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Bogomolov restriction theorem for Higgs bundles

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Abstract

Let (E,θ) be a stable Higgs bundle of rank r on a smooth complex projective surface X equipped with a polarization H. Let $C \subset X$ be a smooth complete curve with $[C] = n \cdot H$. If

$$2n > \frac{R}{r}\left(2rc_2(E) - (r-1)c_1(E)^2\right),$$

where $R = \max\{\binom{r}{s}\binom{r-1}{s-1}\colon 1 \leqslant s \leqslant r-1\}$, then we prove that the restriction of (E,θ) to C is a stable Higgs bundle. This is a Higgs bundle analog of Bogomolov's restriction theorem for stable vector bundles.

MSC: 14F05

Keywords: Bogomolov restriction theorem; Higgs bundles; Stability

1. Introduction

Let X be a smooth irreducible complex projective surface. Fix a very ample line bundle H over X. Let E be a vector bundle over X. If there is a positive integer n_0, and a smooth closed curve $C \subset X$ lying in the linear system $|H^{\otimes n_0}|$, such that the restriction $E|_C$ is stable (respectively, semistable), then using the openness of the stability (respectively, semistability) condition, it is easy to deduce that E itself is stable (respectively, semistable). There are various results in the converse direction; see [8]. One of them is the following celebrated theorem of Bogomolov:

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Theorem 1.1 *(Bogomolov). Let E be a stable vector bundle on X. Let $C \subset X$ be a smooth complete curve with $[C] = n \cdot H$. If*

$$2n > \frac{R}{r}\big(2rc_2(E) - (r-1)c_1(E)^2\big),$$

where $R = \max\{\binom{r}{s}\binom{r-1}{s-1}\colon 1 \leqslant s \leqslant r-1\}$, then the restriction $E|_C$ is stable.

A Higgs vector bundle on X is pair of the form (E, θ), where $E \longrightarrow X$ is a vector bundle, and θ is a section of $End(E) \otimes \Omega_X^1$ satisfying the integrability condition $\theta \wedge \theta = 0$ [7,9]. Higgs bundles play crucial role in diverse topics. Our aim here is to prove an analog of Theorem 1.1 for Higgs bundles.

We prove the following (see Theorem 3.3):

Theorem 1.2. *Let (E, θ) be a stable Higgs bundle of rank r on X. Let $C \subset X$ be a smooth complete curve with $[C] = n \cdot H$. If*

$$2n > \frac{R}{r}\big(2rc_2(E) - (r-1)c_1(E)^2\big),$$

where $R = \max\{\binom{r}{s}\binom{r-1}{s-1}\colon 1 \leqslant s \leqslant r-1\}$, then the restriction of (E, θ) to C is a stable Higgs bundle.

The proof of Theorem 1.2 is modeled on the proof of Theorem 1.1 given in [8].

In [2], the Grauert–Mülich and Flenner's restriction theorems were generalized to principal Higgs bundles. It will be interesting to prove a principal Higgs bundle analog of Theorem 1.2.

2. Preliminaries

2.1. Higgs sheaf

Let X be an irreducible smooth projective surface over $\mathbb{C}$. The holomorphic cotangent bundle of X will be denoted by Ω_X^1.

A *Higgs sheaf* on X is a pair of the form (E, θ), where $E \longrightarrow X$ is a torsionfree sheaf, and

$$\theta : E \longrightarrow E \otimes \Omega_X^1$$

is an $\mathcal{O}_X$-linear homomorphism such that $\theta \wedge \theta = 0$ [10]. The homomorphism θ is called a *Higgs field* on E. A coherent subsheaf F of E is called θ-*invariant* if

$$\theta(F) \subset F \otimes \Omega_X^1.$$

A θ-invariant subsheaf will also be called a *Higgs subsheaf*.

Fix a very ample line bundle $H = \mathcal{O}_X(1)$ on X. The *degree* of a torsionfree coherent sheaf V on X is defined to be the degree of the restriction of V to the general complete intersection curve $D \in |\mathcal{O}_X(1)|$. So,

$$\mathrm{degree}(V) = \big(c_1(V) \cup c_1(H)\big) \cap [X].$$

The quotient $\mathrm{degree}(V)/\mathrm{rank}(V) \in \mathbb{Q}$ is called the *slope* of V, and it is denoted by $\mu(V)$.

