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A Birth-Death Process Suggested by a Chain Sequence

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Abstract—We consider a birth-death process whose birth and death rates are suggested by a chain sequence. We use an elegant transformation to find the transition probabilities in a simple closed form. We also find an explicit expression for time-dependent mean. We find parallel results in discrete time. Finally, we show that the processes under investigation are transient, and hence, the stationary distribution does not exist.

Keywords—Chain sequence, Chebyshev polynomials, Transient behaviour.

1. INTRODUCTION

In this paper, we consider a birth-death process on an infinite state space whose birth and death rates are suggested by a chain sequence. Using the interesting results of constant term chain sequences [1–3], we find expressions for the birth and death rates.

The problem of solving the Chapman-Kolmogorov (C-K) equations explicitly for a specific set of birth and death rates has been approached in the literature in many different ways and still remains a topic of great interest (see, e.g., [4,5]). This problem has given rise to many intricate and interesting special functions and orthogonal polynomials (see, e.g., [6,7]). We use an elegant transformation, which is simple, direct, and does not involve Laplace transforms, to find the transition probabilities in a simple closed form and the time-dependent mean. We also carry out this time-dependent analysis for the same process on a discrete parameter space. Finally, we show that the processes under investigation are transient and hence the stationary distribution does not exist.

Let us recall the definition and some basic results of chain sequences, see [1, Section III.5; 2; 3] for proofs and developments.

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DEFINITION. *A sequence $\{a_n\}_{n=1}^{\infty}$ is a chain sequence if there exists a second sequence $\{g_n\}_{n=0}^{\infty}$ such that*

$$\begin{aligned} &\text{(i)} \quad 0 \le g_0 < 1, \quad 0 < g_n < 1, \qquad n = 1, 2, \dots, \\ &\text{(ii)} \quad a_n = (1 - g_{n-1}) g_n, \qquad n = 1, 2, \dots. \end{aligned} \tag{1}$$

The sequence $\{g_n\}$ is called a parameter sequence for $\{a_n\}$.

If both $\{g_n\}$ and $\{h_n\}$ are parameter sequences for $\{a_n\}$, then

$$g_n < h_n, \quad n = 1, 2, \dots, \qquad \text{if and only if } g_0 < h_0. \tag{2}$$

Every chain sequence $\{a_n\}$ has a *minimal* parameter sequence $\{m_n\}$ uniquely determined by the condition $m_0 = 0$, and it has a *maximal* parameter sequence $\{M_n\}$ characterized by the fact that $M_0 > g_0$ for any other parameter sequence $\{g_n\}$. For every x, $0 \le x \le M_0$, there is a unique parameter sequence $\{g_n\}$ for $\{a_n\}$ such that $g_0 = x$.

A simple example is the constant term sequence $1/4, 1/4, 1/4, \dots$, which is a chain sequence with parameters $g_n = 1/2$, $n = 0, 1, \dots$.

The following results of Wall [3] characterise a constant term chain sequence.

THEOREM 1.1. *A constant term sequence $\beta, \beta, \beta, \dots$ is a chain sequence if and only if $0 < \beta \le 1/4$.*

THEOREM 1.2. *For a constant term chain sequence $\beta, \beta, \beta, \dots$ the minimal parameter m_n is given by*

$$m_n = \frac{1 - \sqrt{1 - 4\beta}}{2} \left[1 - \left\{ \sum_{r=0}^{n} \left(\frac{1 - \sqrt{1 - 4\beta}}{1 + \sqrt{1 - 4\beta}} \right)^r \right\}^{-1} \right], \qquad n = 0, 1, 2, \dots.$$

In Theorem 2.1 we will prove that this m_n can be written as

$$m_n = \frac{\alpha U_{n-1}(1/\alpha)}{2 U_n(1/\alpha)}, \qquad n = 1, 2, 3, \dots,$$

where $U_n(\cdot)$ is the Chebyshev polynomial of second kind of order n and $\alpha = 2\sqrt{\beta}$. These polynomials are defined recursively by

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, \quad n = 0, 1, 2, \dots; \qquad \cos(\theta) = x.$$

They satisfy the recurrence relation

$$U_n(x) = 2x U_{n-1}(x) - U_{n-2}(x),$$

with $U_{-1}(x) = 0$ and $U_0(x) = 1$.

