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An optimal result for sampling density in shift-invariant spaces generated by Meyer scaling function

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ABSTRACT

For a class of continuously differentiable function ϕ satisfying certain decay conditions, it is shown that if the maximum gap $\delta := \sup_i (x_{i+1} - x_i)$ between the consecutive sample points is smaller than a certain number B_0, then any $f \in V(\phi)$ can be reconstructed uniquely and stably. As a consequence of this result, it is shown that if $\delta < 1$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i \in \mathbb{Z}\}$, where $w_i = (x_{i+1} - x_{i-1})/2$ and ϕ is the scaling function associated with Meyer wavelet. Further, the maximum gap condition $\delta < 1$ is sharp.

1. Introduction

In [8], Gröchenig proved that if $\{x_i : i \in \mathbb{Z}\}$ is a sample set with $\sup_i (x_{i+1} - x_i) < 1$, then $\{x_i\}$ is a stable set of sampling for $V(\text{sinc})$ with respect to certain weight $\{w_i : i \in \mathbb{Z}\}$. In [2], Aldroubi and Gröchenig proved that if $X \subset \mathbb{R}$ is a separated set such that $\sup_i (x_{i+1} - x_i) < 1$, then any function in a shift-invariant space with B-spline as a generator can be reconstructed stably and uniquely from its samples $\{f(x_i) : x_i \in X\}$. In that paper, they conjectured that the theorem remains true for a much larger class of shift-invariant spaces. Since their methods use special properties of spline functions, they mentioned that it is not clear how to extend their result to shift-invariant spaces with even a compactly supported generator. Recently in [10], it is shown that for a class of totally positive functions ϕ of finite type, if $\delta := \sup_i (x_{i+1} - x_i) < h$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V_h(\phi)$, where $V_h(\phi) = \overline{\text{span}}\{\phi(\cdot - hk) : k \in \mathbb{Z}\}$.

For sampling in shift-invariant spaces, the oscillation method dates back to the work of [1] where in they used the oscillation function $osc_\delta s$, defined by $(osc_\delta s)(x) = \sup_{|y| \le \delta} |s(x) - s(x + y)|$ in order to ob-

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tain a reconstruction method for spline-like spaces. In [12], an average sampling theorem was given for shift-invariant spaces with equally spaced sample points and arbitrary averaging functions. In [13], average sampling theorems were studied for spline subspaces with standard averaging functions $\chi_{[x_k-1/2, x_k+1/2]}$. In [14], an average sampling theorem was presented for shift-invariant spline spaces along with the optimal upper bound for the support length of averaging functions. In [16], explicit bound expression for sampling inequalities was obtained for shift-invariant spline spaces.

In this paper, we consider a class of continuously differentiable functions satisfying certain decay assumptions. We also assume that $\{T_n\phi : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$ and $\operatorname{esssup}_{w\in[0,1]} \sum_{l\in\mathbb{Z}} (w+l)^2 |\widehat{\phi}(w+l)|^2 < \infty$. Then we show that if $\sup_i (x_{i+1} - x_i)$ is smaller than a certain number B_0, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i\}$, where $w_i = (x_{i+1} - x_{i-1})/2$. In order to prove the above result, first we prove a Bernstein-type inequality, namely, $\|f'\|_2 \leq 2\pi\sqrt{B}\|f\|_2$, for every $f \in V(\phi)$. This helps us to get the required bound B_0 as $\frac{1}{2\sqrt{B}}$, towards the maximum gap condition. As a consequence of sampling theorem for Meyer scaling function ϕ, we show that if $\delta < 1$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i \in \mathbb{Z}\}$. Further, we show that the maximum gap condition $\delta < 1$ is sharp. We wish to emphasize that many of the earlier papers with explicit sampling bound conditions available in the literature were far away from "sharpness" results and up to our knowledge, the present "sharp" result could not be obtained from the available conditions in the literature.

Further, we notice that in the case of Bernstein's inequality for $V(\phi)$, it is not always possible to find the exact value of B except may be in the case of functions whose Fourier transform has compact support. So finally, we show that if ϕ is a differentiable function with support $[a, b]$, then one can explicitly calculate a bound instead of B. In due course, we also provide a sampling formula for reconstructing a function f belonging to $V(\phi)$, where ϕ satisfies the above condition, from its nonuniform samples. We refer to a recent paper of the authors [4], where an explicit sampling formula using complex analysis technique is provided for reconstructing a function f belonging to $V(\phi)$, where ϕ is a compactly supported even function, from its uniform samples.