For any nonzero subsheaf E' of a torsionfree sheaf E, define

$$\xi_{E',E} := \frac{\operatorname{rank}(E)c_1(E') - \operatorname{rank}(E')c_1(E)}{\operatorname{rank}(E)\operatorname{rank}(E')}, \tag{2.1}$$

which is an element of $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

A subsheaf E' of a torsionfree sheaf E is called *normal* if E/E' is torsionfree.

A Higgs sheaf (E, θ) is said to be *stable* (respectively, *semistable*) if for every normal Higgs subsheaf $F \subset E$, the inequality

$$\mu(F) < \mu(E) \quad \bigl(\text{respectively, } \mu(F) \leqslant \mu(E)\bigr)$$

holds.

A Higgs sheaf (E, θ) is said to be a *Higgs bundle* if the underlying coherent sheaf E is locally free. A semistable Higgs bundle (E, θ) is said to be *polystable* if it is a direct sum of stable Higgs bundles of same slope $\mu(E)$.

A semistable Higgs bundle satisfies the Bogomolov inequality. More precisely, if (E, θ) is a semistable Higgs bundle over X, then the discriminant

$$\Delta(E) := 2rc_2(E) - (r-1)c_1(E)^2 \geqslant 0 \tag{2.2}$$

(see [9, Proposition 3.3, Proposition 3.4, Theorem 1]), where $c_j(E)$ is the j-th Chern class of E.

2.2. *The positive cone K^+*

We will briefly recall some basic facts on line bundles X which will be needed later (the details can be found in [1]).

Let $\mathrm{Pic}(X)$ be the abelian group of isomorphism classes of line bundles with the operation of tensor product. The Néron–Severi group $\mathrm{NS}(X)$ is defined to be the quotient of $\mathrm{Pic}(X)$ by the numerical equivalence. Let $\mathrm{NS}_{\mathbb{R}}(X)$ denote the tensor product $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The image of $\mathrm{Pic}(X)$ in $\mathrm{NS}_{\mathbb{R}}(X)$ is a sub-lattice which coincides with the $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. There is a natural nondegenerate pairing on $\mathrm{NS}_{\mathbb{R}}(X)$ given by the cup product that is integral on $H^2(X, \mathbb{Z})$. In $\mathrm{NS}_{\mathbb{R}}(X)$, the domain $x^2 > 0$ breaks up into two cones; a cone of a real vector space V is a subset $C \subset V$ such that all linear combinations elements of C with nonnegative coefficients lie in C. Let K^+ be the component defined by

$$K^+ = \bigl\{D \in \mathrm{NS}_{\mathbb{R}}(X) \bigm| D^2 > 0,\ D \cdot H > 0 \text{ for all ample divisors } H\bigr\}. \tag{2.3}$$

For any $\xi \in K^+$, define $|\xi| = \sqrt{\xi^2}$. Note that the condition $D \cdot H > 0$ in (2.3) is added only to pick one of the two components of the set of all D with $D^2 > 0$. If D is a divisor on X such that $D^2 > 0$ and $D \cdot H_0 > 0$ for one ample divisor H_0, then $D \cdot H > 0$ for all ample divisors H. We have,

$$D \in K^+ \quad \text{if and only if} \quad D \cdot L > 0 \quad \text{for all } L \in K^+ - \{0\}. \tag{2.4}$$

For any nonzero $\xi \in \mathrm{NS}_{\mathbb{R}}(X)$, define the cone

$$C(\xi) := \bigl\{x \in K^+ \bigm| x \cdot \xi > 0\bigr\}. \tag{2.5}$$

From (2.4), (2.5),

$$C(\xi) = K^+ \quad \text{if and only if} \quad \xi \in K^+. \tag{2.6}$$

3. Restriction of Higgs bundles

The following lemma is a straightforward computation.

