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BIFURCATION AND MULTIPLICITY RESULTS FOR A CLASS OF $n \times n$ p-LAPLACIAN SYSTEM

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Abstract. In this paper we study the positive solutions to the $n \times n$ p-Laplacian system:

$$\begin{cases} -\left(\varphi_{p_1}(u_1')\right)' = \lambda h_1(t)\left(u_1^{p_1-1-\alpha_1} + f_1(u_2)\right), & t \in (0,1), \\ -\left(\varphi_{p_2}(u_2')\right)' = \lambda h_2(t)\left(u_2^{p_2-1-\alpha_2} + f_2(u_3)\right), & t \in (0,1), \\ \quad\vdots \qquad = \qquad \vdots & \\ -\left(\varphi_{p_n}(u_n')\right)' = \lambda h_n(t)\left(u_n^{p_n-1-\alpha_n} + f_n(u_1)\right), & t \in (0,1), \\ \quad u_j(0) = 0 = u_j(1);\ j = 1,2,\dots,n, \end{cases}$$

where λ is a positive parameter, $p_j > 1$, $\alpha_j \in (0, p_j - 1)$, $\varphi_{p_j}(w) = |w|^{p_j-2}w$, and $h_j \in C((0,1),(0,\infty)) \cap L^1((0,1),(0,\infty))$ for $j = 1,2,\dots,n$. Here $f_j : [0,\infty) \to [0,\infty)$, $j = 1,2,\dots,n$ are nontrivial nondecreasing continuous functions with $f_j(0) = 0$ and satisfy a combined sublinear condition at infinity. We discuss here a bifurcation result, an existence result for $\lambda > 0$, and a multiplicity result for a certain range of λ. We establish our results through the method of sub-super solutions.

1. Introduction.

Study of positive solutions to the 2×2 system:

$$\begin{cases} -\Delta_{p_1} u_1 = \lambda\left(u_1^{p_1-1-\alpha_1} + f_1(u_2)\right),\ x \in \Omega, \\ -\Delta_{p_2} u_2 = \lambda\left(u_2^{p_2-1-\alpha_2} + f_2(u_1)\right),\ x \in \Omega, \\ \quad u_j = 0,\ x \in \partial\Omega;\ j = 1,2 \end{cases} \tag{1}$$

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was discussed in [5] where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary $\partial\Omega$, $\lambda > 0$, $p_j > 1$, $\alpha_j \in (0, p_j - 1)$, $j = 1,2$, and $\Delta_m w := \mathrm{div}(|\nabla w|^{m-2}\nabla w)$, $m > 1$ is the m-Laplacian operator of w. Assuming $f_j : [0,\infty) \to [0,\infty)$, $j = 1,2$ are nondecreasing continuous functions with $f_j(0) = 0$, it was established that for $\lambda \approx 0$ there exist positive solutions of (1) bifurcating from the trivial branch $(\lambda, u_1 \equiv 0, u_2 \equiv 0)$ at $(0,0,0)$. Further, under additional assumptions on f_j for $j = 1,2$, the existence result for all $\lambda > 0$ and a multiplicity result for a certain range of λ were proven.

Extending the above study to domains exterior of a ball and to $n \times n$ systems, we encounter systems of the form:

$$\begin{cases} -\Delta_{p_1} u_1 = \lambda K_1(|x|)\left(u_1^{p_1-1-\alpha_1} + f_1(u_2)\right), \ x \in B_E, \\ -\Delta_{p_2} u_2 = \lambda K_2(|x|)\left(u_2^{p_2-1-\alpha_2} + f_2(u_3)\right), \ x \in B_E, \\ \quad \vdots \quad = \quad \vdots \\ -\Delta_{p_n} u_n = \lambda K_n(|x|)\left(u_n^{p_n-1-\alpha_n} + f_n(u_1)\right), \ x \in B_E, \\ \quad u_j(x) = 0 \text{ on } |x| = r_0;\ j = 1,2,\dots,n, \\ \quad u_j(x) \to 0 \text{ as } |x| \to \infty;\ j = 1,2,\dots,n, \end{cases} \tag{2}$$

where $B_E := \{x \in \mathbb{R}^N \| x| > r_0 > 0\}$, $p_j > 1$, $\alpha_j \in (0, p_j - 1)$, $f_j : [0,\infty) \to [0,\infty)$ are nontrivial nondecreasing continuous functions with $f_j(0) = 0$, and $K_j \in C([r_0,\infty),(0,\infty))$ are class of functions that satisfy $K_j(|x|) \to 0$ as $|x| \to \infty$ for $j = 1,2,\dots,n$. Restricting the analysis to radial solutions and to the case $p_1 = p_2 = \cdots = p_n = p$ where $1 < p < N$, by the Kelvin type transformation, $r = |x|$ and $t = \left(\frac{r}{r_0}\right)^{\frac{N-p}{1-p}}$, (2) reduces to

$$\begin{cases} -\left(\varphi_p(u_1')\right)' = \lambda \tilde{h}_1(t)\left(u_1^{p-1-\alpha_1} + f_1(u_2)\right), \ t \in (0,1), \\ -\left(\varphi_p(u_2')\right)' = \lambda \tilde{h}_2(t)\left(u_2^{p-1-\alpha_2} + f_2(u_3)\right), \ t \in (0,1), \\ \quad \vdots \quad = \quad \vdots \\ -\left(\varphi_p(u_n')\right)' = \lambda \tilde{h}_n(t)\left(u_n^{p-1-\alpha_n} + f_n(u_1)\right), \ t \in (0,1), \\ \quad u_j(0) = 0 = u_j(1);\ j = 1,2,\dots,n, \end{cases}$$