In the forthcoming sections, we analyze birth-death process whose rates are suggested by a minimal parameter chain sequence. First we will analyze the continuous parameter case.

2. BIRTH-DEATH PROCESS WITH CONTINUOUS PARAMETER SPACE

A birth-death process is a Markov process $\mathcal{X} = \{X(t),\ t \ge 0\}$, say, in which a population initially of size m changes to size n after time t by births and deaths. We assume that the size of the population can build up without any restriction. That is, our process $\mathcal{X}$ will always be taking values in $\mathcal{N} \equiv \{0, 1, \dots\}$ and $X(t)$ denotes the population size at time t. We assume that in an interval $(t, t + \delta t)$ each individual in the population has a probability $\lambda_n \delta t + o\{(\delta t)^2\}$ of giving birth to a new individual and a probability $\mu_n \delta t + o\{(\delta t)^2\}$ of dying. The parameters λ_n and μ_n are, respectively, called the birth rate and death rate when the population has size n, and define

$$P_{mn}(t) \equiv \Pr\{X(t) = n \mid X(0) = m\}, \qquad n, m \in \mathcal{N}, \quad t \geq 0,$$

the conditional probability that the population has size n at time t given its size was m at $t = 0$. For the sake of brevity we denote $P_{mn}(t)$ by $P_n(t)$. By considering $P_n(t + \delta t)$ in terms of $P_{n-1}(t)$, $P_n(t)$, and $P_{n+1}(t)$, the following set of differential-difference equations known as forward Chapman-Kolmogorov (C-K) equations can be obtained:

$$\begin{aligned} P_0'(t) &= -\lambda_0 P_0(t) + \mu_1 P_1(t), \\ P_n'(t) &= \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t), \end{aligned} \tag{3}$$

for $n = 1, 2, 3, \dots$, and whence $0 \leq P_n(t) \leq 1$ and $\sum_{n=0}^{\infty} P_n(t) = 1$, subject to the initial condition $P_n(0) = \delta_{n,m}$ (Kronecker delta) for some $m \in \mathcal{N}$. Also, $\lambda_0 > 0$, $\mu_0 = 0$, and $\lambda_n, \mu_n > 0$, for $n = 1, 2, 3, \dots$. Specifically, we consider a birth-death process with birth and death rates λ_n and μ_n, satisfying the conditions

$$\begin{aligned} \lambda_n + \mu_n &= 1, \\ \lambda_{n-1}\mu_n &= \beta, \qquad n = 1, 2, 3, \dots, \\ \text{i.e.,} \quad (1 - \mu_{n-1})\mu_n &= \beta, \qquad n = 1, 2, 3, \dots, \end{aligned} \tag{4}$$

with $\lambda_0 = 1$ and $\mu_0 = 0$ so that $\{\mu_n\}$ is the minimal parameter sequence for the constant term chain sequence $\{\beta, \beta, \beta, \dots\}$, and hence, by Theorem 1.1, $0 < \beta \leq 1/4$ so that $\lambda_n, \mu_n > 0$, $n = 1, 2, 3, \dots$. In the following result, we find the explicit expressions for the birth and death rates, which are defined recursively in (4).

THEOREM 2.1. *For the process $\mathcal{X}$ with birth and death rates satisfying (4),*

$$\begin{aligned} \lambda_n &= \frac{\alpha U_{n+1}(1/\alpha)}{2U_n(1/\alpha)}, \qquad n = 0, 1, 2, \dots, \\ \mu_n &= \frac{\alpha U_{n-1}(1/\alpha)}{2U_n(1/\alpha)}, \qquad n = 1, 2, 3, \dots, \end{aligned} \tag{5}$$

where $U_n(\cdot)$ is the Chebyshev polynomial of second kind of order n and $\alpha = 2\sqrt{\beta}$.