2. Notations and background

Definition 2.1. A sequence of vectors $\{f_n : \ n \in \mathbb{Z}\}$ in a separable Hilbert space $\mathcal{H}$ is said to be a *Riesz basis* if $\overline{span\{f_n\}} = \mathcal{H}$ and there exist constants $0 < c \leq C < \infty$ such that

$$c \sum_{n\in\mathbb{Z}} |d_n|^2 \leq \Big\| \sum_{n\in\mathbb{Z}} d_n f_n \Big\|^2_{\mathcal{H}} \leq C \sum_{n\in\mathbb{Z}} |d_n|^2, \tag{2.1}$$

for all $(d_n) \in \ell^2(\mathbb{Z})$. Equivalently, a Riesz basis is an image of an orthonormal basis under a bounded invertible operator.

Definition 2.2. A sequence of vectors $\{f_n : \ n \in \mathbb{Z}\}$ in a separable Hilbert space $\mathcal{H}$ is said to be a *frame* if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2_{\mathcal{H}} \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_{\mathcal{H}}|^2 \leq B\|f\|^2_{\mathcal{H}}, \tag{2.2}$$

for every $f \in \mathcal{H}$.

For each $x,\ w \in \mathbb{R}$ and $s > 0$, let $T_x f(t) := f(t-x)$, $M_w f(t) := e^{2\pi iwt} f(t)$, $f \in L^2(\mathbb{R})$.

Definition 2.3. Given a non-zero function $g \in L^2(\mathbb{R})$ and $a, b > 0$, the set of time-frequency shifts

$$\mathcal{G}(g, a, b) := \{T_{am} M_{bk} g : m, k \in \mathbb{Z}\}$$

is called a Gabor system. If $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$, then it is called a Gabor frame or Weyl–Heisenberg frame.

Definition 2.4. A closed subspace M of $L^2(\mathbb{R})$ is called a *shift invariant space* if $T_n\phi \in M$, for every $\phi \in M$ and $n \in \mathbb{Z}$, where T_n is the translation operator defined by $T_n\phi(x) = \phi(x-n)$, for all $x \in \mathbb{R}$. For $\phi \in L^2(\mathbb{R})$, $\overline{span\{T_n\phi : n \in \mathbb{Z}\}}$ is called shift invariant space generated by ϕ and denoted by $V(\phi)$.

Every Riesz basis is a frame. It is well known that $\{T_n\phi : n \in \mathbb{Z}\}$ is a Riesz basis for $V(\phi)$ if and only if

$$0 < c \leq G_\phi(w) \leq C < \infty \qquad \text{a.e. } w \in \mathbb{R}, \tag{2.3}$$

where $G_\phi(w) := \sum_{n\in\mathbb{Z}} |\widehat{\phi}(w+n)|^2$. Here $\widehat{\phi}$ denotes the Fourier transform of ϕ, defined by $\widehat{\phi}(w) := \int_{-\infty}^{\infty} \phi(x)e^{-2\pi iwx}\, dx$. Moreover, $\|G_\phi\|_0 := \operatorname*{essinf}_{w\in[0,1]} G_\phi(w)$ and $\|G_\phi\|_\infty := \operatorname*{esssup}_{w\in[0,1]} G_\phi(w)$ are the optimal Riesz bounds for $\{T_n\phi : n \in \mathbb{Z}\}$. For a detailed study of sampling and reconstruction in shift-invariant spaces, we refer to [3].

Definition 2.5. Let $\{x_n : n \in \mathbb{Z}\}$ be a sequence of real or complex numbers. Then

(*i*) $\{x_n : n \in \mathbb{Z}\}$ is separated if there exists $\gamma > 0$ such that $\inf_{m\neq n} |x_n - x_m| \geq \gamma$.
(*ii*) $\{x_n\}$ is δ-dense if $\sup_n |x_{n+1} - x_n| = \delta$.
(*iii*) $\{x_n : n \in \mathbb{Z}\}$ is said to be a set of uniqueness for $V(\phi)$ if $f(x_n) = 0$, for all n, implies $f \equiv 0$.
(*iv*) $\{x_n : n \in \mathbb{Z}\}$ is said to be a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_n \in \mathbb{R}^* : n \in \mathbb{Z}\}$ if there exist constants $r,\ R > 0$ such that

$$r\|f\|_{L^2(\mathbb{R})} \leq \Big(\sum_{n\in\mathbb{Z}} w_n |f(x_n)|^2\Big)^{1/2} \leq R\|f\|_{L^2(\mathbb{R})}, \tag{2.4}$$

for every $f \in V(\phi)$.

Now we shall state a few well known theorems, which will be useful later in order to prove our main results.