Lemma 3.1. *Let* $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ *be a short exact sequence of nonzero torsionfree sheaves.*

(1) *Let* $G \subset F'$ *be a proper subsheaf. Then*

$$\xi_{G,F} = \xi_{G,F'} + \xi_{F',F},$$

where $\xi_{-,-}$ *is defined in* (2.1).

(2) *Let* $G'' \subset F''$ *be a proper subsheaf of rank* s*, and let* G *be the kernel of the surjective map* $F \longrightarrow F''/G''$*. Then we have*

$$\xi_{G,F} = \frac{r'(r''-s)}{(r'+s)r''}\xi_{F',F} + \frac{s}{r+s}\xi_{G'',F''},$$

where r', r'' *and* r *are ranks of* F', F'' *and* F *respectively.*

(3)

$$\frac{\Delta(F')}{r'} + \frac{\Delta(F'')}{r''} = \frac{\Delta(F)}{r} + \frac{rr'}{r''}(\xi_{F',F})^2,$$

where Δ *is the discriminant defined in* (2.2).

The details of the proof of Lemma 3.1 are omitted.

Proposition 3.2. *Let* (E, θ) *be a Higgs bundle on* X *of rank* $r \geqslant 2$ *with discriminant* $\Delta(E) < 0$*. Then there exists a Higgs normal subsheaf* $E' \subset E$ *such that*

(1) $\xi_{E',E} \in K^+$*, and*

(2)

$$\xi^2_{E',E} \geqslant -\frac{\Delta(E)}{r^2(r-1)}.$$

Proof. Both statements will be proved by using induction on r.

Proof of (1): Suppose that $r = 2$. Since $\Delta(E) < 0$, there exists a normal Higgs subsheaf $L \otimes \mathcal{I}_W \subset E$ of rank one, such that

$$\left(c_1(L) - \frac{1}{2}c_1(E)\right) \cdot H > 0, \tag{3.1}$$

where L is a line bundle on X and W is a zero cycle on X (see (2.2)). We have the following short exact sequence

$$0 \longrightarrow L \otimes \mathcal{I}_W \longrightarrow E \longrightarrow \det(E) \otimes L^{-1} \otimes \mathcal{I}_Z \longrightarrow 0,$$

where Z is a zero cycle, and $\det(E)$ is the determinant line bundle $\bigwedge^2 E$. We have

$$c_2(E) = c_1(L)\big(c_1(E) - c_1(L)\big) + n,$$

where n is some nonnegative integer. The discriminant $\Delta(E)$ is given by

$$\Delta(E) = 4c_2(E) - c_1(E)^2 = -4\left(c_1(L) - \frac{c_1(E)}{2}\right)^2 + n = -4\xi_{L,E}^2 + n. \tag{3.2}$$

Since $\Delta(E) < 0$, we have $\xi_{L,E}^2 > 0$. From (3.1) it follows that $\xi_{L,E}$ has a positive intersection with the ample divisor H. Hence $\xi_{L,E} \in K^+$.

Now assume that $r = \operatorname{rank}(E) > 2$. We impose the induction hypothesis that for every Higgs sheaf (F, θ_0) of rank not greater than $r - 1$, and $\Delta(F) < 0$, there is some normal Higgs subsheaf $F' \subset F$ such that $\xi_{F',F} \in K^+$.

Since (E, θ) is not semistable (see (2.2)), there is a normal Higgs subsheaf E' of (E, θ) such that $\xi_{E',E} \cdot H > 0$. Fix such a subsheaf E'. The quotient E/E' will be denoted by E''. Denote $\Delta' := \Delta(E')$, $\Delta'' := \Delta(E'')$ and $\Delta := \Delta(E)$. Then by Lemma 3.1(3), we have

$$\frac{\Delta'}{r'} + \frac{\Delta''}{r''} = \frac{\Delta}{r} + \frac{rr'}{r''}\xi_{E',E}^2,$$

where r' and r'' are the ranks of E' and E'' respectively.

If $\xi_{E',E}^2 > 0$, then the assertion in part (1) of the proposition holds, because $\xi_{E',E} \cdot H > 0$. So we assume that $\xi_{E',E}^2 \leqslant 0$. Then one of Δ' and Δ'' is negative, and $\xi_{E',E} \notin K^+$.