PROOF. Since $\{\mu_n\}_{n=0}^{\infty}$ is the minimal parameter sequence of the constant term chain sequence $\{\beta, \beta, \beta, \dots\}$, by Theorem 1.2 we have

$$\begin{aligned} \mu_n &= \frac{1 - \sqrt{1 - 4\beta}}{2} \left[1 - \left\{ \sum_{r=0}^{n} \left(\frac{1 - \sqrt{1 - 4\beta}}{1 + \sqrt{1 - 4\beta}} \right)^r \right\}^{-1} \right], \qquad n = 0, 1, 2, \dots \\ &= \left\{ \frac{2}{1 + \sqrt{1 - 4\beta}} \left[\frac{1 + A + A^2 + \cdots + A^n}{1 + A + A^2 + \cdots + A^{n-1}} \right] \right\}^{-1}, \end{aligned}$$

where

$$A = \frac{1 + \sqrt{1 - 4\beta}}{1 - \sqrt{1 - 4\beta}},$$

so that

$$\frac{1}{\mu_1 \mu_2 \dots \mu_n} = \left(\frac{2}{1 + \sqrt{1 - 4\beta}} \right)^n \left[1 + A + A^2 + \cdots + A^n \right].$$

Hence,

$$\begin{aligned} \lambda_0 \lambda_1 \dots \lambda_{n-1} &= \left(\frac{2\beta}{1 + \sqrt{1 - 4\beta}} \right)^n \left[1 + A + A^2 + \cdots + A^n \right] \\ &= \frac{\left(1 + \sqrt{1 - \alpha^2}\right)^{n+1} - \left(1 - \sqrt{1 - \alpha^2}\right)^{n+1}}{2^{n+1}\sqrt{1 - \alpha^2}}, \end{aligned}$$

where $\alpha^2 = 4\beta$. Since $U_n(x)$ can be written as

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right],$$

we have

$$\lambda_0\lambda_1\dots\lambda_{n-1} = \frac{\alpha^n}{2^n}U_n\left(\frac{1}{\alpha}\right).$$

Therefore,

$$\lambda_n = \frac{\alpha U_{n+1}(1/\alpha)}{2U_n(1/\alpha)}, \qquad n = 0,1,2,\dots,$$

and hence,

$$\mu_n = \frac{\alpha U_{n-1}(1/\alpha)}{2U_n(1/\alpha)}, \qquad n = 1,2,3,\dots. \qquad \blacksquare$$

For notational convenience, we use U_n instead of $U_n(1/\alpha)$ throughout the analysis.

Now we find an explicit expression for the transition probabilities when there are $m(\geq 0)$ individuals in the system initially.

THEOREM 2.2. *For the process $\mathcal{X}$, whose birth and death rates are given by (5), with $X(0) = m$, the transition probabilities are given by*

$$P_n(t) = \frac{U_n}{U_m}[I_{n-m}(\alpha t) - I_{n+m+2}(\alpha t)]e^{-t}, \qquad n \in \mathcal{N},$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind of order n.

PROOF. Define

$$Q(s,t) = \sum_{n=-\infty}^{\infty} q_n(t)s^n,$$

where

$$q_n(t) = \begin{cases} \dfrac{P_n(t)}{U_n}e^t, & n = 0,1,\dots, \\ 0, & n = -1,-2,\dots. \end{cases}$$

Then the system of equations (3), after substituting for λ_n and μ_n given by (5), yields

$$\frac{\partial Q(s,t)}{\partial t} = \sqrt{\beta}\left(s+\frac{1}{s}\right)Q(s,t) - \frac{\sqrt{\beta}}{s}q_0(t), \tag{6}$$

with $Q(s,0) = s^m/U_m$.

