathrm{Pic}(\mathbb{F}) = \mathbb{Z}^{N_0}, \tag{5.5}$$

where $N_0 = \sum_{i=1}^{\ell}(a_i - 1)$.

Since $\mathbb{F}$ in (5.4) is a smooth rational projective variety, we have

$$H^2(\mathbb{F}_{\acute{e}tale}, \mathbb{G}_m) = 0\,. \tag{5.6}$$

In view of this and the above description of $R^1\pi_*\mathbb{G}_m$, we may repeat the argument in the proof of Lemma 3.1 for (3.13) to conclude that

$$(R^2\pi_*\mathbb{G}_m)_{\text{torsion}} = 0\,. \tag{5.7}$$

Now from (3.11) we conclude that the Brauer class of $\mathbb{P}'$ generates $H^2(M_{\acute{e}tale}, \mathbb{G}_m)$, where $\mathbb{P}'$ is constructed as in (3.8).

Therefore, from (3.11) we have

$$\longrightarrow \operatorname{Pic}(\mathbb{F}) \xrightarrow{\theta} \operatorname{Br}(\mathcal{N}) \xrightarrow{\theta_1} \operatorname{Br}(M) \longrightarrow 0\,. \tag{5.8}$$

For each $i \in [1\,,\ell]$, consider P_i defined in (5.3). Let $\{\zeta^i_j\}_{2\leq j\leq a_i}$ be the generators of $\operatorname{Pic}(\mathrm{SL}(n,\mathbb{C})/P_i)$ (see (3.18) for the description of these generators), and let

$$\eta:\, \mathbb{Z}^{\oplus N_0} \longrightarrow \bigoplus_{i=1}^{\ell} \operatorname{Pic}(\mathrm{SL}(n,\mathbb{C})/P_i) \,=\, \operatorname{Pic}(\mathbb{F}) \tag{5.9}$$

be the isomorphism defined by these generators (see (3.19) and (5.5)).

For each $i \in [1\,,\ell]$ and $j \in [2\,,a_i]$, define

$$c^i_j \,:=\, \sum_{k=j}^{a_i} r_{i,k}\,.$$

The homomorphism θ in (5.8) satisfies the following:

$$\theta(\zeta^i_j) \,=\, c^i_j \cdot \beta\,, \tag{5.10}$$

where $\beta \in \operatorname{Br}(\mathcal{N})$ is the Brauer class of the Brauer–Severi variety $\mathbb{P}^0_{x_0} \longrightarrow \mathcal{N}$ constructed as in (3.7). As before, define

$$\mathbb{P}' \,:=\, \pi^*\mathbb{P}^0_{x_0}\,. \tag{5.11}$$

The order of β is g.c.d.(n,d) [3, p. 267, Theorem 1.8]. Hence from (5.10) we conclude that the order of $\theta_1(\beta)$ is

$$m' \,:=\, \text{g.c.d.}(\text{g.c.d.}(n,d),\, c^1_2\,,\cdots,c^1_{a_1}\,,\cdots,c^i_j\,,\cdots,c^\ell_2\,,\cdots,c^\ell_{a_\ell})\,, \tag{5.12}$$

where θ_1 is the homomorphism in (5.8).

Consider m defined in Theorem 1.1. Using (1.2) it follows that m coincides with m' defined in (5.12). Therefore, we have the following proposition:

Proposition 5.1. *For the variety M defined in* (5.1)*,*

$$\operatorname{Br}(M) \,=\, \mathbb{Z}/m\mathbb{Z}\,,$$

and $\operatorname{Br}(M)$ *is generated by the Brauer class of the Brauer–Severi variety $\mathbb{P}' \longrightarrow M$ constructed in* (5.11)*.*

Lemma 5.2. *Consider the Zariski open subset $M \subset \mathcal{PM}^\alpha_s$ defined in* (5.1)*. The codimension of the complement*

$$\mathcal{PM}^\alpha_s \setminus M \,\subset\, \mathcal{PM}^\alpha_s$$

is at least two.

Proof. The proof of Lemma 3.2 goes through without any change. □

From Proposition 5.1 and Lemma 5.2 we now conclude that Theorem 1.1 holds under the assumption that the parabolic weights are sufficiently small.

For arbitrary parabolic weights, Theorem 1.1 is now deduced from the above special case using Lemma 2.1 (as done in Section 4). This completes the proof of Theorem 1.1.

6. Existence of universal bundle

As an application of Theorem 1.1, we will show that there is a universal parabolic vector bundle over $X \times \mathcal{PM}_s^\alpha$ if and only if

$$\text{g.c.d.}(d\,,n\,,r_{1,1}\,,\cdots\,,r_{1,a_1}\,,\cdots\,,r_{\ell,a_1}\,,\cdots\,,r_{\ell,a_\ell})\,=\,1\,.$$

This was proved earlier by N. Hoffmann [10, Corollary 6.3].

If $\text{g.c.d.}(d\,,n\,,r_{1,1}\,,\cdots\,,r_{1,a_1}\,,\cdots\,,r_{\ell,a_1}\,,\cdots\,,r_{\ell,a_\ell})\,\neq\,1$, then from Theorem 1.1 it follows immediately that there is no universal vector bundle over $X \times \mathcal{PM}_s^\alpha$; in particular, there is no universal parabolic bundle.

To prove the converse, assume that

$$\text{g.c.d.}(d\,,n\,,r_{1,1}\,,\cdots\,,r_{1,a_1}\,,\cdots\,,r_{\ell,a_1}\,,\cdots\,,r_{\ell,a_\ell})\,=\,1\,. \tag{6.1}$$

Take any parabolic vector bundle

$$E_*\,:=\,(E\,,\{F_{i,1}\supsetneq\cdots\supsetneq F_{i,a_i}\}_{i=1}^{\ell})\,\in\,\mathcal{PM}_s^\alpha\,.$$

Consider the complex lines $\bigwedge^{\text{top}} F_{i,j}$, where $i\,\in\,[1\,,\ell]$ and $j\,\in\,[1\,,a_i]$, together with the complex line

$$\text{Det}\,E\,:=\,\bigwedge\nolimits^{\text{top}} H^0(X,\,E)\bigotimes\bigwedge\nolimits^{\text{top}} H^1(X,\,E)^*\,.$$

Let $\{L_i\}_{i=1}^{m_0}$ be this collection of complex lines.

Automorphisms of a stable parabolic vector bundle are scalar multiplications. In other words,

$$\mathbb{G}_m\,=\,\text{Aut}(E_*)\,.$$

Note that $\text{Aut}(E_*)$ acts on the line L_i for each $i\,\in\,[1\,,m_0]$.

From (6.1) it follows that there are integers $e_i\,\in\,\mathbb{Z}$, $i\,\in\,[1\,,m_0]$, with some e_i nonzero, such that the group $\text{Aut}(E_*)$ acts trivially on the line

$$\bigotimes_{i=1}^{m_0} L_i^{\otimes e_i}\,.$$

Using this it can be shown that in the geometric invariant theoretic construction of the moduli space $\mathcal{PM}_s^\alpha$, a suitable twist of the universal parabolic vector bundle descends to $X \times \mathcal{PM}_s^\alpha$. (See [5, p. 465, Proposition 3.2] for the details.)

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