Consider the infinite system

$$\sum_{k\in\mathbb{Z}} d_k \phi(x_j - k) = f(x_j), \quad j \in \mathbb{Z}.$$

Let U be the infinite matrix with entries

$$U_{jk} = [w_j \phi(x_j - k)], \quad j, k \in \mathbb{Z}.$$

Then the following result gives an equivalent condition for a stable set of sampling for $V(\phi)$ in terms of the operator U.

Theorem 2.1. *(cf. [3]) Let $V(\phi)$ be a reproducing kernel Hilbert space such that ϕ satisfies (2.3). Then the following statements are equivalent.*

(1) *The set $X = \{x_j : j \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_j : j \in \mathbb{Z}\}$.*
(2) *There exist A, $B > 0$ such that*

$$A\|D\|^2_{\ell^2(\mathbb{Z})} \leq \|UD\|^2_{\ell^2(\mathbb{Z})} \leq B\|D\|^2_{\ell^2(\mathbb{Z})},$$

for every $D \in \ell^2(\mathbb{Z})$.

The following result gives a basic connection between Gabor frames and stable set of sampling in a shift-invariant space.

Theorem 2.2. *(cf. [10]) The Gabor system $\mathcal{G}(g, 1, 1)$ is a frame for $L^2(\mathbb{R})$ if and only if each set $x + \mathbb{Z}$ is a stable set of sampling for $V(g)$ with uniform constants independent of $x \in \mathbb{R}$.*

The following result shows that a function g generating a Gabor frame cannot be well localized in both time and frequency.

Theorem 2.3 *(Balian–Low). (cf. [9]) If $\mathcal{G}(g, 1, 1)$ is a frame for $L^2(\mathbb{R})$, then either $xg \notin L^2(\mathbb{R})$ or $w\widehat{g} \notin L^2(\mathbb{R})$.*

The following inequality brings a relation between the L^2-norm of the function and its derivative.

Theorem 2.4 *(Wirtinger's inequality). (cf. [6]) Let f be a complex valued function defined on the interval $[a, b]$. If $f \in C^1\,[a, b]$ with $f(a) = f(b) = 0$, then*

$$\int_a^b |f(x)|^2 \, \mathrm{d}x \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 \, \mathrm{d}x. \tag{2.5}$$

The equality holds iff

$$f(x) = c \sin \frac{\pi(x-a)}{(b-a)}, \quad c \in \mathbb{C}.$$

3. Sampling density for shift-invariant spaces generated by functions having moderate decay

As mentioned earlier, it is well known that if $X \subset \mathbb{R}$ is a separated set such that $\sup_i (x_{i+1} - x_i) < 1$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for a shift-invariant space generated by a B-spline function. Further, for a class of totally positive functions of finite type ≥ 2, if $\sup_i (x_{i+1} - x_i) < h$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V_h(\phi) = \overline{\mathrm{span}}\{\phi(\cdot - hk) : k \in \mathbb{Z}\}$. We wish to look at the sampling density for shift-invariant spaces associated with a class of continuously differentiable functions having certain decay. In fact, let $\mathcal{A}$ denote the class of continuously differentiable functions ϕ such that

(i) $|\phi(x)| \leq \dfrac{C_1}{|x|^{0.5+\epsilon}}$ and $|\phi'(x)| \leq \dfrac{C_2}{|x|^{0.5+\epsilon}}$, for sufficiently large x, where C_1, C_2, ϵ are positive constants,
(ii) $\operatorname*{esssup}_{w \in [0,1]} \sum_{l \in \mathbb{Z}} (w + l)^2 |\widehat{\phi}(w + l)|^2 < \infty$.

Clearly if ϕ is a continuously differentiable function with compact support, then $\phi \in \mathcal{A}$. Let us define, for $\phi \in \mathcal{A}$,

$$B(w) := \frac{\sum_{l\in\mathbb{Z}} (w+l)^2 |\widehat{\phi}(w+l)|^2}{\sum_{l\in\mathbb{Z}} |\widehat{\phi}(w+l)|^2}, \ w \in [0,1], \tag{3.1}$$

$B = \operatorname*{esssup}_{w\in[0,1]} B(w)$ and $B_0 = \dfrac{1}{2\sqrt{B}}$. Then it was shown in [11] that $B \geq \dfrac{1}{4}$, under the assumption that $\{T_n\phi : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$.

Theorem 3.1. *Let $\phi \in \mathcal{A}$ be such that $\{T_n\phi : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$. Then we have the Bernstein-type inequality*

$$\|f'\|_2 \leq 2\pi\sqrt{B}\|f\|_2, \tag{3.2}$$

for every $f \in V(\phi)$.