The solution of this differential equation is easily obtained as

$$Q(s,t) = Q(s,0)e^{\{\sqrt{\beta}(s+1/s)t\}} - \int_0^t \frac{\sqrt{\beta}}{s}q_0\xi e^{\{\sqrt{\beta}(s+1/s)(t-\xi)\}}\,d\xi. \tag{7}$$

It is known that if $\alpha = 2\sqrt{\beta}$, then

$$e^{\{\sqrt{\beta}(s+1/s)t\}} = \sum_{n=-\infty}^{\infty} s^n I_n(\alpha t).$$

Using this in (7) and comparing the coefficients of s^n on both sides, we get

$$q_n(t) = \frac{I_{n-m}(\alpha t)}{U_m} - \int_0^t \sqrt{\beta}q_0(\xi)I_{n+1}(\alpha(t-\xi))\,d\xi, \qquad n \in \mathcal{N}. \tag{8}$$

Comparing the coefficients of s^{-n-2} on both sides of (6) and using $I_n(\cdot) = I_{-n}(\cdot)$, we get

$$0 = \frac{I_{n+m+2}(\alpha t)}{U_m} - \int_0^t \sqrt{\beta} q_0(\xi) I_{n+1}(\alpha(t-\xi))\, d\xi. \tag{9}$$

Subtracting (9) from (8), we obtain

$$q_n(t) = \frac{1}{U_m}\left[I_{n-m}(\alpha t) - I_{n+m+2}(\alpha t)\right], \qquad n \in \mathcal{N}.$$

Hence,

$$P_n(t) = \frac{U_n}{U_m}[I_{n-m}(\alpha t) - I_{n+m+2}(\alpha t)]e^{-t}, \qquad n \in \mathcal{N}.$$ ∎

In the following theorem, we obtain an explicit expression for mean number of units in the system at time t.

THEOREM 2.3. *For the process $\mathcal{X}$, whose birth and death rates are given by (5), with $X(0) = m$, the mean population size at time t is given by*

$$E[X(t)] = m + t - \frac{\alpha U_{m-1}}{U_m} t - \frac{\alpha(m+1)}{U_m} t - \frac{\alpha(m+1)}{U_m} \int_0^t u I_{m+1}(\alpha(t-u)) e^{-(t-u)}\, du, \qquad t \geq 0.$$

PROOF. By Theorem 2.2 we have

$$P_n(t) = \frac{\alpha U_n}{U_m}[I_{n-m}(\alpha t) - I_{n+m+2}(\alpha t)]e^{-t}, \qquad n \in \mathcal{N}.$$

Let us denote the Laplace transform of $P_n(t)$ by $\hat{P}_n(s)$ and define

$$G(\xi, s) \equiv \sum_{n=0}^{\infty} \hat{P}_n(s)\xi^n.$$

After considerable simplifications, we get the following expression for $G(\xi, s)$:

$$G(\xi,s) = \frac{1}{U_m\sqrt{p^2-\alpha^2}}\left[\frac{B^m}{\alpha^m} + \frac{\alpha}{B}U_{m-1}\xi^{m+1} - U_m\xi^m\right]\left(\frac{B^2}{B^2 - 2\xi B + \xi^2\alpha^2}\right)$$
$$- \frac{1}{U_m\sqrt{p^2-\alpha^2}}\left[\frac{B^{m+2}}{\alpha^{m+2}} + \frac{B}{\alpha}U_{m-1}\xi^{m+1} - U_m\xi^m\right]\left(\frac{\alpha^2}{\alpha^2 - 2\xi B + B^2\xi^2}\right),$$

where $p = s + 1$ and $B = [p - \sqrt{p^2 - \alpha^2}]$. We have achieved the above expression by using the generating function of $U_n(\cdot)$, given by

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1 - 2tx - t^2},$$

and the Laplace transform of $I_n(\alpha t)$, given by

$$\frac{\left[s - \sqrt{s^2-\alpha^2}\right]^a}{a^n\sqrt{s^2-\alpha^2}}.$$

Therefore,

$$\hat{m}(p) = \left.\frac{dG}{d\xi}\right|_{\xi=1} = \frac{1}{U_m\sqrt{p^2-\alpha^2}\,U_m\sqrt{p^2-\alpha^2}}\left[\frac{(m+1)\alpha}{B}U_{m-1} - mU_m\right]\left(\frac{B^2}{B^2 - 2B + \alpha^2}\right)$$
$$+ \frac{1}{U_m\sqrt{p^2-\alpha^2}}\left[\frac{B^m}{\alpha^m} + \frac{\alpha}{B}U_{m-1} - U_m\right]\left(\frac{2B^2(B-\alpha^2)}{(B^2 - 2B + \alpha^2)^2}\right)$$
$$- \frac{1}{U_m\sqrt{p^2-\alpha^2}}\left[\frac{(m+1)B}{\alpha}U_{m-1} - mU_m\right]\left(\frac{\alpha^2}{\alpha^2 - 2B + B^2}\right)$$
$$+ \frac{1}{U_m\sqrt{p^2-\alpha^2}}\left[\frac{B^{m+2}}{\alpha^{m+2}} + \frac{B}{\alpha}U_{m-1} - U_m\right]\left(\frac{2\alpha^2 B(1-B)}{(B^2 - 2B + \alpha^2)^2}\right).$$