where $\tilde{h}_j(t) := \left(\frac{p-1}{N-p}\right)^p r_0^p \, t^{\frac{p(1-N)}{N-p}} K_j\left(r_0 t^{\frac{1-p}{N-p}}\right)$ for $j = 1,2,\dots,n$. Clearly, $\tilde{h}_j \in C((0,1],(0,\infty))$ for $j = 1,2,\dots,n$. If we assume that $K_j(r) \leq \frac{1}{r^{N+\sigma}}$ for $r \gg 1$ and $\sigma > 0$, then $\tilde{h}_j(t) \to \infty$ as $t \to 0+$ for $j = 1,2,\dots,n$. However, $\tilde{h}_j \in L^1((0,1],(0,\infty))$ for $j = 1,2,\dots,n$.

Motivated by the aforementioned observations, in this paper, we study the positive solutions to a more general singular $n \times n$ system:

$$\begin{cases} -\left(\varphi_{p_1}(u_1')\right)' = \lambda h_1(t)\left(u_1^{p_1-1-\alpha_1} + f_1(u_2)\right), \ t \in (0,1), \\ -\left(\varphi_{p_2}(u_2')\right)' = \lambda h_2(t)\left(u_2^{p_2-1-\alpha_2} + f_2(u_3)\right), \ t \in (0,1), \\ \quad \vdots \quad = \quad \vdots \\ -\left(\varphi_{p_n}(u_n')\right)' = \lambda h_n(t)\left(u_n^{p_n-1-\alpha_n} + f_n(u_1)\right), \ t \in (0,1), \\ \quad u_j(0) = 0 = u_j(1);\ j = 1,2,\dots,n, \end{cases} \tag{3}$$

where $p_j > 1$, $\alpha_j \in (0, p_j - 1)$, $\varphi_{p_j}(w) = |w|^{p_j-2}w$, and $h_j \in C((0,1),(0,\infty)) \cap L^1((0,1),(0,\infty))$ for $j = 1,2,\ldots,n$. Here $f_j : [0,\infty) \to [0,\infty)$, $j = 1,2,\ldots,n$ are nontrivial nondecreasing continuous functions with $f_j(0) = 0$. By a positive solution $\underline{u} = (u_1, u_2, \ldots, u_n)$ we mean $u_j \in C^1[0,1]$ with $u_j > 0$ on $(0,1)$ for $j = 1,2,\ldots,n$. We first establish a bifurcation result at $(0,\underline{0})$ from the trivial branch $(\lambda, \underline{u} \equiv \underline{0})$. We prove:

Theorem 1.1. *There exists $\lambda_0 > 0$ such that for all $\lambda \in (0,\lambda_0)$, (3) has a positive solution $\underline{u} = (u_1, u_2 \ldots, u_n)$ and $\|u_j\|_\infty \to 0$ as $\lambda \to 0$ for all $j = 1,2,\ldots,n$ (see figure 1).*

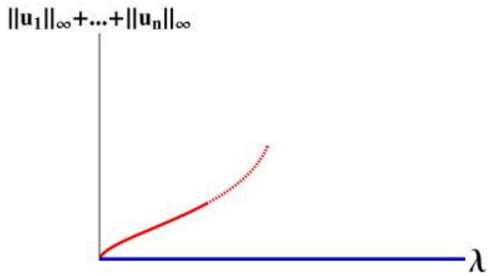

FIGURE 1. Bifurcation of solution from the origin.

Assuming a combined sublinear condition at infinity:

$$(H_1) \quad \lim_{s\to\infty} \frac{[f_1^{[M]} \circ f_2^{[M]} \circ \cdots \circ f_{n-1}^{[M]} \circ (f_n(s))^{\frac{1}{p_n-1}}]^{p_1-1}}{s^{p_1-1}} = 0 \text{ for every } M > 0,$$

where $f_j^{[M]}(s) := f_j(Ms)^{\frac{1}{p_j-1}}$ for $j = 1,2,\ldots,n$, we establish:

Theorem 1.2. *Assume (H_1) holds. Then (3) has a positive solution for all $\lambda > 0$ (see figure 2).*

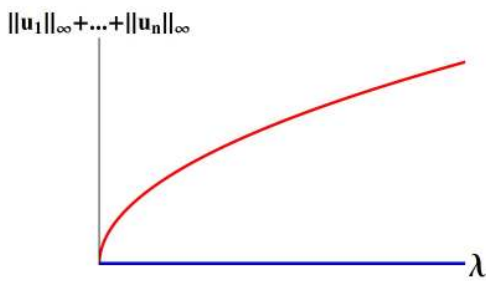

FIGURE 2. Bifurcation for all $\lambda > 0$.