First assume that $\Delta' < 0$. By the induction hypothesis, there exists a normal Higgs subsheaf

$$G \subset E'$$

such that $\xi_{G,E'} \in K^+$. By Lemma 3.1(1), the cone $C(\xi_{G,E})$ (defined in (2.5)) contains the cone $C(\xi_{E',E})$ properly, and $\xi_{G,E} \cdot H > 0$.

Next assume that $\Delta'' < 0$. By the induction hypothesis, there exists a normal Higgs subsheaf

$$G'' \subset E''$$

such that $\xi_{G'',E''} \in K^+$. Let G be the kernel of the composition

$$E \longrightarrow E'' \longrightarrow E''/G''.$$

By Lemma 3.1(2), the cone $C(\xi_{G,E})$ contains the cone $C(\xi_{E',E})$ properly, and $\xi_{G,E} \cdot H > 0$.

Therefore, in both cases we have a Higgs subsheaf $G \subset E$ such that $\xi_{G,E} \cdot H > 0$, and $C(\xi_{G,E})$ strictly contains $C(\xi_{E',E})$.

For any subcone $C(\xi_{E',E})$ containing a nontrivial polarization, there exist finitely many subcones $C(\xi_{G,E})$ containing $C(\xi_{E',E})$, where G is a subsheaf of E (see [4, Lemma 3.4]). Hence by repeating this process, in finitely many steps, we get a normal Higgs subsheaf $E' \subset E$, such that $\xi_{E',E}^2 > 0$ with $\xi_{E',E} \cdot H > 0$, or equivalently, $\xi_{E',E} \in K^+$. This completes the proof of part (1) of the proposition.

Proof of (2): The proof uses induction on r, and follows the steps in [8, Theorem 7.3.4].

If $r = 2$, the inequality follows from (3.2).

Now suppose that $r > 2$. Let E' be a Higgs normal subsheaf of (E, θ) of rank r' such that $\xi_{E',E} \in K^+$. The Hodge Index theorem implies that

$$\xi_{E',E}^2 \leqslant \frac{(\xi_{E',E} \cdot H)^2}{H^2} \leqslant \frac{(\mu_{\max}(E) - \mu(E))^2}{H^2},$$

where $\mu_{\max}(E)$ is the maximum among the slopes of Higgs subsheaves of (E, θ), or equivalently, it is the slope of the smallest subsheaf in the Harder–Narasimhan filtration of (E, θ). Let E' be a Higgs subsheaf such that $\xi_{E',E}^2$ attains the maximum value.

By an argument identical to the one in the proof of [8, p. 174, Theorem 7.3.3], we have $\Delta' = \Delta(E') \geqslant 0$.

Suppose that

$$\frac{\Delta(E)}{r} < -r(r-1)\xi^2_{E',E}. \tag{3.3}$$

Let r'' be the rank of the quotient Higgs sheaf $E'' := E/E'$. The discriminant of E'' will be denoted by Δ''. We have by Lemma 3.1(3) and (3.3),

$$\frac{\Delta''}{r''} = \frac{\Delta}{r} - \frac{\Delta'}{r'} + \frac{rr'}{r''}\xi^2_{E',E} < -\frac{rr''(r-1) - rr''}{r''}\xi^2_{E',E} = -r^2\frac{r''-1}{r''}\xi^2_{E',E} < 0.$$

So, by induction hypothesis, there exists a normal Higgs subsheaf $G'' \subset E''$ such that $\xi_{G'',E''} \in K^+$, and

$$\xi^2_{G'',E''} \geqslant -\frac{\Delta''}{r''^2(r''-1)} > \frac{r^2}{r''^2}\xi^2_{E',E''} \tag{3.4}$$

by the previous inequality.