After considerable simplifications we get

$$\hat{m}(p) = \frac{m}{p-1} + \frac{1}{(p-1)^2} - \frac{1}{(p-1)^2}\frac{\alpha U_{m-1}}{U_m} - \frac{\left[p - \sqrt{p^2-\alpha^2}\right]^{m+1}}{(p-1)^2}\frac{1}{\alpha^m U_m}.$$

That is,

$$\hat{m}(s) = \frac{m}{s} + \frac{1}{s^2} - \frac{1}{s^2}\frac{\alpha U_{m-1}}{U_m} - \frac{\left[(s+1) - \sqrt{(s+1)^2-\alpha^2}\right]^{m+1}}{s^2}\frac{1}{\alpha^m U_m}.$$

On inverting, we obtain

$$E[X(t)] = m + t - \frac{\alpha U_{m-1}}{U_m}t - \frac{\alpha(m+1)}{U_m}t - \frac{\alpha(m+1)}{U_m}\int_0^t u I_{m+1}(\alpha(t-u))e^{-(t-u)}\,du. \quad \blacksquare$$

In the next section, we analyse the discrete analogue of the above process and find explicit expressions for the transition probabilities and the population mean.

3. BIRTH-DEATH PROCESS WITH DISCRETE PARAMETER SPACE

In this section, we consider a discrete birth-death process $\tilde{\mathcal{X}} = \{X_n,\ n = 0,1,2,\dots\}$, say, whose state space is $\mathcal{N}$ and during any time slot births occur according to a Bernoulli process with probability of a birth being λ_j and deaths occur according to geometric distribution with probability of a death being μ_j when the population has size j [8]. These assumptions are different from the one used in [9]. In our process λ_n, μ_n are given by (5) and X_n denotes the population size at discrete time epoch n. Define

$$P_{m,j}(n) \equiv \Pr(X_n = j \mid X_0 = m), \qquad m,j \in \mathcal{N}, \quad n = 0,1,2\dots$$

the conditional probability that the population has size j during the time slot n given its size was m during the slot 0. For brevity, let $P_j(n)$ denote $P_{m,j}(n)$. Then these probabilities satisfy

$$\begin{aligned}
P_0(n+1) &= (1-\lambda_0)P_0(n) + \mu_1 P_1(n),\\
P_j(n+1) &= \lambda_{j-1}P_{j-1}(n) + (1-\lambda_j-\mu_j)P_j(n) + \mu_{j+1}P_{j+1}(n),\\
&\qquad j = 1,2,\dots,n+m-1,\\
P_{n+m}(n+1) &= \lambda_{n+m-1}P_{n+m-1}(n) + (1-\lambda_{n+m}-\mu_{n+m})P_{n+m}(n),\\
P_{n+m+1}(n+1) &= \lambda_{n+m}P_{n+m}(n),\\
P_j(n+1) &= 0, \qquad \text{for } n+1 < |j-m|.
\end{aligned} \tag{10}$$

In the following theorem, we give explicit expressions for the transition probabilities obtained by using a similar analysis as done in Theorem 2.2.