Next let $h^*(t) := \max_{j=1,2,\ldots,n}\{h_j(t)\}$, $h_*(t) := \min_{j=1,2,\ldots,n}\{h_j(t)\}$, $\underline{h}_* := \inf_{t\in(0,1)} h_*(t)$, $L_{ij} := \frac{(2p_i)^{p_j}}{(p_i-1)^{p_j-1}}$ and $w_{p_j} \in C^1[0,1]$ be the unique solution of boundary value problem:

$$\begin{cases} -(\varphi_{p_j}(w'))' = h^*(t), \ t \in (0,1), \\ \quad w(0) = 0 = w(1) \end{cases}$$

(see [3]). Now if $\underline{h}_* > 0$ and f_j satisfy:
(H_2) there exist positive constant a and b $(> a)$ such that

$$\min_{j=1,2,\dots,n}\left\{\frac{1}{2\|w_{p_j}\|_\infty^{p_j-1}}\min\left\{a^{\alpha_j},\frac{a^{p_j-1}}{f_j(a)}\right\}\right\} > \min_{i=1,2,\dots,n}\left\{\max_{j=1,2,\dots,n}\left\{L_{ij}\frac{b^{p_j-1}}{\underline{h}_* f_j(b)}\right\}\right\},$$

then we prove:

Theorem 1.3. *Assume $\underline{h}_* > 0$ and $(H_1)-(H_2)$ hold. Then (3) has at least three positive solutions for $\lambda \in (\lambda_*, \lambda^*)$ where*

$$\lambda_* := \min_{i=1,2,\dots,n}\left\{\max_{j=1,2,\dots,n}\left\{L_{ij}\frac{b^{p_j-1}}{\underline{h}_* f_j(b)}\right\}\right\}$$

and

$$\lambda^* := \min_{j=1,2,\dots,n}\left\{\frac{1}{2\|w_{p_j}\|_\infty^{p_j-1}}\min\left\{a^{\alpha_j},\frac{a^{p_j-1}}{f_j(a)}\right\}\right\}$$

(see figure 3).

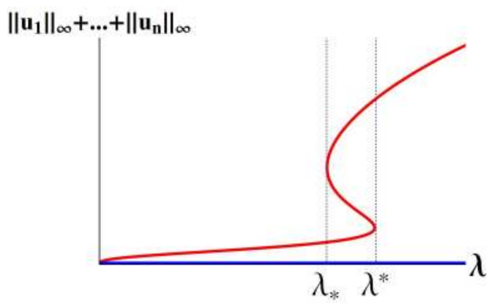

FIGURE 3. Multiplicity results for certain range of λ.

We establish Theorems 1.1 - 1.3 by the method of sub-super solution. By a subsolution of (3) we mean a function $(\psi_1, \psi_2, \dots, \psi_n) \in C^1[0,1]\times C^1[0,1]\times\cdots\times C^1[0,1]$ such that $\psi_j(0) = 0 = \psi_j(1)$ for $j = 1, 2, \dots, n$ and

$$\int_0^1 \varphi_{p_1}(\psi_1'(s))\zeta'(s)ds \le \int_0^1 \lambda h_1(s)\left(\psi_1^{p_1-1-\alpha_1}(s) + f_1(\psi_2(s))\right)\zeta(s)ds \text{ for all } \zeta \in W,$$
$$\int_0^1 \varphi_{p_2}(\psi_2'(s))\zeta'(s)ds \le \int_0^1 \lambda h_2(s)\left(\psi_2^{p_2-1-\alpha_2}(s) + f_2(\psi_3(s))\right)\zeta(s)ds \text{ for all } \zeta \in W,$$
$$\vdots \qquad \le \qquad \vdots$$
$$\int_0^1 \varphi_{p_n}(\psi_n'(s))\zeta'(s)ds \le \int_0^1 \lambda h_n(s)\left(\psi_n^{p_n-1-\alpha_n}(s) + f_n(\psi_1(s))\right)\zeta(s)ds \text{ for all } \zeta \in W.$$

By a supersolution of (3) we mean a function $(\phi_1, \phi_2, \dots, \phi_n) \in C^1[0,1]\times C^1[0,1]\times\cdots\times C^1[0,1]$ such that $\phi_j(0) = 0 = \phi_j(1)$ for $j = 1, 2, \dots, n$ and

$$\int_0^1 \varphi_{p_1}(\phi_1'(s))\zeta'(s)ds \ge \int_0^1 \lambda h_1(s)\left(\phi_1^{p_1-1-\alpha_1}(s) + f_1(\phi_2(s))\right)\zeta(s)ds \text{ for all } \zeta \in W,$$
$$\int_0^1 \varphi_{p_2}(\phi_2'(s))\zeta'(s)ds \ge \int_0^1 \lambda h_2(s)\left(\phi_2^{p_2-1-\alpha_2}(s) + f_2(\phi_3(s))\right)\zeta(s)ds \text{ for all } \zeta \in W,$$

$$\vdots \qquad \geq \qquad \vdots$$

$$\int_0^1 \varphi_{p_n}(\phi_n'(s))\zeta'(s)ds \geq \int_0^1 \lambda h_n(s)\left(\phi_n^{p_n-1-\alpha_n}(s) + f_n(\phi_1(s))\right)\zeta(s)ds \text{ for all } \zeta \in W,$$

where $W := \{h \in C_0^\infty(0,1) | h \geq 0 \text{ in } (0,1)\}$. By a strict subsolution of (3) we mean a subsolution which is not a solution. By a strict supersolution of (3) we mean a supersolution which is not a solution. Then the results in [1] and [4] can be extended to such singular systems and the following lemmas hold:

Lemma 1.4. *Let $(\psi_1, \psi_2, \ldots, \psi_n)$ be a subsolution and $(\phi_1, \phi_2, \ldots, \phi_n)$ be a supersolution of (3). If $\psi_j \leq \phi_j$ for $j = 1, 2, \ldots, n$, then (3) has at least one solution $(u_1, u_2, \ldots, u_n)$ such that $u_j \in C^1[0,1]$ and $\psi_j \leq u_j \leq \phi_j$ for $j = 1, 2, \ldots, n$.*

Lemma 1.5. *Let f_j be nonnegative and nondecreasing for $j = 1, 2, \ldots, n$, and suppose there exist a subsolution $(\psi_1, \psi_2, \ldots, \psi_n)$, a strict subsolution $(\bar\psi_1, \bar\psi_2, \ldots, \bar\psi_n)$, a strict supersolution $(\bar\phi_1, \bar\phi_2, \ldots, \bar\phi_n)$ and a supersolution $(\phi_1, \phi_2, \ldots, \phi_n)$ of (3) such that $\psi_j \leq \bar\psi_j \leq \phi_j$, $\psi_j \leq \bar\phi_j \leq \phi_j$ for $j = 1, 2, \ldots, n$ and $(\bar\psi_1, \bar\psi_2, \ldots, \bar\psi_n) \not\leq (\bar\phi_1, \bar\phi_2, \ldots, \bar\phi_n)$. Then (3) has at least three distinct solutions $(u_1, u_2, \ldots, u_n)$, $(u_1^*, u_2^*, \ldots, u_n^*)$ and $(\bar u_1, \bar u_2, \ldots, \bar u_n)$ such that*

$$(u_1, u_2, \ldots, u_n) \in A := [(\psi_1, \psi_2, \ldots, \psi_n), (\bar\phi_1, \bar\phi_2, \ldots, \bar\phi_n)],$$
$$(u_1^*, u_2^*, \ldots, u_n^*) \in B := [(\bar\psi_1, \bar\psi_2, \ldots, \bar\psi_n), (\phi_1, \phi_2, \ldots, \phi_n)],$$
$$(\bar u_1, \bar u_2, \ldots, \bar u_n) \in [(\psi_1, \psi_2, \ldots, \psi_n), (\phi_1, \phi_2, \ldots, \phi_n)] \setminus (A \cup B).$$

We will establish Theorem 1.1 in Section 2 and Theorems 1.2 - 1.3 in Section 3. Finally, in Section 4, we discuss a simple example satisfying hypotheses of Theorems 1.1 - 1.3.

Remark 1. Note that the study of positive radial solution of (2) when $p_i \neq p_j$ for some $i, j \in \{1, 2, \ldots, n\}$ remains open.

2. **Proof of Theorem 1.1.** Let $\gamma > 0$ be such that $\gamma\alpha_j < 1$ and $\gamma(p_j - 1) < 1$ for $j = 1, 2, \ldots, n$. Let $\lambda_0 > 0$ be such that

$$\lambda_0^{1-\gamma\alpha_j}\|w_{p_j}\|_\infty^{p_j-1-\alpha_j} + \lambda_0^{1-\gamma(p_j-1)} f_j(\lambda_0^\gamma \|w_{p_{j+1}}\|_\infty) < 1 \text{ for } j = 1, 2, \ldots, n-1 \quad (4)$$

and

$$\lambda_0^{1-\gamma\alpha_n}\|w_{p_n}\|_\infty^{p_n-1-\alpha_n} + \lambda_0^{1-\gamma(p_n-1)} f_n(\lambda_0^\gamma \|w_{p_1}\|_\infty) < 1. \quad (5)$$

Define $\phi_j := \lambda^\gamma w_{p_j}$ for $j = 1, 2, \ldots, n$. For $\lambda < \lambda_0$ and $j = 1, 2, \ldots, n-1$ using (4) we have

$$\begin{aligned} -(\varphi_{p_j}(\phi_j'))' &= \lambda^{\gamma(p_j-1)} h^*(t) \\ &\geq \lambda^{\gamma(p_j-1)} h^*(t)\left(\lambda_0^{1-\gamma\alpha_j}\|w_{p_j}\|_\infty^{p_j-1-\alpha_j} + \lambda_0^{1-\gamma(p_j-1)} f_j(\lambda_0^\gamma\|w_{p_{j+1}}\|_\infty)\right) \\ &\geq \lambda h_j(t)\left(\left(\lambda^\gamma w_{p_j}\right)^{p_j-1-\alpha_j} + f_j\left(\lambda^\gamma w_{p_{j+1}}\right)\right) \\ &= \lambda h_j(t)\left(\phi_j^{p_j-1-\alpha_j} + f_j\left(\phi_{j+1}\right)\right). \end{aligned}$$

Similarly for $\lambda < \lambda_0$ using (5) we have

$$-(\varphi_{p_n}(\phi_n'))' \geq \lambda h_n(t)\left(\phi_n^{p_n-1-\alpha_n} + f_n(\phi_1)\right).$$

Further, $\phi_j(0) = 0 = \phi_j(1)$ for $j = 1, 2, \ldots, n$. Hence $(\phi_1, \phi_2, \ldots, \phi_n)$ is a supersolution of (3) for $\lambda < \lambda_0$.