Proof. Clearly $B = \operatorname*{esssup}_{w\in[0,1]} B(w) < \infty$. Now let $f \in V(\phi)$. Then it is easy to show that $f(x) = \sum_{k\in\mathbb{Z}} c_k\phi(x-k)$, and $f'(x) = \sum_{k\in\mathbb{Z}} c_k\phi'(x-k)$, $(c_k) \in \ell^2(\mathbb{Z})$. (The equality holds pointwise as ϕ satisfies (i) in the definition of $\mathcal{A}$.) Let $m_f(w) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi ikw}$. Then using Plancherel identity, it follows that

$$\begin{aligned}
\|f'\|_2^2 = \|\widehat{f'}\|_2^2 &= \Big\|\sum_{k\in\mathbb{Z}} c_k\widehat{\phi'}(\cdot - k)\Big\|_2^2 \\
&= \int_{-\infty}^{\infty} \Big|\sum_{k\in\mathbb{Z}} 2\pi i c_k w\widehat{\phi}(w)e^{-2\pi ikw}\Big|^2 \, dw \\
&= 4\pi^2 \int_{-\infty}^{\infty} \big|m_f(w)w\widehat{\phi}(w)\big|^2 \, dw \\
&= 4\pi^2 \int_0^1 |m_f(w)|^2 \sum_{l\in\mathbb{Z}} (w+l)^2|\widehat{\phi}(w+l)|^2 \, dw \\
&= 4\pi^2 \int_0^1 B(w)|m_f(w)|^2 \sum_{l\in\mathbb{Z}} |\widehat{\phi}(w+l)|^2 \, dw \\
&\leq 4\pi^2 B\|f\|_2^2. \quad \square
\end{aligned}$$

Remark 3.1. The constant $2\pi\sqrt{B}$ cannot be improved. The proof of this statement follows similar lines as in the proof of Theorem 1 in [5].

We observe the following.

Remark 3.2. If $\{x_i\}$ is a separated set of zeros of $f \in V(\phi)$ such that $x_i < x_{i+1}$, $i \in \mathbb{Z}$, then $\bigcup_i [x_i, x_{i+1}] = \mathbb{R}$.

Remark 3.3. If f is of the form

$$f(x) = c \sin \frac{\pi(x - x_i)}{(x_{i+1} - x_i)}, \quad c \in \mathbb{C},$$

in the interval $[x_i, x_{i+1}]$, then

$$\begin{aligned}
\|f\|_2^2 &= \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} \left| c \sin \frac{\pi(x - x_i)}{(x_{i+1} - x_i)} \right|^2 \, \mathrm{d}x \\
&= |c|^2 \sum_{i \in \mathbb{Z}} \frac{x_{i+1} - x_i}{\pi} \int_0^{\pi} \sin^2 u \, \mathrm{d}u \\
&= \frac{|c|^2}{2} \sum_{i \in \mathbb{Z}} (x_{i+1} - x_i).
\end{aligned}$$

Since $\{x_i : i \in \mathbb{Z}\}$ is separated, there exists $\gamma > 0$ such that $x_{i+1} - x_i \geq \gamma$ from which it follows that $\|f\|_2 = \infty$.

Theorem 3.2. *Let $\phi \in \mathcal{A}$ be such that $\{T_n \phi : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$. If a non-zero function $f \in V(\phi)$ has infinitely many zeros on the real axis which are separated, then there exists at least one pair of consecutive zeros whose distance apart is greater than B_0.*

The proof of this theorem is similar to the proof of Theorem 2.1 in [15], where in we make use of Wirtinger's inequality for the interval $[x_i, x_{i+1}]$ and the Remarks 3.2 and 3.3.

Corollary 3.1. *Let $f \in V(\phi)$ be such that $f(x_i) = 0$ for all i. If $\{x_i : i \in \mathbb{Z}\}$ is a separated set such that $\sup_i (x_{i+1} - x_i) \leq B_0$, then $f \equiv 0$.*

Let $\{x_i : i \in \mathbb{Z}\}$, $\cdots < x_{i-1} < x_i < x_{i+1} < \cdots$, be a sample set of density δ. Let $y_i = \frac{x_i + x_{i+1}}{2}$. Consider the approximation operator

$$Af = P\left(\sum_{i \in \mathbb{Z}} f(x_i) \chi_{[y_i, y_{i+1}]} \right),$$

where P is an orthogonal projection of $L^2(\mathbb{R})$ onto $V(\phi)$. Then one has the following

Theorem 3.3. *Let $\phi \in \mathcal{A}$ be such that $\{T_n \phi : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$. If $\sup_i (x_{i+1} - x_i) < B_0$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i \in \mathbb{Z}\}$, where $w_i = \frac{x_{i+1} - x_{i-1}}{2}$.*

We refer to [8] and Theorem 8.14 of [7] for the proof.