THEOREM 3.1. *For the process $\tilde{\mathcal{X}}$, whose birth and death rates are given by (5), with $X_0 = m$, the transition probabilities are given by the following.*

For $n = 0,1,2,\dots$, if $n-m$ is even,

$$P_{2j}(n) = \begin{cases} \dfrac{U_{2j}}{U_m}\left(\dfrac{\alpha}{2}\right)^n \left[\dbinom{n}{\frac{n-m}{2}+j} - \dbinom{n}{\frac{n-m}{2}-j-1}\right], & j = 0,1,\dots,\dfrac{n-m}{2}-1,\\[2ex] \dfrac{U_{2j}}{U_m}\left(\dfrac{\alpha}{2}\right)^n \dbinom{n}{\frac{n-m}{2}+j}, & j = \dfrac{n-m}{2},\dots,\dfrac{n+m}{2}, \end{cases}$$

and if $n-m$ *is odd,*

$$P_{2j+1}(n) = \begin{cases} \frac{U_{2j+1}}{U_m}\left(\frac{\alpha}{2}\right)^n \left[\binom{n}{\frac{n-m+1}{2}+j} - \binom{n}{\frac{n-m-1}{2}-j-1}\right], & j = 0,1,\dots,\frac{n-m-1}{2}-1, \\ \frac{U_{2j+1}}{U_m}\left(\frac{\alpha}{2}\right)^n \binom{n}{\frac{n-m+1}{2}+j}, & j = \frac{n-m-1}{2},\dots,\frac{n+m-1}{2}. \end{cases}$$

In the following theorem, we derive explicit expressions for the mean population when there are $m(\geq 0)$ individuals in the population initially.

THEOREM 3.2. *For the process* $\tilde{\mathcal{X}}$, *whose birth and death rates are given by (5), with* $X_0 = m$, *the population mean is given by*

$$M_m(n) = \begin{cases} B - \frac{\alpha^n}{2^n U_m} \sum_{k=0}^{(n-m-1)/2-1} \binom{n}{k} (2k+m+1-n) U_{2k+m-1-n}, & \text{if } n-m \text{ is odd}, \\ B - \frac{\alpha^n}{2^{n-1} U_m} \sum_{k=0}^{(n-m)/2-1} \binom{n}{k} (2k+m+1-n) U_{2k+m-n}, & \text{if } n-m \text{ is even}, \end{cases}$$

where $B = m - n + n\alpha + n\alpha(U_{m+1}/U_m)$.

PROOF. We prove the theorem for the case when $n-m$ is odd. A similar argument will hold for the case when $n-m$ is even.

Define

$$G_o(s,n) \equiv \sum_{j=0}^{(n+m-1)/2} P_{2j+1}(n) s^{2j+1}.$$

Substituting for P_{2j+1} from the previous theorem and doing some algebraic calculations we get

$$G_o(s,n) = D\left\{\sum_{j=0}^{(n+m-1)/2} \binom{n}{\frac{n-m+1}{2}+j} U_{2j+1}s^{2j+1} - \sum_{j=0}^{(n-m-1)/2-1} \binom{n}{\frac{n-m-1}{2}-j-1} U_{2j+1}s^{2j+1}\right\},$$

where $D = \beta^{n/2}/U_m$. Let $n-m = 2k+1$, then

$$\begin{aligned} G_o(s,n) &= D\left\{\sum_{j=0}^{k+m} \binom{2k+m+1}{k+j+1} U_{2j+1}s^{2j+1} - \sum_{j=0}^{k-1} \binom{2k+m+1}{k-j-1} U_{2j+1}s^{2j+1}\right\} \\ &= \frac{D}{\sin\theta}\operatorname{Im}\left\{\sum_{j=0}^{k+m} \binom{2k+m+1}{k+j+1} \Theta^{2j+2}s^{2j+1} - \sum_{j=0}^{k-1} \binom{2k+m+1}{k-j-1} \Theta^{2j+2}s^{2j+1}\right\}, \end{aligned}$$

where $\operatorname{Im}\{x\}$ stands for the imaginary part of x, $1/\alpha = \cos\theta$, and $\Theta = e^{i\theta}$. After considerable simplifications, we get