Next given $\lambda > 0$, we construct a subsolution of (3). Let $z_{p_j} > 0$ in $(0,1)$ be the eigenfunction with $\|z_{p_j}\|_\infty = 1$ corresponding to the principal eigenvalue λ_{1,p_j} of the problem:

$$\begin{cases} -(\varphi_{p_j}(z'))' = \lambda h_*(t)|z|^{p_j-2}z, \ t \in (0,1), \\ \quad z(0) = 0 = z(1) \end{cases}$$

(see [2]). Choose $m \approx 0$ such that $\lambda_{1,p_j} m^{\alpha_j} \le \lambda$ for $j = 1, 2, \dots, n$. Define $\psi_j := m z_{p_j}$ for $j = 1, 2, \dots, n$. For $j = 1, 2, \dots, n-1$ we have

$$-(\varphi_{p_j}(\psi_j'))' \quad = \quad \lambda_{1,p_j} h_*(t)(m z_{p_j})^{p_j-1} \le \lambda h_j(t)\left(\psi_j^{p_j-1-\alpha_j} + f_j(\psi_{j+1})\right)$$

and similarly we have

$$-(\varphi_{p_n}(\psi_n'))' \le \lambda h_n(t)\left(\psi_n^{p_n-1-\alpha_n} + f_n(\psi_1)\right).$$

Further, $\psi_j(0) = 0 = \psi_j(1)$ for $j = 1, 2, \dots, n$. Hence $(\psi_1, \psi_2, \dots, \psi_n)$ is a subsolution of (3). We can also choose $m \approx 0$ such that $(\psi_1, \psi_2, \dots, \psi_n) \le (\phi_1, \phi_2, \dots, \phi_n)$ since $w_{p_j}'(0) > 0$ and $w_{p_j}'(1) < 0$ for $j = 1, 2, \dots, n$. Hence by Lemma 1.4 there exists a solution $(u_1, u_2, \dots, u_n)$ such that $(\psi_1, \psi_2, \dots, \psi_n) \le (u_1, u_2, \dots, u_n) \le (\phi_1, \phi_2, \dots, \phi_n)$. Moreover $\|u_j\|_\infty \to 0$ as $\lambda \to 0$ since $\|\phi_j\|_\infty \to 0$ as $\lambda \to 0$ for $j = 1, 2, \dots, n$. ☐

3. Proofs of Theorems 1.2 - 1.3.

3.1. Proof of Theorem 1.2.

Let $(\psi_1, \psi_2, \dots, \psi_n)$ be as in Theorem 1.1. Then $(\psi_1, \psi_2, \dots, \psi_n)$ is a subsolution of (3) for $\lambda > 0$. Next we construct a supersolution of (3). Let $M > 1$ be such that for $j = 2, 3, \dots, n$

$$\left(M f_j^{[\beta_j]} \circ f_{j+1}^{[\beta_{j+1}]} \circ \cdots \circ f_n^{[\beta_n]}\left(M\|w_{p_1}\|_\infty\right)\right)^{\alpha_j} \ge \left((2\lambda)^{\frac{1}{p_j-1}}\|w_{p_j}\|_\infty\right)^{p_j-1-\alpha_j}, \quad (6)$$

where

$$\beta_j := \begin{cases} (2\lambda)^{\frac{1}{p_{j+1}-1}} M\|w_{p_{j+1}}\|_\infty; & j = 1, 2, \dots, n-1, \\ 1; & j = n. \end{cases}$$

Let $\beta := \max_{j=1,2,\dots,n}\{\beta_j\}$. By (H_1) we can choose $M^* \gg 1$ such that $M^* > M$,

$$\frac{1}{2\lambda\|w_{p_1}\|_\infty^{p_1-1}} \ge \frac{\left(f_1^{[\beta]} \circ f_2^{[\beta]} \circ \cdots \circ f_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)\right)^{p_1-1}}{(M^*\|w_{p_1}\|_\infty)^{p_1-1}} \quad (7)$$

and

$$\frac{{M^*}^{\alpha_1}}{2} \ge \lambda\|w_{p_1}\|_\infty^{p_1-1-\alpha_1}. \quad (8)$$

From (6), we obtain

$$\left(M f_j^{[\beta]} \circ f_{j+1}^{[\beta]} \circ \cdots \circ f_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)\right)^{\alpha_j} \ge \left((2\lambda)^{\frac{1}{p_j-1}}\|w_{p_j}\|_\infty\right)^{p_j-1-\alpha_j} \quad (9)$$

since f_j are nondecreasing functions for $j = 1, 2, \dots, n$. Now we define

$$\hat{\phi}_j := \begin{cases} M^* w_{p_1}; & j = 1, \\ \left((2\lambda)^{\frac{1}{p_j-1}} M f_j^{[\beta]} \circ f_{j+1}^{[\beta]} \circ \cdots \circ f_{n-1}^{[\beta]} \circ f_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)\right) w_{p_j}; & j = 2, \dots, n. \end{cases}$$