Example 3.1. Consider the Littlewood–Paley wavelet

$$\phi(x) := \frac{\sin \pi x}{\pi x}(2 \cos \pi x - 1).$$

Its Fourier transform is given by

$$\widehat{\phi}(w) = \begin{cases} 1 & \text{if } 1/2 \leq |w| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The collection $\{T_n\phi : n \in \mathbb{Z}\}$ is an orthonormal basis for $V(\phi)$ and $\sum_{l\in\mathbb{Z}} |\widehat{\phi}(w+l)|^2 = 1$ almost everywhere. Therefore,

$$B(w) = \sum_{l\in\mathbb{Z}} (w+l)^2 |\widehat{\phi}(w+l)|^2.$$

But one can easily see that $B(w) = 2$ if $w = 0$ or $1/2$, $B(w) = (w-1)^2$ on $(0, 1/2)$ and $B(w) = w^2$ on $(1/2, 1]$. Hence $B = \operatorname*{ess\,sup}_{w\in[0,1]} B(w) = 1$ and $B_0 = \frac{1}{2}$. Thus if $\sup_i (x_{i+1} - x_i) < \frac{1}{2}$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i \in \mathbb{Z}\}$, where $w_i = \frac{x_{i+1} - x_{i-1}}{2}$.

Remark 3.4. The Theorem 3.3 does not give the sharp bound for all shift-invariant spaces $V(\phi)$ generated by functions belonging to $\mathcal{A}$. For example, if ϕ is a B-spline function of order ≥ 2, then the constant value $B_0 \approx 0.9$. (cf. [5]). However, Aldroubi and Gröchenig proved that the best constant value of the maximum gap is 1. Thus it becomes a natural question to investigate whether we can find a function belonging to $\mathcal{A}$ for which Theorem 3.3 gives the sharp bound. If ϕ belongs to the class of functions proposed by Meyer, then we answer the question affirmatively. This is established in Theorem 4.1, in fact, which is the essence of the paper.

Now, by Theorem 3.1, the matrix $U_{jk} = [w_j \phi(x_j - k)]$ with $w_j = \frac{x_{j+1} - x_{j-1}}{2}$, $j, k \in \mathbb{Z}$, is bounded above and below. Hence the pseudo inverse of U exists and is given by $U^\dagger = (U^*U)^{-1}U^*$. Then we have the following sampling formula.

Theorem 3.4. *Let $\phi \in \mathcal{A}$ be such that $\{T_n\phi : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$. Then every $f \in V(\phi)$ can be reconstructed uniquely from the sample set $\{x_j : j \in \mathbb{Z}\}$ by the formula*

$$f(x) = \sum_{j\in\mathbb{Z}} f(x_j)\psi_j(x), \tag{3.3}$$

where $\psi_j(x) := \sum_{k\in\mathbb{Z}} (U^\dagger)_{kj} \phi(x - k)$.

Proof. Let $f \in V(\phi)$. Then there exists a unique sequence $D = (d_k) \in \ell^2(\mathbb{Z})$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} d_k \phi(x-k), \qquad x \in \mathbb{R}.$$

Notice that $UD = \{w_j f(x_j)\}_{j\in\mathbb{Z}}$. Define $L(f) := \{w_j f(x_j)\}_{j\in\mathbb{Z}}$. Since the matrix U^*U is invertible, we have

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} \left[U^\dagger L(f)\right]_k \phi(x-k) \\ &= \sum_{k\in\mathbb{Z}} \Big[\sum_{j\in\mathbb{Z}} (U^\dagger)_{kj} w_j f(x_j)\Big] \phi(x-k) \end{aligned}$$

$$= \sum_{j\in\mathbb{Z}} w_j f(x_j) \sum_{k\in\mathbb{Z}} (U^\dagger)_{kj}\phi(x-k)$$
$$= \sum_{j\in\mathbb{Z}} w_j f(x_j)\psi_j(x),$$

where $\psi_j(x) = \sum_{k\in\mathbb{Z}} (U^\dagger)_{kj}\phi(x-k)$. □

As mentioned earlier, in the case of Bernstein's inequality for $V(\phi)$, it is not always possible to find the exact value of B except may be in the case of functions whose Fourier transform has compact support. In the next theorem, we show that if ϕ is a differentiable function with support $[a, b]$, then one can explicitly calculate a bound instead of B.