Let G denote the kernel of the composition $E \longrightarrow E'' \longrightarrow E''/G''$. By Lemma 3.1(2),

$$\xi_{G,E} = \frac{r'(r''-s)}{(r'+s)r''}\xi_{E',E} + \frac{s}{r'+s}\xi_{G'',E''}.$$

Since K^+ is convex, and both $\xi_{E',E}$ and $\xi_{G'',E}$ are in K^+, we conclude that $\xi_{G,E} \in K^+$. Furthermore,

$$\begin{aligned}|\xi_{G,E}| &\geqslant \frac{r'(r''-s)}{(r'+s)r''}|\xi_{E',E}| + \frac{s}{r+s}|\xi_{G'',E}| \\ &> \frac{r'(r''-s)}{(r'+s)r''}|\xi_{E',E}| + \frac{s}{r'+s}\cdot\frac{r}{r''}|\xi_{E',E}| = |\xi_{E',E}|.\end{aligned}$$

But this contradicts the maximality of $|\xi_{E',E}|$. This completes the proof of the proposition. □

Theorem 3.3. *Let (E,θ) be a stable Higgs vector bundle of rank $r \geqslant 2$ with respect to the polarization H. Let $R = \max\{\binom{r}{s}\binom{r-1}{s-1};\ 1 \leqslant s \leqslant r-1\}$, and let $C \subset X$ be a smooth curve with $[C] = nH$. If*

$$2n > \frac{R}{r}\Delta(E) + 1, \tag{3.5}$$

then the restriction $(E,\theta)|_C$ is a stable Higgs bundle.

Proof. Suppose that $(E,\theta)|_C$ is not a stable Higgs bundle. Let F be a Higgs quotient bundle of $E|_C$, with $\operatorname{rank}(F) \leqslant r-1$, such that

$$\mu(E|_C) \geqslant \mu(F). \tag{3.6}$$

Let s be the rank of F.

We will first reduce the proof to the case where $s = 1$.

Suppose that $s > 1$. By taking s-th exterior power, we get

$$\bigwedge^s E \xrightarrow{f} \bigwedge^s E|_C \xrightarrow{g} \bigwedge^s F = L. \tag{3.7}$$

The discriminant of $\bigwedge^s E$ is

$$\Delta\Big(\bigwedge^s E\Big) = \binom{r-1}{s-1}\binom{r}{s}\frac{\Delta(E)}{r} \tag{3.8}$$

(see [8, p. 175, line 11]).

From (3.5) and (3.8),

$$2n > \Delta\Big(\bigwedge^s E\Big) + 1. \tag{3.9}$$

The Higgs field θ on E induces a Higgs field on $\bigwedge^s E$; this induced Higgs field will be denoted by $\bigwedge^s \theta$. The Higgs bundle $(\bigwedge^s E, \bigwedge^s \theta)$ is a Higgs polystable (see [3, Lemma 4.4]). Let

$$\Big(\bigwedge^s E, \bigwedge^s \theta\Big) = \bigoplus_{i=1}^{\ell} (E_i, \theta_i)$$

be the Jordan–Holder filtration of $(\bigwedge^s (E), \bigwedge^s \theta)$, where each (E_i, θ_i) is a Higgs stable bundle with $\mu(E_i) = \mu(\bigwedge^s E)$. By [8, Corollary 7.3.2],

$$\Delta(E_i) \leqslant \Delta(E) \tag{3.10}$$

for all $i \in [1, \ell]$.

Define

$$\phi := g \circ f.$$

Without loss of generality, we can assume image $\phi(E_1) =: L' \neq 0$. We note that

$$\deg\big(L'\big) \leqslant \deg(L). \tag{3.11}$$

We assume that $\operatorname{rank}(E_1) > 1$. The case $\operatorname{rank}(E_1) = 1$ will be treated separately.

We have

$$\mu(E|_C) \geqslant \mu(F), \tag{3.12}$$

and

$$\mu(E_1|_C) = \mu\Big(\bigwedge^s E|_C\Big) = \frac{\binom{r-1}{s-1} c_1(E) \cdot C}{\binom{r}{s}} = s\mu(E|_C), \tag{3.13}$$

because $c_1(\bigwedge^s E) = \binom{r-1}{s-1} c_1(E)$ (see [5, p. 55]), and $\operatorname{rank}(\bigwedge^s E) = \binom{r}{s}$. From (3.12) and (3.13),

$$\mu(E_1|_C) = s\mu(E|_C) \geqslant s\mu(F) = \mu(L). \tag{3.14}$$

Since $\phi(E_1) \neq 0$, we reduced the proof to the case where the rank of the quotient F is one. We assume that $s = \operatorname{rank}(F) = 1$.