Then using (7) we have

$$\begin{aligned}
&-(\varphi_{p_1}(\hat{\phi}_1'))' \\
&= h^*(t)\left(\frac{M^{*p_1-1}}{2}+\frac{M^{*p_1-1}}{2}\right) \\
&\geq h_1(t)\left(M^{*p_1-1-\alpha_1}\frac{M^{*\alpha_1}}{2}+\lambda\left(f_1^{[\beta]}\circ f_2^{[\beta]}\circ\cdots\circ f_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)\right)^{p_1-1}\right) \\
&\geq \lambda h_1(t)\left((M^*\|w_{p_1}\|_\infty)^{p_1-1-\alpha_1}+f_1\big((2\lambda)^{\frac{1}{p_2-1}}Mf_2^{[\beta]}\circ\cdots\circ f_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)w_2\big)\right) \\
&\geq \lambda h_1(t)\left(\hat{\phi}_1^{p_1-1-\alpha_1}+f_1(\hat{\phi}_2)\right)
\end{aligned}$$

and for $j=2,3,\ldots,n-1$ using (9) we have

$$\begin{aligned}
&-(\varphi_{p_j}(\hat{\phi}_j'))' \\
&= h^*(t)\left((2\lambda)^{\frac{1}{p_j-1}}Mf_j^{[\beta]}\circ f_{j+1}^{[\beta]}\circ\cdots\circ f_n^{[\beta]}(M^*\|w_{p_1}\|_\infty)\right)^{p_j-1} \\
&\geq \lambda h_j(t)\left(Mf_j^{[\beta]}\circ f_{j+1}^{[\beta]}\circ\cdots\circ f_n^{[\beta]}(M^*\|w_{p_1}\|_\infty)\right)^{p_j-1} \\
&\quad + \lambda h_j(t)\left(Mf_j^{[\beta]}\circ f_{j+1}^{[\beta]}\circ\cdots\circ f_n^{[\beta]}(M^*\|w_{p_1}\|_\infty)\right)^{p_j-1} \\
&\geq \lambda h_j(t)\left((2\lambda)^{\frac{1}{p_j-1}}Mf_j^{[\beta]}\circ f_{j+1}^{[\beta]}\circ\cdots\circ f_n^{[\beta]}(M^*\|w_{p_1}\|_\infty)\|w_{p_j}\|_\infty\right)^{p_j-1-\alpha_j} \\
&\quad + \lambda h_j(t)f_j\left((2\lambda)^{\frac{1}{p_{j+1}-1}}Mf_{j+1}^{[\beta]}\circ\cdots\circ f_n^{[\beta]}(M^*\|w_{p_1}\|_\infty)w_{p_{j+1}}\right) \\
&\geq \lambda h_j(t)\left(\hat{\phi}_j^{p_j-1-\alpha_j}+f_j(\hat{\phi}_{j+1})\right).
\end{aligned}$$

Similarly using (9) we have

$$\begin{aligned}
-(\varphi_{p_n}(\hat{\phi}_n'))' &= h^*(t)\left((2\lambda)^{\frac{1}{p_n-1}}Mf_n^{[\beta]}(M^*\|w_{p_1}\|_\infty)\right)^{p_n-1} \\
&\geq \lambda h_n(t)\left(Mf_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)\right)^{\alpha_n}\left(Mf_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)\right)^{p_n-1-\alpha_n} \\
&\quad + \lambda h_n(t)\left(Mf_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)\right)^{p_n-1} \\
&\geq \lambda h_n(t)\left((2\lambda)^{\frac{1}{p_n-1}}\|w_{p_n}\|_\infty\right)^{p_n-1-\alpha_n}\left(Mf_n^{[\beta]}\left(M^*\|w_{p_1}\|_\infty\right)\right)^{p_n-1-\alpha_n} \\
&\quad + \lambda h_n(t)f_n\left(M^*w_{p_1}\right) \\
&\geq \lambda h_n(t)\left(\hat{\phi}_n^{p_n-1-\alpha_n}+f_n(\hat{\phi}_1)\right).
\end{aligned}$$

Hence $(\hat{\phi}_1,\hat{\phi}_2,\ldots,\hat{\phi}_n)$ is a supersolution of (3). Since $z_{p_j}'(0)>0$ and $z_{p_j}'(1)<0$ for $j=1,2,\ldots,n$, we can again choose $M^*\gg 1$ such that $(\psi_1,\psi_2,\ldots,\psi_n)\leq(\hat{\phi}_1,\hat{\phi}_2,\ldots,\hat{\phi}_n)$. By Lemma 1.4 there exist a positive solution $(u_1,u_2,\ldots,u_n)$ such that $(\psi_1,\psi_2,\ldots,\psi_n)\leq(u_1,u_2,\ldots,u_n)\leq(\hat{\phi}_1,\hat{\phi}_2,\ldots,\hat{\phi}_n)$. □

3.2. **Proof of Theorem 1.3.** Define $\tilde{\phi}_j := \frac{a}{\|w_{p_j}\|_\infty} w_{p_j}$ for $j = 1, 2, \ldots, n$. For $\lambda < \lambda^*$ and $j = 1, 2, \ldots, n-1$ we have

$$\begin{aligned} -(\varphi_{p_j}(\tilde{\phi}_j'))' =& h^*(t) \left(\frac{a^{p_j-1}}{2\|w_{p_j}\|_\infty^{p_j-1}} + \frac{a^{p_j-1}}{2\|w_{p_j}\|_\infty^{p_j-1}} \right) \\ >& h_j(t) \left(\lambda a^{p_j-1-\alpha_j} + \lambda f_j(a) \right) \\ \geq& \lambda h_j(t) \left(\tilde{\phi}_j^{p_j-1-\alpha_j} + f_j(\tilde{\phi}_{j+1}) \right). \end{aligned}$$

Similarly for $\lambda < \lambda^*$ we have

$$-(\varphi_{p_n}(\tilde{\phi}_n'))' > \lambda h_n(t) \left(\tilde{\phi}_n^{p_n-1-\alpha_n} + f_n(\tilde{\phi}_1) \right).$$