Theorem 3.5. *Let ϕ be a differentiable function with support $[a, b]$ such that $\{T_n\phi : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$. Then for any $f \in V(\phi)$, we have*

$$\|f'\|_2 \le \sqrt{(1+b-a)}\frac{\|\phi'\|_2}{\|G_\phi\|_0}\|f\|_2. \tag{3.4}$$

Proof. Let $f \in V(\phi)$. Then $f(x) = \sum_{k\in\mathbb{Z}} c_k\phi(x-k)$ and $f'(x) = \sum_{k\in\mathbb{Z}} c_k\phi'(x-k)$, where $(c_k) \in \ell^2(\mathbb{Z})$. Consider

$$\widehat{f}\left(\frac{w}{b-a}\right) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} c_k\phi(x-k)e^{-2\pi iwx/b-a}\mathrm{d}x$$
$$= \lambda_f(w) \int_{\mathbb{R}} \phi(y)e^{-2\pi iwy/b-a}\mathrm{d}x$$
$$= (b-a)\lambda_f(w)\widehat{\phi}_{ab}(w), \tag{3.5}$$

where $\lambda_f(w) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi ikw/b-a}$ and $\widehat{\phi}_{ab}(w) = \frac{1}{b-a}\int_a^b \phi(t)e^{-2\pi iwt/b-a}\,\mathrm{d}t$.

Similarly, we can show that

$$\widehat{f'}\left(\frac{w}{b-a}\right) = (b-a)\lambda_f(w)\widehat{\phi'}_{ab}(w), \tag{3.6}$$

where $\widehat{\phi'}_{ab}(w) = \frac{1}{b-a}\int_a^b \phi'(t)e^{-2\pi iwt/b-a}\,\mathrm{d}t$. On the other hand, one can easily show that

$$\sum_{n\in\mathbb{Z}} |\widehat{\phi'}_{ab}(w+n)|^2 = \frac{1}{b-a}\|\phi'\|_2^2. \tag{3.7}$$

Consider

$$\|f'\|_2^2 = \|\widehat{f'}\|_2^2 = \frac{1}{b-a}\left\|\widehat{f'}\left(\frac{1}{b-a}\cdot\right)\right\|_2^2$$
$$= (b-a)\int_{\mathbb{R}} |\lambda_f(w)|^2|\widehat{\phi'_{ab}}(\omega)|^2\,\mathrm{d}w$$

$$= (b-a) \int_a^b |\lambda_f(w)|^2 \sum_{l\in\mathbb{Z}} |\widehat{\phi'_{ab}}(w+(b-a)l)|^2 \, \mathrm{d}w. \tag{3.8}$$

Since any interval of length 1 contains at most $1 + \frac{1}{b-a}$ distinct points of minimum distance $b-a$ between each other, we have

$$\begin{aligned} \|f'\|_2^2 &\leq (b-a) \int_a^b |\lambda_f(w)|^2 \left(1+\frac{1}{b-a}\right) \sup_{w\in[0,1]} \sum_{l\in\mathbb{Z}} |\widehat{\phi'_{ab}}(w+l)|^2 \, \mathrm{d}w \\ &= \left(1+\frac{1}{b-a}\right) \|\widehat{\phi'}\|_2^2 \int_a^b |\lambda_f(w)|^2 \, \mathrm{d}w, \end{aligned} \tag{3.9}$$

using (3.7). (We refer to Lemma 6.1.2 of [9] in this context.) Therefore,

$$\begin{aligned} \|f'\|_2^2 &\leq \left(1+\frac{1}{b-a}\right) \|\widehat{\phi'}\|_2^2 (b-a) \|c\|_{\ell^2(\mathbb{Z})}^2 \\ &\leq (1+b-a) \frac{\|\phi'\|_2^2}{\|G_\phi\|_0^2} \|f\|_2^2. \quad \square \end{aligned}$$

4. Sampling density condition for Meyer scaling function

Consider a function $\vartheta(w)$ defined on the interval $0 \leq w \leq 1$ satisfying the following properties:

(P_1) $0 \leq \vartheta(w) \leq 1$,
(P_2) $\vartheta(w) + \vartheta(1-w) = 1$,
(P_3) $\vartheta(w)$ is a monotonically decreasing function,
(P_4) $\vartheta(w) = 1$, $0 \leq w \leq \frac{1}{3}$.

The function ϑ is extended to the real line by setting $\vartheta(w) = \vartheta(-w)$ for $-1 \leq w \leq 0$ and $\vartheta(w) = 0$ for $|w| > 1$. Then the Meyer scaling function is defined as

$$\phi(x) := \int_{-1}^{1} \sqrt{\vartheta(w)} e^{2\pi i w x} \mathrm{d}w.$$

Since collection $\{T_n\phi : n \in \mathbb{Z}\}$ forms an orthonormal basis for $V(\phi)$, we get $\sum_{l\in\mathbb{Z}} |\widehat{\phi}(w+l)|^2 = 1$ a.e., $w \in \mathbb{R}$. Therefore,

$$B(w) = \sum_{l\in\mathbb{Z}} (w+l)^2 |\widehat{\phi}(w+l)|^2.$$

In this case,

$$B(w) = \begin{cases} w^2 & \text{if } 0 \leq w \leq 1/3, \\ (w-1)^2\vartheta(w-1) + w^2\vartheta(w) & \text{if } 1/3 \leq w \leq 2/3, \\ (w-1)^2 & \text{if } 2/3 \leq w \leq 1. \end{cases}$$