$$\begin{aligned} G_o(s,n) = {} & \frac{D}{\sin\theta}\operatorname{Im}\left\{\Theta^{-2m}s^{-2m-1}\left(1+\Theta^2 s^2\right)^{2k+m+1}\binom{2k+m+1}{k}\frac{1}{s}\right\} \\ & - \frac{D}{\sin\theta}\operatorname{Im}\left\{\sum_{l=0}^{k-1}\binom{2k+m+1}{l}\frac{1}{s}\right\}\left\{\Theta^{2(l-k)}s^{2(l-k)-1} + \left(\Theta^{2(l-k)}s^{2(l-k)-1}\right)^{-1}\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} M_m(n) &= \left.\frac{dG_o}{ds}\right|_{s=1} \\ &= \frac{D}{\sin\theta}\left\{(-2k-1)(\sin(m+1)\theta)\left(2\cos\theta\right)^{2k+m+1}\right. \\ &\quad \left. + 2(2k+m+1)\sin((m+2)\theta)\left(2\cos\theta\right)^{2k+m}\right\} \\ &\quad - \frac{2D}{\sin\theta}\left\{\sum_{l=0}^{k-1}\binom{2k+m+1}{l}(l-k)(\sin(2(l-k)\theta)\right\}. \end{aligned}$$

Using $U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta$, we get

$$M_m(n) = \frac{D}{\sin\theta}\left\{(-2k-1)\left(\frac{2}{\alpha}\right)^{2k+m+1} U_m + 2(2k+m+1)\left(\frac{2}{\alpha}\right)^{2k+m} U_{m+1}\right\} - 2D\left\{\sum_{l=0}^{k-1}\binom{2k+m+1}{l}(l-k)U_{2(l-k)-1}\right\}.$$

Substituting $2k = n-m-1$, we get, for $n-m$ odd,

$$M_m(n) = B - \frac{\alpha^n}{2^n U_m}\sum_{j=0}^{(n-m-1)/2-1}\binom{n}{j}(2j+m+1-n)U_{2j+m-1-n}. \qquad \blacksquare$$

4. TRANSIENT PROCESSES

In this section, we will show that the processes studied in the last two sections are transient.

THEOREM 4.1. *The processes $\mathcal{X}$ and $\tilde{\mathcal{X}}$ under consideration are transient.*

PROOF. We introduce the following notations:

$$A = \sum_{n=0}^{\infty}\frac{1}{\lambda_n\pi_n}, \qquad B = \sum_{n=0}^{\infty}\pi_n, \qquad C = \sum_{n=0}^{\infty}\frac{1}{\lambda_n\pi_n}\sum_{i=0}^{n}\pi_i.$$

The process will be nonexplosive if and only if $C = \infty$, in which case there is a unique birth-death process with the given transition rates. This process is recurrent if and only if $A = \infty$, and then positive recurrent if and only if $B < \infty$. More details about these quantities can be had from [10]. In our case, the rates λ_n are bounded above, and since the series C represents the expected passage time of the process from 0 to ∞, it follows that $C = \infty$. Therefore, we conclude that the process is nonexplosive.

For $0 < \beta \le 1/4$,

$$B = \sum_{n=0}^{\infty}\pi_n,$$

where

$$\pi_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} = U_n^2 \qquad \text{(using (5))}.$$

It follows that $\pi_n \to \infty$, hence $B = \infty$ and positive recurrent is impossible. In particular, for $\beta = 1/4$ we have $\alpha = 1$ and

$$B = \sum_{n=0}^{\infty}(n+1)^2 = \infty.$$

Now,

$$A = \sum_{n=0}^{\infty}\frac{1}{\lambda_n\pi_n} = \frac{2}{\alpha}\sum_{n=0}^{\infty}\frac{1}{U_nU_{n+1}} \qquad \text{(using (5))}$$

$$= \frac{4}{\alpha^2}\sum_{n=0}^{\infty}\frac{\lambda_n}{U_{n+1}^2} \le \sum_{n=0}^{\infty}\frac{1}{U_{n+1}^2} \qquad (\text{since } \lambda_n \le 1,\ \forall n).$$

It follows that $A < \infty$, so null-recurrence is also impossible. In particular, for $\beta = 1/4$ we have $\alpha = 1$ and

$$A \le \sum_{n=0}^{\infty}\frac{1}{(n+2)^2},$$

hence $A < \infty$. Hence, the processes $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are transient. Therefore, the stationary distribution does not exist. $\blacksquare$

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