Hence $(\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_n)$ is a strict supersolution of (3) for $\lambda < \lambda^*$.
Next we construct a strict subsolution for $\lambda > \lambda_*$. Let $\epsilon \in (0, \frac{1}{2})$ and $\kappa, \eta \in (1, \infty)$. Define $\rho : [0,1] \to [0,1]$ by

$$\rho(t) := \begin{cases} \hat{\rho}(t), & 0 \leq t \leq \frac{1}{2}, \\ \hat{\rho}(1-t), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

where

$$\hat{\rho}(t) := \begin{cases} 1 - \left(1 - \left(\frac{t}{\epsilon}\right)^\eta\right)^\kappa, & 0 \leq t \leq \epsilon, \\ 1, & \epsilon < t \leq \frac{1}{2}. \end{cases}$$

Let $d(t) = b\rho(t)$ and $\underline{h}_*$ as before. For $j = 1, 2, \ldots, n$ we define ψ_j^* as the $C^2[0, \frac{1}{2}]$ solution of the problem:

$$\begin{cases} -(\varphi_{p_j}(\psi'))' = \lambda \underline{h}_* f_j(d), \ t \in (0, \frac{1}{2}), \\ \quad \psi(0) = 0 = \psi'(\frac{1}{2}). \end{cases}$$

Now extend ψ_j^* to $[\frac{1}{2}, 1]$ as

$$\tilde{\psi}_j(t) := \begin{cases} \psi_j^*(t), & 0 \leq t \leq \frac{1}{2}, \\ \psi_j^*(1-t), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now if

$$\tilde{\psi}_j(t) > d(t), \ t \in \left(0, \frac{1}{2}\right) \tag{10}$$

then for $j = 1, 2, \ldots, n-1$ we have

$$-(\varphi_{p_j}(\tilde{\psi}_j'))' = \lambda \underline{h}_* f_j(d) \leq \lambda h_j(t) f_j(d) < \lambda h_j(t) \left(\tilde{\psi}_j^{p_j-1-\alpha_j} + f(\tilde{\psi}_{j+1}) \right), \ t \in (0,1)$$

and similarly

$$-(\varphi_{p_n}(\tilde{\psi}_n'))' < \lambda h_n(t) \left(\tilde{\psi}_n^{p_n-1-\alpha_n} + f(\tilde{\psi}_1) \right), \ t \in (0,1),$$

which implies $(\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_n)$ will be a strict subsolution. However, (10) follows if $\tilde{\psi}_j'(t) > d'(t)$ for $t \in (0, \frac{1}{2})$ since $\psi_j(0) = 0 = d(0)$. Note that for $j = 1, 2, \ldots, n$ we obtain $\tilde{\psi}_j'(t) > d'(t)$ for $\epsilon \leq t \leq \frac{1}{2}$ because $d'(t) = 0$ and $\tilde{\psi}_j'(t) > 0$ for $\epsilon \leq t \leq \frac{1}{2}$. For $t \in (0, \epsilon)$ we have

$$\tilde{\psi}_j'(t) \geq \varphi_{p_j}^{-1} \left(\int_\epsilon^{\frac{1}{2}} \lambda \underline{h}_* f_j(d(s)) ds \right) = \varphi_{p_j}^{-1} \left(\lambda \underline{h}_* f_j(b) \left(\frac{1}{2} - \epsilon \right) \right).$$

Since $|d'(t)| \leq \frac{b\kappa\eta}{\epsilon}$, it is easy to see that $\tilde{\psi}_j'(t) > d'(t)$ for $t \in (0,\epsilon)$ provided

$$\varphi_{p_j}^{-1}\left(\lambda \underline{h}_* f_j(b)\Big(\frac{1}{2}-\epsilon\Big)\right) > \frac{\kappa\eta b}{\epsilon} \text{ for } j=1,2,\dots,n$$

or equivalently

$$\lambda > \max_{j=1,2,\dots,n}\left\{(\kappa\eta)^{p_j-1}\frac{1}{\epsilon^{p_j-1}(\frac{1}{2}-\epsilon)}\frac{b^{p_j-1}}{\underline{h}_* f_j(b)}\right\}. \tag{11}$$

Since $\lambda_* = \min_{i=1,2,\dots,n}\left\{\max_{j=1,2,\dots,n}\left\{L_{ij}\frac{b^{p_j-1}}{\underline{h}_* f_j(b)}\right\}\right\} = \max_{j=1,2,\dots,n}\left\{L_{\theta j}\frac{b^{p_j-1}}{\underline{h}_* f_j(b)}\right\}$ for some $\theta \in \{1,2,\dots,n\}$, taking $\epsilon = \frac{p_\theta-1}{2p_\theta}$ in the definition of ρ, (11) reduces to showing

$$\lambda > \max_{j=1,2,\dots,n}\left\{(\kappa\eta)^{p_j-1}L_{\theta j}\frac{b^{p_j-1}}{\underline{h}_* f_j(b)}\right\}. \tag{12}$$