Since $B \geq \frac{1}{4}$, $B(w)$ attains its maximum in the interval $[1/3, 2/3]$. Therefore,

$$B = \sup_{1/3 \leq w \leq 2/3} (w-1)^2 \vartheta(w-1) + w^2 \vartheta(w)$$
$$= \sup_{1/3 \leq w \leq 2/3} (w-1)^2 + (2w-1)\vartheta(w).$$

The function ϑ satisfies $1/2 \leq \vartheta(w) \leq 1$ and $2w - 1 \leq 0$ in the interval $[1/3, 1/2]$. Further, $0 \leq \vartheta(w) \leq 1/2$ and $2w - 1 \geq 0$ in the interval $[1/2, 2/3]$. Hence we get

$$B(w) = (w-1)^2 + (2w-1)\vartheta(w) \leq (w-1)^2 + \frac{1}{2}(2w-1)$$
$$= w^2 - w + \frac{1}{2} \leq \frac{5}{18},$$

in $[1/3, 2/3]$, using standard calculus techniques. Consequently, $\frac{1}{4} \leq B \leq \frac{5}{18}$ and $B_0 \geq 0.9487$. In addition, if the function $\vartheta(w)$ satisfies the property

$$(P_5) : \vartheta(w) \geq 2 - 3w \text{ in } [1/3, 1/2] \text{ and } \vartheta(w) \leq 2 - 3w \text{ in } [1/2, 2/3],$$

then

$$B(w) = (w-1)^2 + (2w-1)\vartheta(w) \leq (w-1)^2 + (2w-1)(2-3w)$$
$$= -5w^2 + 5w - 1 \leq \frac{1}{4},$$

in $[1/3, 2/3]$. In this case, $B = \frac{1}{4}$ and $B_0 = 1$. Thus, if ϑ is a C^∞ function satisfying the properties (P_1) to (P_5), then we have the following: If $\sup_i (x_{i+1} - x_i) < 1$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i \in \mathbb{Z}\}$, where $w_i = (x_{i+1} - x_{i-1})/2$. We arrive at this conclusion using Theorem 3.3.

There are many functions satisfying the properties (P_1) to (P_5). We now confine ourselves to the class of functions proposed by Meyer. Consider the function

$$\widehat{\phi}(w) = \begin{cases} 1 & |w| \leq \frac{1}{3}, \\ \cos\left[\frac{\pi}{2}\nu(3|w|-1)\right] & \frac{1}{3} \leq |w| \leq \frac{2}{3}, \\ 0 & \text{otherwise,} \end{cases}$$

where ν is a C^∞ function satisfying

$$\nu(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1, \end{cases}$$

with the additional property

$$\nu(x) + \nu(1-x) = 1.$$

For ν, one can take the Meyer polynomials, ν_r $(r \in \mathbb{N})$ which is defined using beta function. More precisely,

$$\nu_r(x) = \frac{1}{B(r,r)} \int_0^x t^{r-1}(1-t)^{r-1} dt, \quad 0 \leq x \leq 1,$$

Table 1
Meyer polynomials.

r	$\nu_r(x)$
1	x
2	$x^2(3-2x)$
3	$x^3(10-15x+6x^2)$
4	$x^4(35-84x+70x^2-20x^3)$
5	$x^5(126-420x+540x^2-315x^3+70x^4)$
6	$x^6(462-1980x+3465x^2-3080x^3+1386x^4-252x^5)$

where $B(r,s)=\int\limits_0^1 t^{r-1}(1-t)^{s-1}dt$ is the beta function. It is clear that

$$\begin{aligned}1-\nu_r(1-x) &= \frac{1}{B(r,r)}\Big[\int\limits_0^1 t^{r-1}(1-t)^{r-1}dt - \int\limits_0^{1-x} t^{r-1}(1-t)^{r-1}dt\Big]\\ &= \frac{1}{B(r,r)}\int\limits_{1-x}^1 t^{r-1}(1-t)^{r-1}\mathrm{d}t = \nu_r(x).\end{aligned}$$

The first six Meyer ν_r-polynomials are given in the Table 1.

The functions ϕ and $\widehat{\phi}$ are rapidly decreasing functions. It is easy to show that $[\widehat{\phi}(w)]^2$ satisfies the properties (P_1) to (P_4).