We have

$$2n > \Delta(E) + 1 \quad \text{and} \quad C^2 \geqslant \Delta(E) \geqslant \frac{\Delta(E)}{r-1},$$

and the destabilizing quotient line bundle L satisfies the inequality

$$c_1(E) \cdot C - r\deg(L) \geqslant 0 \tag{3.15}$$

(see (3.14)).

We have an exact sequence of Higgs sheaves

$$0 \longrightarrow G \longrightarrow E \longrightarrow \iota_*(L) \longrightarrow 0, \tag{3.16}$$

where $\iota : C \hookrightarrow X$ is the inclusion map. Therefore,

$$\begin{aligned} &\operatorname{rank}(G) = r, \qquad c_1(G) = c_1(F) - C, \\ &c_2(G) = c_2(E) - C \cdot c_1(E) + \frac{1}{2} C \cdot (C + K_X) + \deg(L) + (1 - g_C) \end{aligned} \tag{3.17}$$

where g_C is the genus of the curve C.

By using adjunction formula and (3.15),

$$\Delta(G) = \Delta(E) - 2\big(c_1(E) \cdot C - r \cdot \deg(L)\big) - (r-1)C^2 < 0. \tag{3.18}$$

Hence by Proposition 3.2, there exists a normal Higgs subsheaf $G' \subset G$ of rank $t < r$ such that $\xi_{G',G} \in K^+$, and

$$\xi_{G',G}^2 \geqslant -\frac{\Delta(G)}{r^2(r-1)}. \tag{3.19}$$

By (3.17),

$$\xi_{G',E} := \frac{rc_1(G') - tc_1(E)}{rt} = \xi_{G',G} - \frac{1}{r} C. \tag{3.20}$$

Since E is Higgs stable, and the intersection product takes integer values,

$$\xi_{G',E} \cdot C = \frac{rc_1(G') \cdot C - tc_1(F) \cdot C}{rt} < -\frac{n}{rt}. \tag{3.21}$$

By (3.20) and (3.21),

$$\xi_{G',G} \cdot C \leqslant -\frac{n}{rt} + \frac{n^2 H^2}{r}. \tag{3.22}$$

Now by (3.19) and (3.22),

$$-\frac{\Delta(G)}{r^2(r-1)} n^2 H^2 \leqslant \xi_{G',G}^2 C^2 \leqslant (\xi_{G',G} \cdot C)^2 \leqslant \left(\frac{n^2 H^2}{r} - \frac{n}{rt}\right)^2. \tag{3.23}$$

By (3.18) and (3.23), we have

$$\frac{-\Delta(E)}{r-1} H^2 \leqslant \frac{1}{t^2} - \frac{2nH^2}{t}.$$

Hence

$$2n \leqslant \frac{t}{r-1}\Delta(F) + \frac{1}{tH^2} \leqslant \Delta(E) + 1,$$

which contradicts our assumption in (3.9); note that $\Delta(E) \geqslant 0$ by (3.3).

Now suppose that $\operatorname{rank}(E_1) = 1$.

We have a nonzero homomorphism

$$E_1|_C \xrightarrow{f} L$$

between two line bundles on a curve C with $\deg(E_1|_C) \geqslant \deg(L)$. The key point is that $E_1|_C \cong L \cong \bigwedge^s F$ [6, Chapter IV, p. 295, Lemma 1.2]. Hence we have a rank one quotient $\bigwedge^s(E) \longrightarrow E_1$ with $\mu(\bigwedge^s(E)) = \mu(E_1)$, such that the restriction $\bigwedge^s E|_C \longrightarrow E_1|_C$ is the s-th exterior power of $E|_C \longrightarrow F$. Now by [8, Lemma 7.3.6],

$$F = \bigwedge^{s} \widetilde{E},$$

where $\widetilde{E}$ is a quotient of E of rank s; this lemma of [8] is stated for semistable bundles, but the proof goes through for Higgs bundles without any change. This contradicts the stability of E. □

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