We can choose $\kappa > 1$ and $\eta > 1$ such that (12) is satisfied. Hence (10) holds for $\lambda > \lambda_*$. Thus $(\tilde{\psi}_1,\tilde{\psi}_2,\dots,\tilde{\psi}_n)$ is a strict subsolution of (3) for $\lambda > \lambda_*$.
From Theorem 1.2, we have a sufficiently small subsolution $(\psi_1,\psi_2,\dots,\psi_n)$ and a sufficiently large supersolution $(\hat{\phi}_1,\hat{\phi}_2,\dots,\hat{\phi}_n)$ such that

$$(\psi_1,\psi_2,\dots,\psi_n) \leq (\tilde{\phi}_1,\tilde{\phi}_2,\dots,\tilde{\phi}_n) \leq (\hat{\phi}_1,\hat{\phi}_2,\dots,\hat{\phi}_n)$$

and

$$(\psi_1,\psi_2,\dots,\psi_n) \leq (\tilde{\psi}_1,\tilde{\psi}_2,\dots,\tilde{\psi}_n) \leq (\hat{\phi}_1,\hat{\phi}_2,\dots,\hat{\phi}_n).$$

Since $\|\tilde{\phi}_j\|_\infty = a < b \leq \|\tilde{\psi}_j\|_\infty$ for $j=1,2,\dots,n$, we obtain $(\tilde{\psi}_1,\tilde{\psi}_2,\dots,\tilde{\psi}_n) \not\leq (\tilde{\phi}_1,\tilde{\phi}_2,\dots,\tilde{\phi}_n)$. Hence (3) has at least three distinct positive solution for $\lambda \in (\lambda_*,\lambda^*)$ by Lemma 1.5. □

4. **Example.** Here we discuss an example that satisfies the hypotheses of Theorem 1.1 - 1.3. Consider the system:

$$\begin{cases} -(\varphi_{p_1}(u_1'))' = \lambda\frac{1}{t^{\beta_1}}\left(u_1^{p_1-1-\alpha_1} + e^{\frac{\tau u_2}{\tau+u_2}} - 1\right), & t \in (0,1),\\ -(\varphi_{p_2}(u_2'))' = \lambda\frac{1}{t^{\beta_2}}\left(u_2^{p_2-1-\alpha_2} + u_3^{\zeta_1}\right), & t \in (0,1),\\ \quad\vdots \qquad = \qquad \vdots & \\ -(\varphi_{p_n}(u_n'))' = \lambda\frac{1}{t^{\beta_n}}\left(u_n^{p_n-1-\alpha_n} + u_1^{\zeta_{n-1}}\right), & t \in (0,1),\\ \quad u_j(0) = 0 = u_j(1);\ j=1,2,\dots,n, & \end{cases} \tag{13}$$

where $\tau > 0$, $\beta_j \in (0,1)$, $\zeta_j > 0$ and $h_j(t) = \frac{1}{t^{\beta_j}}$ for $j=1,2,\dots,n$. Here $f_1(s) = e^{\frac{\tau s}{\tau+s}} - 1$ and $f_j(s) = s^{\zeta_{j-1}}$ for $j=2,3,\dots,n$. Clearly $f_j(0)=0$ for $j=1,2,\dots,n$. Further, (H_1) holds since f_1 is bounded for each $\tau > 0$. Hence Theorem 1.1 - 1.2 hold for all $\tau > 0$ and $\zeta_j > 0$ for $j=1,2,\dots,n-1$. Next consider the case when $\zeta_j > p_{j+1}-1$ for $j=1,2,\dots,n-1$ and $\tau \gg 1$. Choosing $a=1$ and $b=\tau$ we have:

$$\min_{j=1,2,\dots,n}\left\{\frac{1}{2\|w_{p_j}\|_\infty^{p_j-1}}\min\left\{a^{\alpha_j},\frac{a^{p_j-1}}{f_j(a)}\right\}\right\} \geq \frac{1}{4}\min_{j=1,2,\dots,n}\left\{\frac{1}{\|w_{p_j}\|_\infty^{p_j-1}}\right\}$$

and

$$\min_{i=1,2,\dots,n}\left\{\max_{j=1,2,\dots,n}\left\{L_{ij}\frac{b^{p_j-1}}{\underline{h}_* f_j(b)}\right\}\right\} \leq \frac{L^*}{\underline{h}_*}\max\left\{\frac{\tau^{p_1-1}}{e^{\frac{\tau}{2}}-1},\tau^{p_2-1-\zeta_1},\dots,\tau^{p_n-1-\zeta_{n-1}}\right\},$$

where $L^* := \max_{i,j=1,2,\ldots,n}\Big\{L_{ij}\Big\}$. It is easy to show that

$$\max\Big\{\frac{\tau^{p_1-1}}{e^{\frac{\tau}{2}}-1}, \tau^{p_2-1-\zeta_1}, \ldots, \tau^{p_n-1-\zeta_{n-1}}\Big\} \to 0 \text{ as } \tau \to \infty.$$

Thus (H_2) is satisfied for $\tau \gg 1$. Hence (13) has at least three positive solution for a certain range of λ. In fact, for a given $\lambda \in (0, \hat{\lambda})$ where

$$\hat{\lambda} := \min\Big\{\frac{1}{2\|w_{p_1}\|_\infty^{p_1-1}(e-1)}, \frac{1}{2\|w_{p_2}\|_\infty^{p_2-1}}, \ldots, \frac{1}{2\|w_{p_n}\|_\infty^{p_n-1}}\Big\},$$

there exists $\tau^* > 0$ such that (13) has at least three positive solution for $\tau > \tau^*$.

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