Now, we show that $[\widehat{\phi}(w)]^2$ satisfies the property (P_5). Since $\nu_r(x)$ is a convex function in the interval $[1/3,1/2]$ and $\cos^2 x$ is a concave decreasing function in the interval $[0,\pi/4]$, the function $[\widehat{\phi}(w)]^2$ is concave in $[1/3,1/2]$. Therefore, each point on the chord between $[1/3,1]$ and $[1/2,1/2]$ is below the graph of $[\widehat{\phi}(w)]^2$. Hence, the function $[\widehat{\phi}(w)]^2$ satisfies the property $[\widehat{\phi}(w)]^2 \geq 2-3w$ in $[1/3,1/2]$. Using property (2), we get

$$\begin{aligned}\widehat{\phi}(-w+\frac{1}{2})^2-\frac{1}{2} &= \cos^2\left[\frac{\pi}{2}\nu(-3w+\frac{1}{2})\right]-\frac{1}{2}\\ &= \cos^2\left[\frac{\pi}{2}\big[1-\nu(3w+\frac{1}{2})\big]\right]-\frac{1}{2}\\ &= \sin^2\left[\frac{\pi}{2}\nu(3w+\frac{1}{2})\right]-\frac{1}{2}\\ &= 1-\cos^2\left[\frac{\pi}{2}\nu(3w+\frac{1}{2})\right]-\frac{1}{2}\\ &= -\big[\widehat{\phi}(w+\frac{1}{2})^2-\frac{1}{2}\big].\end{aligned}$$

Therefore, $[\widehat{\phi}(w)]^2$ is anti-symmetric with respect to the line $y=\frac{1}{2}$. Since $[\widehat{\phi}(w)]^2$ is concave in $[1/3,1/2]$, $[\widehat{\phi}(w)]^2$ is convex in $[1/2,2/3]$. Thus, the function $[\widehat{\phi}(w)]^2$ satisfies the property $[\widehat{\phi}(w)]^2 \leq 2-3w$ in $[1/2,2/3]$. Hence $B=\frac{1}{4}$.

Further, we shall show that the maximum gap condition $(\delta<1)$ is sharp. In other words, we show that if $\delta=1$, then there exists $x\in\mathbb{R}$ such that $x+\mathbb{Z}$ is not a stable set of sampling for $V(\phi)$. Suppose that were not true. Then each set $\{x_i\}$ with $\delta=1$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i\in\mathbb{Z}\}$, where $w_i=(x_{i+1}-x_{i-1})/2$, i.e., there exists $A,B>0$ such that

$$A\|f\|_2^2 \leq \sum_{i\in\mathbb{Z}} w_i|f(x_i)|^2 \leq B\|f\|_2^2,$$

for every $f \in V(\phi)$. In particular, $x+\mathbb{Z}$ is a stable set of sampling for $V(\phi)$ for every $x \in \mathbb{R}$ with bounds A, B. Then it follows from by Theorem 2.2 that $\mathcal{G}(\phi, 1, 1)$ is a frame for $L^2(\mathbb{R})$. Hence by Balian–Low theorem, either $x\phi \notin L^2(\mathbb{R})$ or $\xi\widehat{\phi} \notin L^2(\mathbb{R})$. But both $x\phi, w\widehat{\phi} \in L^2(\mathbb{R})$ as $\phi, \widehat{\phi}$ are rapidly decreasing functions, which is a contradiction. Hence $\delta < 1$ is sharp. Thus we have proved the following

Theorem 4.1. *Let ϑ be a C^∞ function satisfying the properties (P_1) to (P_5). Let*

$$\phi(x) := \int_{-1}^{1} \sqrt{\vartheta(\xi)} e^{2\pi i w x} \mathrm{d}w$$

be the Meyer scaling function. Suppose $\delta = \sup_i (x_{i+1} - x_i) < 1$. Then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i \in \mathbb{Z}\}$, where $w_i = (x_{i+1} - x_{i-1})/2$. Further, the maximum gap condition $\delta < 1$ is sharp.

Remark 4.1. Consider a totally positive function ψ of finite type ≥ 2, i.e., $\widehat{\psi}(w) = \prod_{\nu=1}^{M} (1 + 2\pi i \delta_\nu w)^{-1}$, $\delta_\nu \in \mathbb{R}^*$ and $M \geq 2$. It is clear that the Fourier transform of ψ cannot have compact support. However, the function $\sqrt{\vartheta} = \widehat{\phi}$ considered in Theorem 4.1 has compact support. This means ϕ is not a totally positive function of finite type ≥ 2. Hence the Theorem 4.1 cannot be deduced from Theorem 2 in [10].

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